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The Electric Field at the Ground Plane Near a Top-Loaded Monopole Antenna with Special Regard to Electrically Small L- and T-Antennas

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The present article deals with the calculation of the electric field strength at the ground plane near electrically small top-loaded antennas having a known current distribution, with special reference to L- and T-antennas. The formulas and numerical values obtained here for this component may be used in calculating the contribution to the ground losses around an antenna of the above-mentioned type due to the vertical component of the earth current.

An exact expression involving an integral has been obtained for the electric field strength at the ground plane due to the current in a linear antenna having an arbitrary inclination. If the length and the height of the antenna is small compared to the wavelength, and if the current distribution on the antenna can be expressed by a finite number of terms of a power series, it is theoretically possible to obtain a closed expression for the field at the ground plane. However, only in special cases does this expression become sufficiently simple to be of practical value for numerical calculations.

Working formulas have been obtained and numerical calculations carried out for the near zone field of an electrically small vertical or horizontal antenna with a linear current distribution. Based on these results, a calculation has been made of the electric field strength at the ground plane near electrically small L- and T-antennas. Also the relative contribution to this component due to the top loading has been calculated.

1. Introduction

In calculating the ground losses around an antenna, it is customary to assume that the current is flowing in an infinitely thin layer at the surface of the ground so that the current distribution for the purpose of loss calculation may be completely described by a surface current density. However, as has been pointed out by Wait [1],[2] the vertical component of the ground current may in some cases contribute considerably to the ground losses. This is particularly the case if the antenna has a large top loading so that a large current will flow from the top loading to the ground. If, further, the antenna is placed on an island of the same order of magnitude as the top loading so that the ground losses outside the aperture of the antenna are insignificant, the contribution to the ground losses coming from the vertical component of the current distribution may be expected to become still more predominant.

The vertical component of the current density in the earth at the surface of the ground is proportional to the vertical component of the electric field strength at the surface. (With perfect ground conductivity the electric field strength has only a vertical component, and in the case of finite ground conductivity the electric field may often with good approximation be calculated as if the ground was perfectly conducting.) In order to be able to calculate the contribution to the ground losses arising from the vertical component of the current density, we must therefore know the vertical component of the electric field strength at the surface of the earth. The calculation of this component is the object of the present article.

The calculations are carried out under the assumption that the current distribution on the antenna and on the top loading is known. We shall, in particular, consider the case where the top loading consists of a few inclined, or as a special case, horizontal wires. In this case the current distribution on the antenna and on the top loading may be expected to be approximately linear. Working formulas will be developed for and numerical calculations made of the electric

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[2] Figures in brackets indicate the literature references at the end of this paper.

field strength at the ground plane near electrically small L-antennas, T-antennas, and also antennas with four horizontal top-loading wires.

All of the antennas which are considered in this paper are made up of a vertical wire and a top loading consisting of straight wires. We will, therefore, start by developing a formula for the vertical component of the electric field strength at the ground plane, $(E_z)_{z=0}$, due to the current in an inclined wire placed above the ground plane and having a known current distribution. This formula may then be used for finding $(E_z)_{z=0}$ for any one of the antennas considered in this article.

The models used in developing the formulas in this paper are the same as those used in an earlier paper by Knudsen [2] dealing with the surface current density near a top-loaded monopole. However, whereas in the former paper the vector potential method was used in finding the surface current density, it has here been found expedient to use the Hertz dipole method in calculating the electric field strength. The notation used in the report on the surface current density [2] has been used here to as large an extent as seemed practical.

2. List of Principal Symbols

a = length of radial wires in top loading,
A_m = coefficient in power series expansion for current distribution function $g(\sigma)$,
b = height over ground plane of one end of inclined wire,
$B = -(\rho/c) \sin\alpha \cos(\phi-\beta) - (b/c)\cos\alpha$,
c = length of inclined wire,
$C = (\rho/c)^2 + (b/c)^2$,
d = reference distance (arbitrary),
$\bar{e}$ = normalized electric field strength at ground plane due to current in inclined wire or top-loaded antenna: $\bar{e} = (kd^2/\zeta I_0)\bar{E}$,
$\bar{e}^h$ = normalized electric field strength at ground plane due to current in one horizontal wire,
$\bar{e}^H$ = normalized electric field strength at ground plane due to current in horizontal members of T-antenna,
$\bar{e}^v$ = normalized electric field strength at ground plane due to current in vertical wire,
$\bar{e}^+$ = normalized electric field strength at ground plane due to current in horizontal members of antenna with four top-loading wires,
$f(s)$ = current distribution function for inclined wire in terms of s,
$g(\sigma)$ = current distribution function for inclined wire in terms of σ,
h = length of vertical member of antenna,
$I(s)$ = current on inclined wire,
I_0 = reference current (arbitrary),
I_b = current in vertical member at ground plane,
$k = \omega\sqrt{\mu\epsilon}$, propagation constant,
n = number of wires in top loading,
$N = C - B^2$,
$\bar{R}$ = vector from point of antenna to point of ground plane $(\rho,\phi,0)$,
$\hat{R}$ = unit vector coparallel to $\bar{R}$: $\hat{R} = \bar{R}/R$,
s = coordinate along inclined wire,
$\hat{t}$ = unit vector pointing in positive direction of inclined wire,
$\hat{x}$ etc. = unit vector in x-direction etc.,
$X = (R/c)^2$,
$X_1 = 1 + 2B + C$,
α = angle between downwards vertical direction (negative z-axis) and positive direction $(\hat{t})$ of inclined wire,
β = azimuth of inclined wire,
ϵ = dielectric constant,
ζ = intrinsic impedance of free space $\sqrt{\mu/\epsilon}$,
μ = permeability,
$\xi = \rho/h$, normalized distance from base of antenna,
$\rho,\phi,0$ = coordinates of field point at ground plane,
σ = normalized coordinate s/c along inclined wire,
$\tau = a/h$, normalized length of the top-loading wires,
$\psi = \phi - \beta$.

