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Phase Angle Master Standard for 400 Cycles per Second

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A continuously variable, 0- to 180-degree, phase shift standard for 400 cycles per second is described in detail. It consists of a π-section line made up of twelve 14.6 degree and three 4.3 degree sections to provide for two sizes of coarse steps and an RC circuit at the input to the line to provide for fine steps and a continuous fine control. A method for accurately adjusting the characteristic impedance of all π-sections to the same value, which is used as the termination, was devised. Under these conditions it is shown that the phase shift introduced by each π-section can be accurately computed from a measured value of inductance. The phase shift of each π-section was also determined by an experimental procedure dependent upon a 180-degree phase shift introduced by a toroidal transformer. The values obtained by these two independent methods agree to within 0.01 degree.

1. Introduction

Phase angle in this paper is defined as the quantity used to measure the time phase relation between two sinusoidal voltages of the same frequency. Highly accurate measurements of small angles, i.e., less than 3°, have long been made in connection with instrument transformer calibration testing and phase angle defect measurements of capacitors. Impedance and power factor measurements involving somewhat larger phase shifts but usually with less accuracy have also been performed for some time. However, until quite recently the accurate measurement of phase angle as such over a wide range and at frequencies higher than 60 cps has not received much attention.

Interest in precise phase angle measurements has been aroused as a result of computer and guidance problems. Various methods and devices for measuring phase shift have been developed in solving these problems. In order to obtain accurate comparisons and coordination of the test data obtained in various standardizing laboratories, reference standards must be available which will maintain their calibration when shipped from one laboratory to another. Also, at least one laboratory (probably the National Bureau of Standards) must be capable of checking the absolute calibration of such reference standards.

The calibration of reference standards is most conveniently carried out if a master standard is available which has been thoroughly investigated and accurately calibrated. The purpose of this paper is to describe such a master standard which has been designed and constructed for use at 400 cps at the National Bureau of Standards and to explain the methods used in arriving at its calibration.

2. Requirements of a Master Standard

Conferences with various persons concerned with accurate phase angle measurements indicated that the greatest current need was for measurements from 0° to 180° at 400 cps with an accuracy of 0.01°, if possible. There are numerous methods of obtaining phase shift in this range, and measurement accuracies to within a degree or even a few tenths of a degree can be attained without great difficulty. However, to cover the range from 0° to 180° with a 0.01° accuracy at any point, a master standard must contain several different magnitudes of phase-shift steps which can be switched in or out by selector switches and a continuously variable fine control. To obtain a usable calibration, the phase shift put in by each step on any selector switch or by any given change in the fine dial must be independent of the settings on all other switches and dials. In the case of a resistance decade box a similar requirement is nearly always automatically fulfilled, but for most phase shifting devices this is not true. For a master standard phase shifter it should also be possible to compute phase shift for each step from the measured values of resistance, capacitance, and inductances.

3. Description of Design Used

Resistance-capacitance circuits are commonly used to obtain phase shifts, and the phase shifts inserted by such circuits can, under certain conditions, be accurately computed from measured values of resistance and capacitance. However for such circuits phase shift is proportional to change in capacitance or resistance only up to a few degrees, and above that a fixed change in resistance or capacitance gives a different phase shift depending upon the total values of resistance and capacitance. Thus, decades of resistance or capacitance cannot be used to make up a calibrated phase shifter for large angles but fortunately they can be used for fine control up to about 5°.

To obtain coarse steps of 5° and larger, L–C lumped delay lines (multisection ladder networks) were tried and found to be suitable. The main advantage of such a delay line is that the phase shift added by any individual section is always the same no matter how many sections are already in use, provided the characteristic or surge impedance of all sections is accurately adjusted to the same value and the line is terminated with this same impedance.

Under these conditions the phase shift of each section can be computed from a measured value of inductance. Also the input impedance is independent of the number of sections so that the input can be part of an R–C circuit used to get fine control of phase angle.

The final design consisted of twelve 14.6° π-sections and three 4.3° π-sections with switching arrangements so that any desired combination of sections could be connected as a delay line. The input to this line was used as part of an R–C network which incorporated 10 capacitor steps giving 0.44° phase shift each and a variable air capacitor for continuous fine control.

Several circuit connections and switching arrangements for the complete phase shifter were considered and tried. The one finally selected is shown schematically in figure 1. The "input" or voltage to be shifted is connected to a 4,000-ohm noninductive resistor, shunted by a capacitance-remove type variable air capacitor (1,100 to 100 pf), in series with the input to the delay line. A decade set of mica capacitors (0.004 to 0.044 μf) and a zero adjusting capacitance are connected in parallel with the delay line input. A set of 13 single-pole, two-position, mercury-contact switches are arranged to connect any number from 0 to 12 of the 14.6° π-sections into this delay line. Also a set of 4 switches is arranged to connect any number from 0 to 3 of the 4.3° π-sections into the delay line. The end of the delay line is connected to a resistor of 1,000 ohms shunted by a capacitor of 0.0021 μf which was chosen to be equal to the characteristic impedance of the line. The terminated end of the line is connected to the "output" binding posts of the phase shifter through the "test-check" switch when it is in the "test" position.

4. Theory and Method of Measurements on π-Sections

The circuit diagram in figure 2 shows a generalized π-section delay line which can be used as a phase shifter. To eliminate reflections so that the shift introduced by any one section is constant irrespective of other sections in the line, the terminating impedance, Z_0, must be made equal to the input impedance, Z_{in}. Considering one section only and solving for Z_0 when $Z_0=Z_{\text{in}}$ gives

$$Z_0^2=\frac{Z_1Z_2}{1+\dfrac{Z_1}{4Z_2}}. \tag{1}$$

Also,

$$e_0=\frac{e}{1+\dfrac{Z_1}{2Z_2}+\dfrac{Z_1}{Z_0}}. \tag{2}$$

Equation (2) shows that for each section the output voltage lags the input voltage by the angle represented by the vector, $1+(Z_1/2Z_2)+(Z_1/Z_0)$.

For a properly terminated line made up of pure inductors and capacitors the phase shift per section

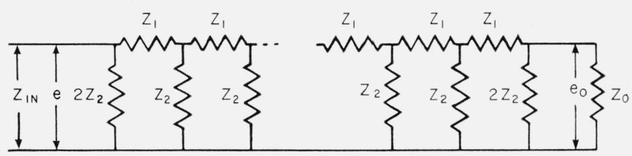

FIGURE 2. *Generalized π-section line.*

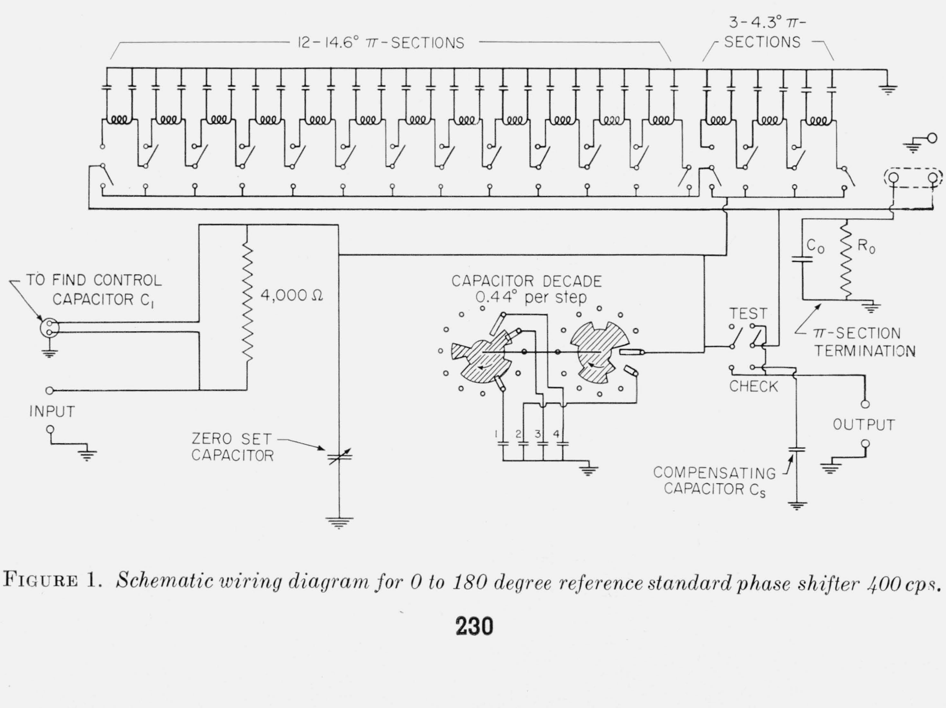

FIGURE 1. *Schematic wiring diagram for 0 to 180 degree reference standard phase shifter 400 cps.*