3. Electric Field at Ground Plane Due to Inclined Wire

Let us consider a straight piece of wire of the length c with one of its end points located on the z-axis of a rectangular coordinate system (x,y,z) at a height b above the horizontal (x,y)-plane, the wire forming an angle α with the negative z-axis, and the vertical plane containing the wire forming the angle β with the (z,x)-plane as shown in figure 1. A coordinate s along the wire is introduced, $s=0$ corresponding to the point of the z-axis. We assume that the wire carries a current $I(s)=I_0 f(s)$ where I_0 is an arbitrary reference current. We introduce the normalized coordinate σ defined by $\sigma=s/c$ and the function $g(\sigma)$ defined by $g(\sigma)=g(s/c)=f(s)$.

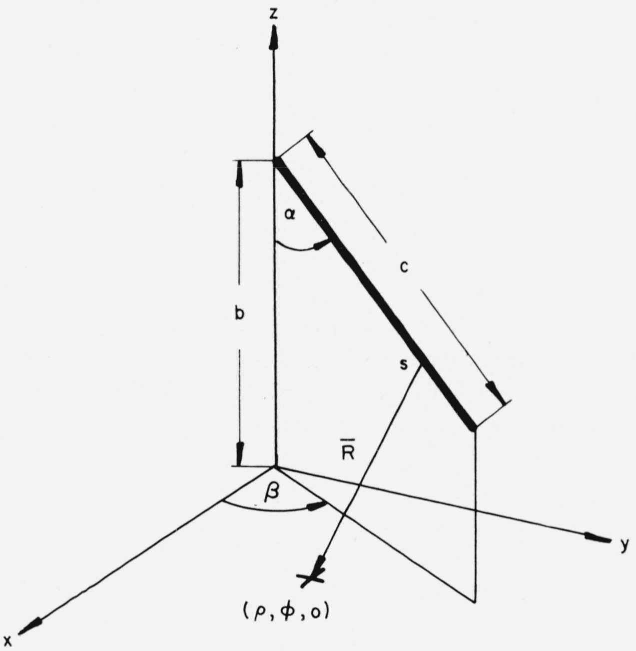

FIGURE 1. *Notation used in calculating the electric field from the current in an inclined wire above a ground plane.*

Let us consider an element ds of the wire located at s. Introducing a unit vector $\hat{t}$ pointing in the positive direction of the wire, this element may be considered a Hertz dipole with the moment $I(s)\, ds\, \hat{t}$. Further, denoting by $\overline{R}$ the vector from a point s on the antenna to a point on the ground plane with the coordinates $(\rho,\phi,0)$, we may express the z-component of the electric field strength $d\overline{E}$ of the Hertz dipole and its image by [3]

$$dE_z=\frac{iI(s)ds e^{ikR}}{2\pi\omega\epsilon}\left[-\frac{k^2}{R}\hat{R}\times(\hat{R}\times\hat{t})-\frac{ik}{R^2}(3\hat{R}\cdot\hat{t}\hat{R}-\hat{t})+\frac{1}{R^3}(3\hat{R}\cdot\hat{t}\hat{R}-\hat{t})\right]_z. \tag{1}$$

In this formula $\hat{R}=\overline{R}/R$ denotes a unit vector coparallel to $\overline{R}$.

Using the reference current I_0 introduced above and further introducing the arbitrary reference distance d; then given an electric field strength $\overline{E}$ we may define a corresponding normalized, dimensionless electric field strength $\overline{e}$ by

$$\overline{e}=\frac{kd^2}{\zeta I_0}\overline{E} \tag{2}$$

where ζ denotes the intrinsic impedance of free space

$$\zeta=\sqrt{\frac{\mu}{\epsilon}}.$$

We thus find the following expression for the vertical component e_z of the normalized electric field strength of the field produced by the wire

$$e_z=\frac{i\left(\frac{d}{c}\right)^2}{2\pi}\int_0\left\{-\frac{(kc)^2}{\left(\frac{R}{c}\right)^3}\left[\frac{R_x}{c}\frac{R_z}{c}t_x-\left(\frac{R_x}{c}\right)^2 t_z-\left(\frac{R_y}{c}\right)^2 t_z+\frac{R_y}{c}\frac{R_z}{c}t_y\right]\right.$$
$$-\frac{ikc}{\left(\frac{R}{c}\right)^2}\left[\frac{3\frac{R_z}{c}}{\left(\frac{R}{c}\right)^2}\left(\frac{R_x}{c}t_x+\frac{R_y}{c}t_y+\frac{R_z}{c}t_z\right)-t_z\right]$$
$$\left.+\frac{1}{\left(\frac{R}{c}\right)^3}\left[\frac{3\frac{R_z}{c}}{\left(\frac{R}{c}\right)^2}\left(\frac{R_x}{c}t_x+\frac{R_y}{c}t_y+\frac{R_z}{c}t_z\right)-t_z\right]\right\}g(\sigma)e^{ikc\frac{R}{c}}\,d\sigma \quad (3)$$

where the subscripts x,y,z indicate the x-, y-, and z-components of the corresponding vectors.