could be readily computed from the known values of inductance and capacitance. However, there are several practical considerations which tend to reduce the accuracy of the computed value. For $Z_1=j\omega L$ and $Z_2=1/j\omega C$, the terminating impedance as determined from eq (1) must be

$$Z_0^2=\frac{L}{C}\,\frac{1}{1-\frac{\omega^2 LC}{4}}. \tag{3}$$

Thus Z_0 is real (made up of resistance only), but its value depends to some extent upon frequency. If sections with different values of inductance are to have the same Z_0, then C must be determined from the above equation and becomes

$$C=\frac{L}{Z_0^2}\left(1+\frac{\omega^2 L^2}{4Z_0^2}+\frac{\omega^4 L^4}{8Z_0^4}\cdots\right). \tag{4}$$

Another practical consideration is the resistance of the inductors. To get a suitable value of terminating resistance (say 1,000 ohms) the inductance for a 15° section must be 100 mh. An air-core inductor of this magnitude would either have too much resistance if wound on a toroidal shaped core or would be too susceptible to nearby magnetic fields if made in the form of a circular coil for optimum [1] time constant. The best practical solution is a toroidal winding on a high permeability core. However, since both the inductance and effective resistance of such a high-Q coil would be dependent to some extent on the current through it, the current must be held to a nearly constant value—chosen to be considerably below that giving saturation of the core. Measurements of resistance to about 1 percent and inductance to a much higher accuracy must also be made at the same current, and these values should be stable.

4.1. Method Used To Measure Inductors

An accurate measurement of both the inductance and the resistance of the toroidal high-Q inductors was obtained by using a Maxwell-Wien bridge with a Wagner ground as shown in figure 3. Here R_2 and R_4 are precision woven wire resistance standards (300 ohms each) for which the time constant is less than 10^{-8} sec. C_1 consists of a 0- to 1.11-μf three-decade mica capacitor in parallel with a 50- to 1,100-pf variable air capacitor, both of which are completely shielded and accurately calibrated. R_1 is a four-decade resistance box (0 to 100,000 ohms) of low time constant woven wire resistors. The bridge balance is obtained by adjusting C_1 and R_1. When both the Wagner arm and the bridge are balanced the following relations hold for L_3 and R_3 (neglecting the second-order term):

$$L_3=C_1R_2R_4 \qquad \text{and} \qquad R_3=\frac{R_2R_4}{R_1}.$$

[1] H. B. Brooks, Design of standards of inductance and the proposed use of models in the design of air-core and iron-core reactors, BS J. Research (1931) RP342.

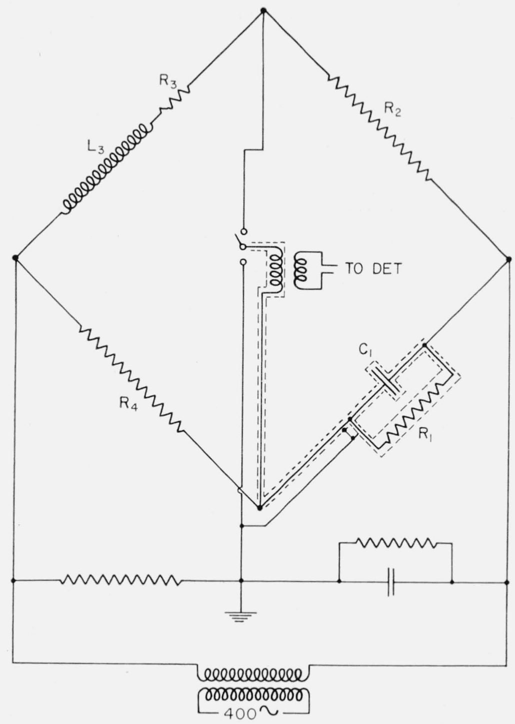

FIGURE 3. *Maxwell-Wien bridge with Wagner ground used to measure inductance and resistance of inductors.*

For the bridge components used, the second order corrections are negligible for an accuracy of 0.01 percent in L_3 and 1 percent in R_3. The values used for R_2 and R_4 should include any lead resistance up to the bridge corners.

An oscillator having an output waveform with total distortion less than 0.1 percent was used to supply this bridge. The frequency was accurately set to 400 cps by forming a Lissajous pattern with a 100-cps standard frequency signal. The voltage appled to the inductor was set by using a voltmeter connected from ground to the L_3 R_4 bridge corner. The actual values of L and r for the inductors used in the π-sections are listed in table 1. These are the values used in computing phase shift for each section.

TABLE 1. *Measured and computed data for inductors*

400 cps measurements using Maxwell-Wien Bridge				Phase shift computed from measured L in degrees	Phase shift from experimental calibration in degrees
Inductor no.	Volts across inductor	mh	ohms		
1	0.25	100.29_0	2.79	14.600	14.600
2		100.34_0	2.82	14.606	14.609
3		100.19_9	2.78	14.587	14.589
4		100.15_4	2.80	14.581	14.581
5		100.51_3	2.80	14.631	14.636
6		100.15_2	2.85	14.580	14.581
7		100.08_5	2.79	14.571	14.571
8		99.89_9	2.87	14.546	14.542
9		100.15_4	2.95	14.580	14.582
10		99.70_6	2.87	14.518	14.515
11		100.40_7	3.02	14.616	14.620
12		100.66_0	3.10	14.651	14.658
1	.075	29.876	1.025	4.307	4.308
2		29.852	1.020	4.304	4.304
3		29.891	0.986	4.309	4.310

Because of the procedure used in matching characteristic impedance for each π-section, as described in the following paragraphs, this computation does not require that the exact value of capacitance for the π-section be known. It does require an exact knowledge of the terminating impedance.

4.2. Procedure Used in Matching the Characteristic Impedance of Each π-Section

By arranging a potentiometer-bridge circuit with the π-section termination (C_0 and R_0 in parallel) as one of the arms and using a sensitive detector to obtain a balance, any small variation in this terminating impedance can readily be detected. If all circuit elements are kept the same after getting an initial balance, except that one π-section is added ahead of the termination, the detector would remain in balance if the characteristic impedance of the π-section were exactly equal to C_0 and R_0 in parallel. If the characteristic impedance is off slightly the detector can be rebalanced by a fine adjustment on the two capacitor legs of the π-section—thus experimentally making the impedance of the π-section exactly equal to C_0 and R_0 in parallel.

The circuit arrangement actually used in making these adjustments is shown schematically in figure 4. Most of the circuit elements shown in this diagram either represent or correspond directly to those used to make up the π-section phase standard (as shown in detail in fig. 1). R_1 is the 4,000-Ωresistor. C_1 is the fine control capacitor. C_2 represents the capacitor decade and the zero set capacitor. The single π-section is used to represent the entire delay line. C_0 and R_0 are the π-section termination. The only additional circuit elements required are the ratio transformer and detector (the detector transformer only being shown). The procedure used was to first balance the detector with the π-section disconnected (switch A up and B open) by adjusting the ratio transformer and capacitor C_2. Then with a π-section connected ahead of C_0 and R_0 (switch A down and B closed) the detector was balanced by adjusting $C'/2$ and $C''/2$, using fine trimmers in parallel with the main capacitance. This last balance is somewhat "tricky" to obtain because the two unbalanced voltages put in by $C'/2$ and $C''/2$ respectively are not 90° apart in phase—but after a little practice it can be obtained quite rapidly. By using a highly selective tuned amplifier as detector this method gives a very sensitive way to adjust the characteristic impedance of the π-section to be exactly equal to R_0 and C_0 in parallel.

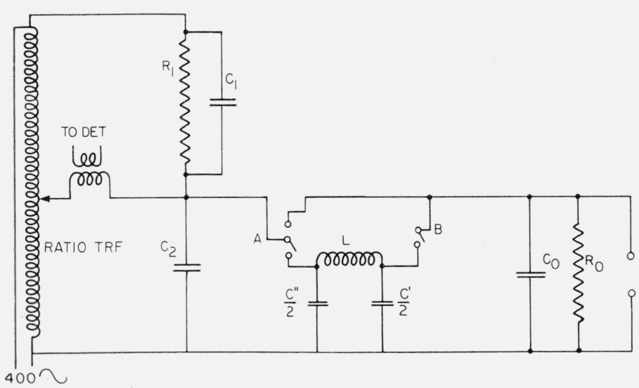

Figure 4. *Schematic diagram of phase standard with ratio transformer and detector as connected for adjustment of capacitor trimmers on each π-section.*