In the remaining part of this report we will assume that the length and the height of the inclined wire are small compared to the wavelength, i.e., $k(b+c)\ll 1$, and we will only consider the near zone field, i.e., we assume $k\rho\ll 1$.

In this case we get

$$e_z\cong\frac{i\left(\frac{d}{c}\right)^2}{2\pi}\int_0^1\frac{1}{\left(\frac{R}{c}\right)^3}\left[\frac{3\frac{R_z}{c}}{\left(\frac{R}{c}\right)^2}\left(\frac{R_x}{c}t_x+\frac{R_y}{c}t_y+\frac{R_z}{c}t_z\right)-t_z\right]g(\sigma)d\sigma. \quad (4)$$

In order to evaluate the integrals occurring in the above expression it is expedient to expand the known current distribution function $g(\sigma)$ in a power series,

$$g(\sigma)=\sum_{m=0}^{\infty}A_m\sigma^m. \quad (5)$$

Approximating $g(\sigma)$ by a finite number of terms in the above series expression, it is possible to express the integral in the formula above by known functions.

We shall here interest ourselves especially for the case where the top loading consists of only a few wires as in the case of the L-antenna and the T-antenna, respectively. The current distribution may then be assumed to be approximately linear, so that the current distribution function $g(\sigma)$ may be expressed by the two first terms in the above power series. As we are interested mainly in the contribution to the field from the top loading, regarding the current in the vertical member, we shall make the further assumption that it is constant.

4. Vertical Wire

The contribution e_z^v to the vertical component of the electric field at the ground plane due to the vertical member in a top-loaded antenna is considered here. The length of the vertical member is denoted by h and the length of each of the top-loading wires by a. As the reference current I_0 occurring in the above formula, we use here and in the following the current I_b at the bottom of the vertical wire. Approximating the current in the vertical wire by a constant current, we then find the following expression for the current distribution function $g(\sigma)$

$$g(\sigma)=\frac{I(0)+I(h)}{2I_b}=\frac{1}{2}\left[1+\frac{a}{h+a}\right]=1-\frac{h}{2(h+a)}. \tag{6}$$

In this section and in the following sections we use the length h of the vertical member as the reference distance d in the formulas developed above. We find the following expression for e_z^v

$$e_z^v=\frac{i}{2\pi}\left[1-\frac{h}{2(h+a)}\right]\int_0^1 \frac{1}{\left(\frac{R}{h}\right)^3}\left[\frac{3\sigma^2}{\left(\frac{R}{h}\right)^2}-1\right]d\sigma, \tag{7}$$

$$\left(\frac{R}{h}\right)^2=X=\sigma^2+\left(\frac{\rho}{h}\right)^2.$$

In evaluating the above integral we use the expressions for integrals of the type

$$\int_0^1 \frac{\sigma^n}{X^m}d\sigma$$

listed in the appendix. We then obtain

$$e_z^v=-\frac{i}{2\pi}\left[1-\frac{1}{2\left(1+\frac{a}{h}\right)}\right]\frac{1}{\left[1+\left(\frac{\rho}{h}\right)^2\right]^{3/2}}. \tag{8}$$

In figure 2, $i\, e_z^v$, i.e., the imaginary unit times the vertical component of the normalized electric field strength at the ground plane, has been plotted as a function of ρ/h and with a/h as a parameter.

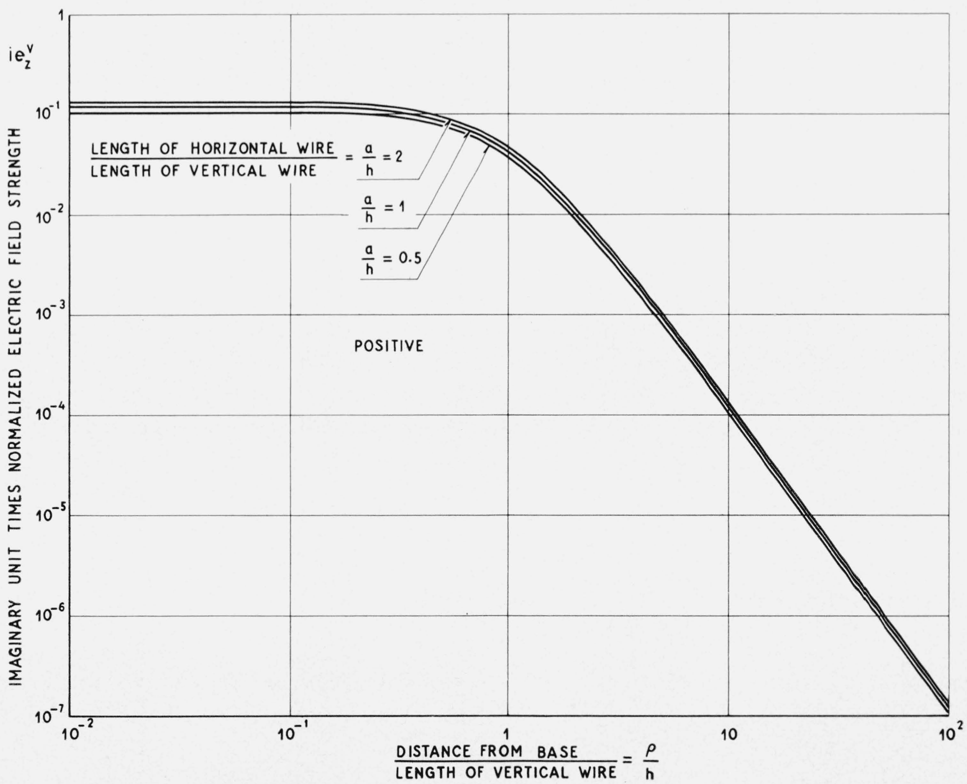

FIGURE 2. *Vertical electric field at ground plane due to vertical current of a top-loaded monopole.*

5. Horizontal Top Loading

We calculate here the contribution e_z^h to the vertical component of the electric field at the ground plane due to one of the top-loading wires of length a in the antenna with a vertical member of length h considered in the last section. As was mentioned before, in applying the formulas developed above, we also here use the current I_b at the bottom of the vertical wire as the reference current I_0 and the length h of the vertical member as the reference distance d. Denoting the number of top-loading wires by $n(n=1,2$ or $4)$ and making use of the assumption of a linear current distribution, we then have the following expression for the current distribution function $g(\sigma)$

$$g(\sigma)=\frac{1}{n}\frac{I(s)}{I_b}=\frac{1}{n}\frac{a}{a+h}\left(1-\sigma\right). \tag{9}$$

We then find the following expression for e_z^h in the near zone field

$$e_z^h=-\frac{i\,3}{2\pi\, n\left(\frac{a}{h}\right)^2\left(1+\frac{a}{h}\right)}\left\{\frac{\rho}{a}\cos\left(\phi-\beta\right)\int_0^1\frac{1}{\left(\frac{R}{a}\right)^5}\left(1-\sigma\right)d\sigma-\int_0^1\frac{\sigma}{\left(\frac{R}{a}\right)^5}\left(1-\sigma\right)d\sigma\right\}, \tag{10}$$

$$\left(\frac{R}{a}\right)^2=X=\sigma^2-2\sigma\frac{\rho}{a}\cos\left(\phi-\beta\right)+\left(\frac{\rho}{a}\right)^2+\left(\frac{h}{a}\right)^2.$$

The integrals occurring in the above formula are evaluated by using the expressions for integrals of the type

$$\int_0^1\frac{\sigma^n}{X^m}d\sigma$$

listed in the appendix.

We then find

$$e_z^h=-\frac{i}{2\pi n\left(\frac{a}{h}\right)^2\left(1+\frac{a}{h}\right)N}\left[\frac{1+B}{X_1^{1/2}}-\frac{N+BC}{C^{3/2}}\right]. \tag{11}$$

For large values of ρ it is difficult to calculate e_z^h from this expression with sufficient accuracy as the two terms in the bracket will be nearly equal. In this case it is expedient to expand the two terms in series.

We then get

$$\begin{aligned}ine_z^h\simeq&\frac{\tau^2}{2\pi(1+\tau)(\xi^2\sin^2\psi+1)\xi^2}\left\{\frac{3}{2}\sin^2\psi\cos\psi\right.\\&+\frac{1}{2}\sin^2\psi(-1+5\cos^2\psi)\tau\frac{1}{\xi}+\frac{1}{8}\cos\psi[-18+30\cos^2\psi+\sin^2\psi(-15+35\cos^2\psi)\tau^2]\frac{1}{\xi^2}\\&+\frac{1}{8}[(6-60\cos^2\psi+70\cos^4\psi)\tau+\sin^2\psi(3-42\cos^2\psi+63\cos^4\psi)\tau^3]\frac{1}{\xi^3}\\&+\frac{1}{16}\cos\psi[45-105\cos^2\psi+(75-350\cos^2\psi+315\cos^4\psi)\tau^2\\&\left.+\sin^2\psi(35-210\cos^2\psi+231\cos^4\psi)\tau^4]\frac{1}{\xi^4}\right.,\end{aligned} \tag{12}$$

where

$$\frac{\rho}{h}=\xi\qquad\frac{a}{h}=\tau\qquad\phi-\beta=\psi.$$

Using this expression for ine_z^h for large values of ρ and the exact expression for the smaller values of ρ, we plot in figures 3a–b the quantity ine_z^h, i.e., the imaginary unit times the number of horizontal wires times the vertical component of the normalized electric field strength at the ground plane, as a function of ρ/h and with a/h as a parameter for $\phi-\beta=0°$, and 180°. It is seen as well from the expression for e_z^h as by direct inspection that

$$e_z^h(-(\phi-\beta))=e_z^h(\phi-\beta). \tag{13}$$

6. L-Antenna

Let us consider an L-antenna, the vertical member of which has the length h and the horizontal member the length a as shown in figure 4. The contribution e_z^v to the vertical component of the normalized electric field strength at the ground plane due to the vertical member may be obtained directly from figure 2. As the number of top-loading wires is equal to 1, the contribution e_z^h due to the horizontal member may be obtained directly from figure 3. The vertical component e_z of the normalized electric field strength at the ground plane due to the whole antenna may be expressed as

$$e_z=e_z^v+e_z^h.$$

It is obvious that e_z satisfies the same symmetry relation as e_z^h.

It is of interest to calculate the quotient between the contribution e_z^h of the normalized, electric field strength due to the top loading and the contribution e_z^v due to the vertical member. In figure 5 the ratio e_z^h/e_z^v has been plotted as a function of ρ/h and with a/h as a parameter for $\phi-\beta=0°$. It is seen, as might have been expected, that the relative magnitude of the contribution from the horizontal member decreases with increasing distance from the antenna.