The derivation of the relations between the π-section circuit constants and the terminating impedance, for which the input and terminating impedance are equal, is given in appendix 1. The procedure used was to derive an expression for the input impedance of the π-section (terminated by C_0 and R_0 in parallel) and equate this to C_0 and R_0 in parallel, giving a complex equation in circuit parameters. The two separate equations obtained from the complex equation can then be used to determine any two of the circuit parameters in terms of the others. First, it is assumed that the capacitors in the two legs of the π-section are equal, i.e., $=C'/2=C''/2=C/2$. The expressions derived for C_0 and $C/2$ are given by eqs (9) and (10), respectively, in appendix 1. These values of C_0 and $C/2$ can now be put in the two original equations by letting $C''/2=C/2\ (1+\lambda'')$ and $C'/2=C/2\ (1+\lambda')$. Equations (11) and (12) obtained in this manner can be solved for actual values of λ' and λ'' which are measures of the adjustments required in the capacitance legs to maintain the same characteristic impedance for slight changes in other π-section parameters. To help clarify the above procedure an example using actual values for the 14.6° π-section is worked out in appendix 1.

The values of $C'/2$, i.e., $C/2\ (1+\lambda')$, obtained in this manner, together with the known or assigned values of the other circuit parameters can be used to compute phase shift as explained in the following section.

4.3. Computation of Phase Shift for a π-Section

The phase shift of a π-section, i.e., phase difference between voltage in and out, does not depend upon the input capacitance leg because it is in parallel with the input voltage. It does depend on all other circuit constants including the terminating impedance. A derivation for phase shift is given in appendix 2, and eq (19) gives an expression for tan θ as a function of these circuit constants. Actually for any π-section whose values of $C''/2$ and $C'/2$ have been adjusted experimentally (as described above) so that its characteristic impedance is equal to the termination (R_0 and C_0 in parallel) the phase shift can be computed from the measured value of inductance L only provided some limits are set for values of r, R', and R''. This can best be illustrated by using a specific example.

Take first the average 14.6° π-section values as given in appendix 1, i.e., $L=100$ mh, $r=2.8\,\Omega$,

$R'=10^7\Omega$. For $R_0=1{,}000\ \Omega$ and $C'/2=C''/2=C/2$, $C_0=0.002099\times10^{-6}$ and $C/2=0.0508191\times10^{-6}$ (as computed in appendix 1). Putting these values in eq (19), the phase shift angle θ, as shown in appendix 2, can be expressed as $\tan\ \theta=0.25968$. For the maximum changes in r, R' and R'' encountered in the twelve 14.6° π-sections as determined from actual measurements, the changes in $\tan\ \theta$, as computed in appendix 2, are given in the following table.

r	R'	R''	Tan θ
2.80	10^7	10^7	0.25968
3.10	10^7	10^7	.25968
2.80	10^8	10^7	.25967
2.80	10^7	10^8	.25969
2.80	10^8	10^8	.25968

Since $\tan\ \theta$ is very nearly the same for all of these changes in r, R', and R'', an expression for $\tan\ \theta$ in terms of L only, which holds for each of the 14.6° π-sections can be derived. As shown in appendix 2 this expression is

$$\tan\ \Theta=\frac{L}{R_0}\,2596.7_8.$$

Using the same procedure, as described for the 14.6° π-section, an expression for $\tan\ \theta$ which holds for each of the 4.3° π-sections was also derived. This expression is

$$\tan\ \Theta=\frac{L}{R_0}\,2520.5.$$

5. Theory and Calibration of R–C Network

By using the special measuring techniques and theoretical relations described in the previous section, a computed value of phase shift for each of the π-sections can be derived, and such values are listed in table 1. The phase shift inserted by the R–C network, connected ahead of the π-sections as shown in figures 1 and 4, may also be computed using known values of R and C. Appendix 3 shows how this can be done, and actual phase shifts for various values of C_1 and C_2 are given in table 2.

One requirement for a standard phase shifter is that the phase shift introduced by any given change in the continuous fine control be independent of the setting of other phase shift dials. From the fourth column in table 2 it can be seen that the phase shift put in by a 500-pf change in the fine control capacitor is the same to within 0.0005° for all settings on C_2 (the 0.46° steps) up to 43,000 pf. Thus, if the π-sections are all matched so that the impedance looking into the line is the same no matter how many sections are connected, the phase shift put in by a given change in the setting of the fine control capacitor will be essentially the same irrespective of phase shift put in by the "capacitor decade" and the π-section line. As shown in table 2 the average value of this phase shift is 0.2305° for a change in C_1 of 500 pf. (0.0046° for 10 pf or 0.01° corresponds to 21.7 pf).