7. T-Antenna

We consider a T-antenna of the height h and with its horizontal member having the length $2a$ as shown in figure 6. The contribution e_z^v to the vertical component of the normalized electric field strength at the ground plane due to the vertical member may be obtained directly from figure 2. Letting β_1 denote the azimuth of one of the two horizontal wires of which the top loading consists, we have for the azimuth of the other radial wire

$$\beta_2=\beta_1+\pi.$$

Utilizing the symmetry properties of the vertical component e_z^h of the normalized electric field strength at the ground plane due to one wire with linearly tapered current distribution, we then find the following expression for the vertical component e_z^H of the normalized electric field strength at the ground plane due to the total top loading by adding the contributions from each of the two wires

$$e_z^H(\phi-\beta_1)=e_z^h(\phi-\beta_1)+e_z^h(\phi-\beta_2)=e_z^h(\phi-\beta_1)+e_z^h(\phi-\beta_1-\pi)=e_z^h(\phi-\beta_1)+e_z^h(\pi-(\phi-\beta_1)). \tag{14}$$

As the number of top-loading wires is equal to 2, we may find $ie_z^h(\phi-\beta_1)$ as ½ times the reading of the curves plotted in figure 3. It is easily verified that $e_z^H(\phi-\beta_1)$ satisfies the following symmetry relations

$$e_z^H(-(\phi-\beta_1))=e_z^H(\phi-\beta_1) \tag{15}$$

$$e_z^H(\pi-(\phi-\beta_1))=e_z^H(\phi-\beta_1). \tag{16}$$

The vertical component e_z of the normalized electric field strength at the ground-plane due to the whole antenna may now be obtained from

$$e_z=e_z^v+e_z^H.$$

It is obvious that e_z satisfies the same symmetry relations as e_z^H.

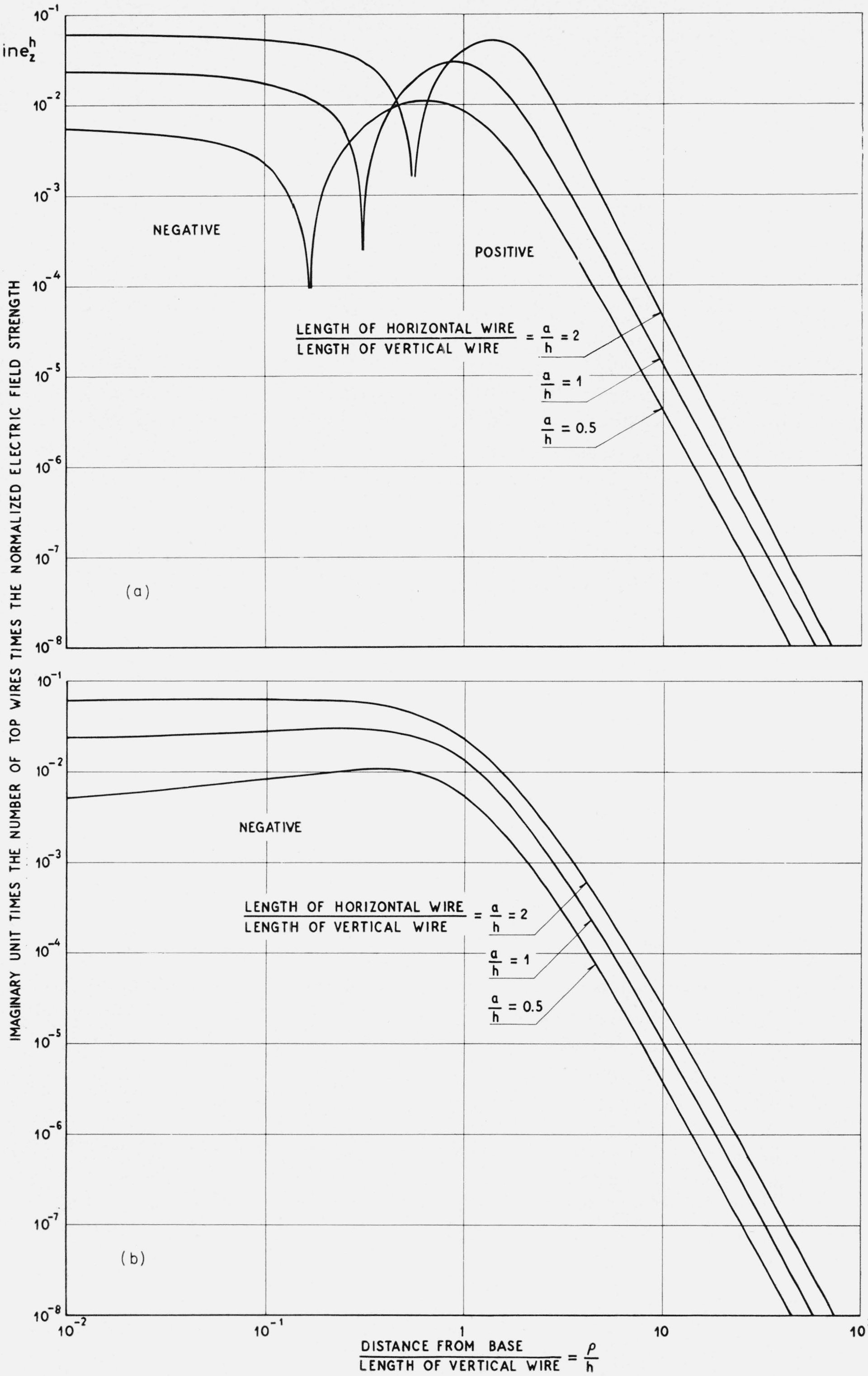

FIGURE 3. *Vertical electric field at ground plane due to horizontal current in one top wire of a top-loaded monopole.*

a: $\phi-\beta=0°$, b: $\phi-\beta=180°$.

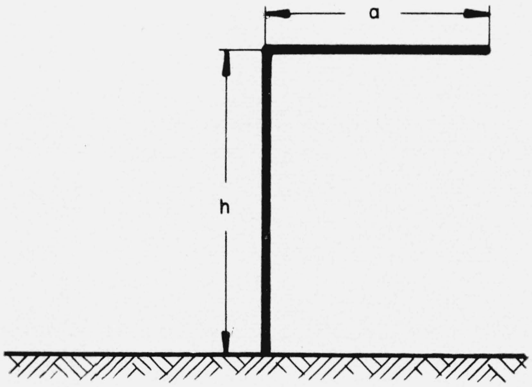

FIGURE 4. L-*antenna*

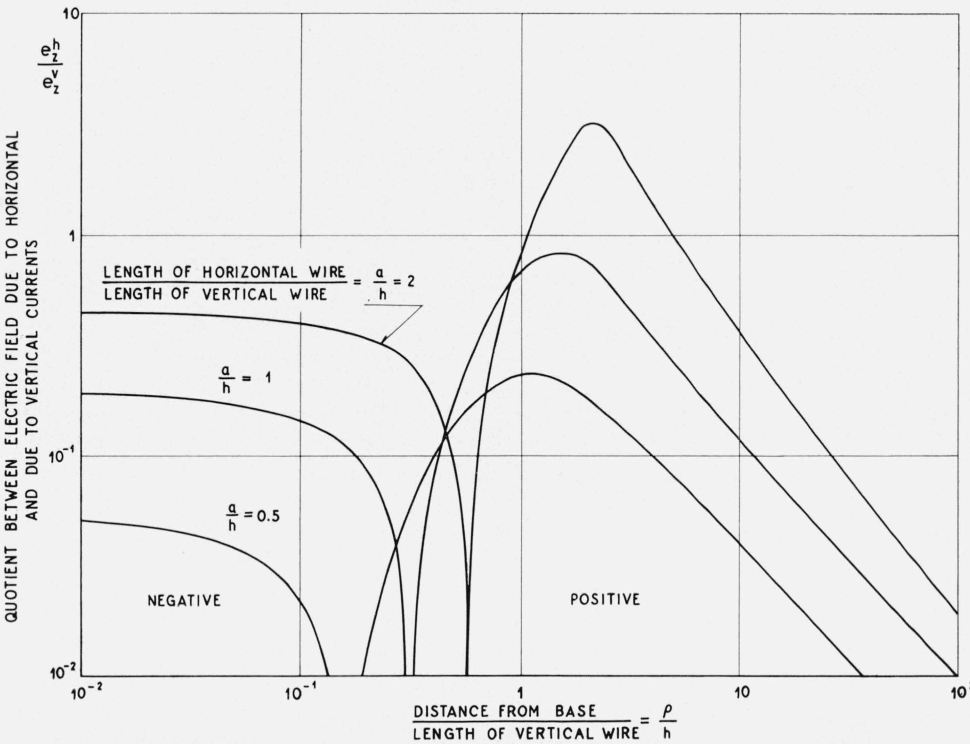

FIGURE 5. *Quotient between the contributions to the vertical electric field at ground plane due to the top loading and due to the vertical member of an* L-*antenna*, $\phi-\beta=0°$.

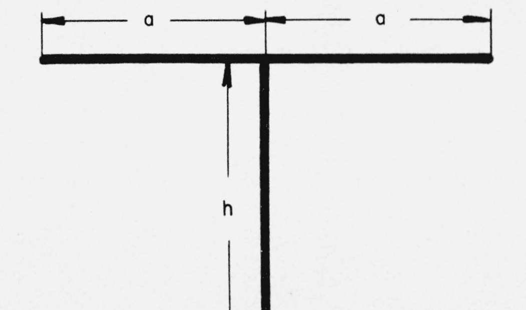

FIGURE 6. T-*antenna.*

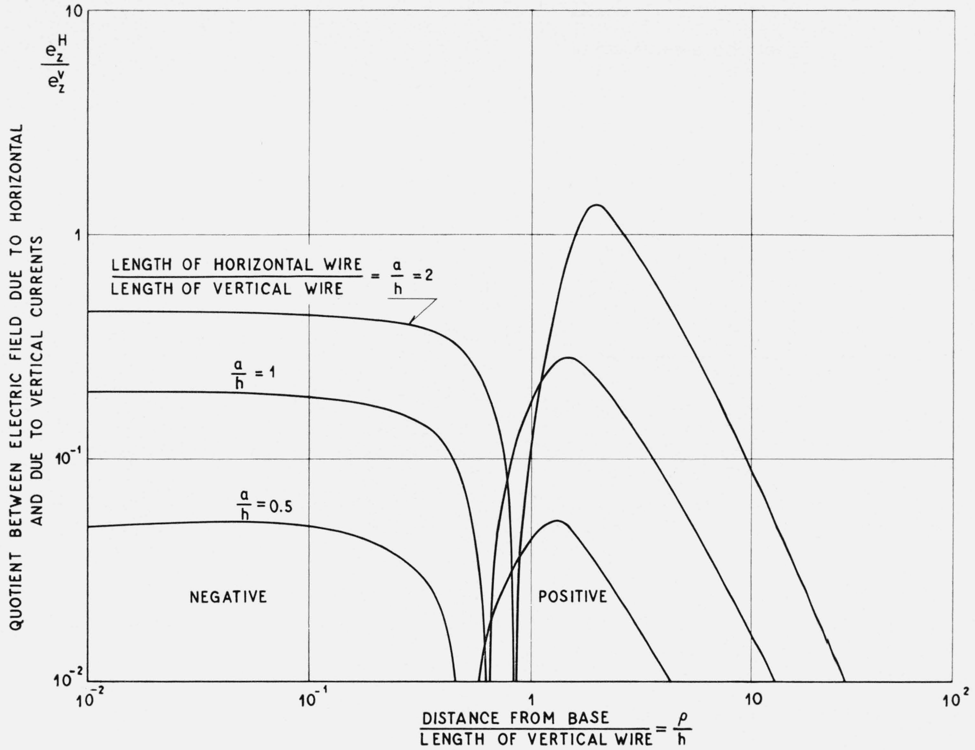

FIGURE 7. *Quotient between the contributions to the vertical electric field at ground plane due to the top loading and due to the vertical member of a* T*-antenna,* $\phi-\beta_1=0°$.

In figure 7 the ratio e_z^H/e_z^v has been plotted as a function of ρ/h and with a/h as a parameter for $\phi-\beta_1=0°$. We see that also in the case of the T-antenna the relative magnitude of the contribution from the top loading decreases with increasing distance from the antenna.

8. Antenna With Four Top-Loading Wires

We consider here an antenna consisting of a vertical wire and four equiangularly spaced, horizontal wires as shown in figure 8. We include this case in the present investigation because it approximates the case of a disk-loaded antenna, so that we are hereby enabled to compare fairly well the formulas and numerical results derived in this paper with the corresponding formulas and numerical results that might be derived later on the basis of the model of a disk-loaded antenna. We assume also in the present case the current distribution on the antenna including the top loading is linear. Whereas this current distribution probably fairly well approximates the actual current distribution in the case of L- and T-antennas, it is questionable whether it does so in the case of the antenna with four top-loading wires. However, we use the assumption of a linear current distribution for the sake of simplicity, remembering that the main emphasis in this paper is on L- and T-antennas.

The length of the vertical wire is denoted by h and the length of each of the radial wires by a. The contribution e_z^v to the vertical component e_z of the electric field strength at the ground plane may be obtained directly from figure 2. Letting β_1, β_2, β_3, and β_4 denote the azimuth of the four wires of which the top loading consists, we have

$$\beta_2=\beta_1+\frac{\pi}{2}$$

$$\beta_3=\beta_1+\pi$$

$$\beta_4=\beta_1+\frac{3\pi}{2}.$$

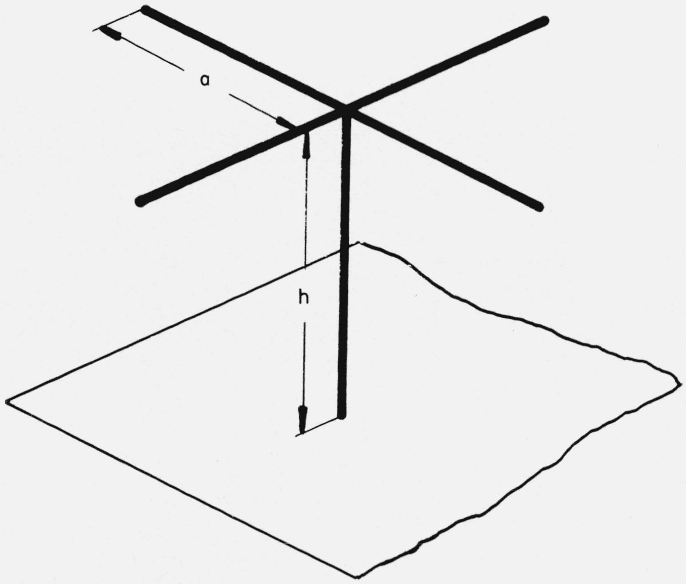

FIGURE 8. *Antenna with four top-loading wires.*

Using this notation and the symmetry properties of the vertical component e_z^h of the normalized electric field strength at the ground plane due to one wire, we then find the following expression for the vertical component e_z^+ of the normalized electric field strength at the ground plane due to the total top loading.

$$e_z^+(\phi-\beta_1)=e_z^h(\phi-\beta_1)+e_z^h\left(\phi-\beta_1-\frac{\pi}{2}\right)+e_z^h(\phi-\beta_1-\pi)+e_z^h\left(\phi-\beta_1-\frac{3\pi}{2}\right)$$

$$=e_z^h(\phi-\beta_1)+e_z^h\left(\frac{\pi}{2}-(\phi-\beta_1)\right)+e_z^h(\pi-(\phi-\beta_1))+e_z^h\left(\phi-\beta_1+\frac{\pi}{2}\right). \quad (17)$$

As the number of top-loading wires is equal to 4, $ie_z^h(\phi-\beta_1)$ may be found as 1/4 times the value obtained from the curves plotted in figure 3. We may easily verify that $e_z^+(\phi-\beta_1)$ satisfies the following symmetry relations

$$e_z^+(-(\phi-\beta_1))=e_z^+(\phi-\beta_1), \quad (18)$$

$$e_z^+(\pi-(\phi-\beta_1))=e_z^+(\phi-\beta_1), \quad (19)$$

$$e_z^+\left(\frac{\pi}{2}-(\phi-\beta_1)\right)=e_z^+(\phi-\beta_1). \quad (20)$$

The vertical component e_z of the normalized electric field strength at the ground plane due to the whole antenna is then obtained from

$$e_z=e_z^v+e_z^+.$$

The component e_z is seen to satisfy the same symmetry relations as e_z^+.

In figure 9 the ratio e_z^+/e_z^v has been plotted as a function of ρ/h and with a/h as a parameter for $\phi-\beta=0°$.

9. Conclusion

Formulas have been derived and numerical calculations carried out for the vertical component of the electric field strength at the ground plane in the near zone field of a vertical or horizontal linear antenna having a length and height above the ground plane that is small

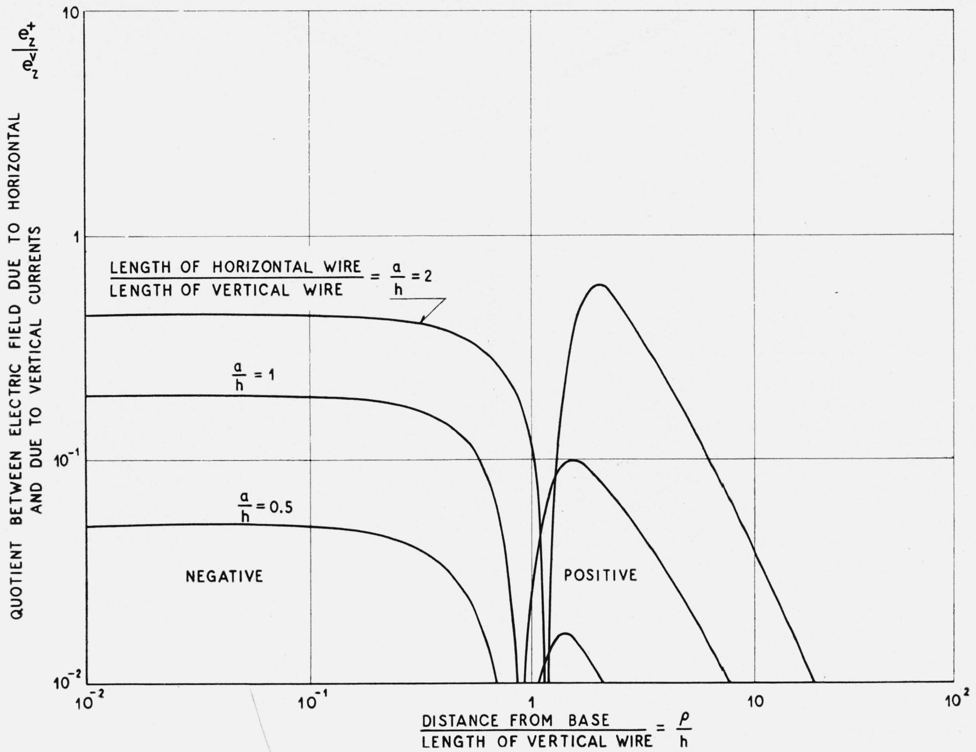

FIGURE 9. *Quotient between the contributions to the vertical electric field at ground plane due to the top loading and due to the vertical member of an antenna with four top-loading wires,* $\phi-\beta_1=0°$.

compared to the wavelength. These results have been used in obtaining the above-mentioned component and the relative magnitude of the contribution to this component due to the horizontal top loading in the near zone field of electrically small L-antennas, T-antennas, and antennas with four horizontal, top-loading wires. The results obtained here should be useful in calculating the contribution to the ground losses due to the vertical component of the earth currents.

It is seen from the curves that there is a reduction of the relative vertical electric field when symmetrical top loading is used. This seems to be an advantage of the T- and four top hot wire antenna over the L-antenna.

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Our thanks are due to James R. Wait for having suggested the problem and for helpful comments, to Erik B. Hansen for having checked the formulas with the result that one formula was considerably condensed, and to Anabeth C. Murphy for having made the numerical computations.

10. Appendix

Some Integrals of the Type $\int \frac{\sigma^n}{(\sigma^2+2B\sigma+C)^m} d\sigma$

In calculating the vertical component of the electric field strength at the ground plane in the near zone field of a vertical or horizontal linear antenna having a linear current distribution, the length and height of which is small compared to the wavelength, we have use for closed expressions for some integrals of the type

$$\int_0^1 \frac{\sigma^n}{X^m} d\sigma$$

where

$$X=\sigma^2+2B\sigma+C,$$

B and C being constants satisfying the inequality

$$N=C-B^2 \geq 0.$$

The corresponding indefinite integrals have partly been obtained from Gröbner and Hofreiter's table of integrals [4].

Introducing the following abbreviation

$$X_1=(X)_{\sigma=1}=1+2B+C$$

we have for the pertinent definite integrals

$$\int_0^1 \frac{1}{X^{3/2}} d\sigma = \frac{1}{N}\left[\frac{1+B}{X_1^{1/2}}-\frac{B}{C}\right], \tag{21}$$

$$\int_0^1 \frac{1}{X^{5/2}} d\sigma = \frac{1}{3N}\left[\frac{1+B}{X_1^{3/2}}-\frac{B}{C^{3/2}}\right]+\frac{2}{3N^2}\left[\frac{1+B}{X_1^{1/2}}-\frac{B}{C^{1/2}}\right], \tag{22}$$

$$\int_0^1 \frac{\sigma}{X^{5/2}} d\sigma = \frac{1}{3N}\left[-\frac{B+C}{X_1^{3/2}}+\frac{1}{C^{1/2}}\right]+\frac{2B}{3N^2}\left[-\frac{1+B}{X_1^{1/2}}+\frac{B}{C^{1/2}}\right], \tag{23}$$

$$\int_0^1 \frac{\sigma^2}{X^{5/2}} d\sigma = \frac{1}{3N}\left[\frac{2B^2-C+BC}{X_1^{3/2}}-\frac{B}{C^{1/2}}\right]+\frac{B^2+C}{3N^2}\left[\frac{1+B}{X_1^{1/2}}-\frac{B}{C^{1/2}}\right]. \tag{24}$$

11. References

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