The last column in table 2 indicates that the phase shift put in by a 3,900-pf change in C_2, i.e., by one step on the decade capacitor dial, depends to a slight extent upon the total values for C_1 and C_2. For any given setting of C_2 the change in this phase shift for settings of C_1 from 1,200 to 200 pf is less than 0.00015°. Thus, the effect of the fine control capacitor setting on the decade capacitor steps is negligible. As C_2 is increased the phase shift for a 3,900-pf change in C_2 decreases somewhat, the maximum change being about 0.003°. The calibration for each step on the capacitor decade can take this change into account, but this does indicate that another capacitor decade could not be added. The phase shift for each step on the decade capacitor could be computed if the value of capacitance added for each step were known to a high accuracy (to about 1 pf for 0.0001°), but it was decided that the experimental method as explained in the following section would be easier to apply.

TABLE 2. *Phase shift for various settings of the capacitors in the R–C network*

(See figure 8)

R_1=4,000 ohms; R_0=1,000 ohms

C_1	C_2	Phase shift, θ	$\Delta\theta$ for 500 pf change in C_1	$\Delta\theta$ for 3,900 pf change in C_2
pf	*pf*	*deg*	*deg*	*deg*
1,200	4,000	$+0.0921_5$	----------	----------
700		$-.1382_3$	0.2303_8	----------
200		$-.3686_2$	$.2303_9$	----------
1,200	7,900	$-.3570_3$		0.4491_8
700		$-.5874_2$	$.2303_9$	$.4491_9$
200		$-.8178_4$	$.2304_2$	$.4492_2$
1,200	11,800	$-.8060_9$		$.4490_6$
700		-1.0365_0	$.2304_1$	$.4490_8$
200		-1.2669_3	$.2304_3$	$.4490_9$
1,200	15,700	-1.2549_6		$.4488_7$
700		-1.4853_9	$.2304_3$	$.4488_9$
200		-1.7158_5	$.2304_6$	$.4489_2$
1,200	19,600	-1.7035_9		$.4486_3$
700		-1.9340_6	$.2304_7$	$.4486_7$
200		-2.1645_5	$.2304_9$	$.4487_0$
1,200	23,500	-2.1519_3		$.4483_4$
700		-2.3824_4	$.2305_1$	$.4483_8$
200		-2.6129_8	$.2305_4$	$.4484_3$
1,200	27,400	-2.5999_2		$.4479_9$
700		-2.8304_9	$.2305_7$	$.4480_5$
200		-3.0610_5	$.2305_6$	$.4480_7$
1,200	31,300	-3.0475_0		$.4475_8$
700		-3.2781_3	$.2306_3$	$.4476_4$
200		-3.5087_6	$.2306_3$	$.4477_1$
1,200	35,200	-3.4946_4		$.4471_4$
700		-3.7253_3	$.2306_9$	$.4472_0$
200		-3.9560_0	$.2306_7$	$.4472_4$
1,200	39,100	-3.9412_6		$.4466_2$
700		-4.1720_1	$.2307_5$	$.4466_8$
200		-4.4027_6	$.2307_5$	$.4467_6$
1,200	43,000	-4.3873_3		$.4460_7$
700		-4.6181_4	$.2308_1$	$.4461_3$
200		-4.8489_8	$.2308_4$	$.4462_2$

6. Experimental Calibrations

An experimental calibration can be devised for any phase shifter which fulfills the requirements given in section 2. In brief, the procedure is first to obtain the phase shift of each step of each dial in terms of settings on the other dials (including the fine control). These relative values can then be converted to absolute values either by using a known reference phase shift such as 180° or by assuming that the calibration of the fine dial is correct. In the procedure actually followed the computed values for the fine control dial were used to get a calibration on each of the decade capacitor steps. Using these, a preliminary calibration of first the small and then the large π-sections was obtained. Then the 180° point was checked using a special toroidal core transformer. Details of the method used will now be described.

A potentiometer-bridge circuit arrangement plus an auxiliary phase shifter (see fig. 5) were used to compare the phase shift settings on the various dials of the π-section standard. The auxiliary phase shifter, consisting of an R-C circuit supplied through a ratio transformer, need not be accurately calibrated, but it must be stable and continuously variable from 0° up to at least 15°. A sensitive amplifier with shielded input transformer was used to indicate when the phase shifts put in by standard and auxiliary phase shifters were equal. The ratio transformer was used to give a magnitude balance.

The phase shift put in by the fine control capacitor as computed in section 5 was assumed to be exactly correct, and then each step on the capacitor decade switch was measured in terms of the fine control capacitor, as follows. All π-sections were disconnected. With the fine control capacitor set on

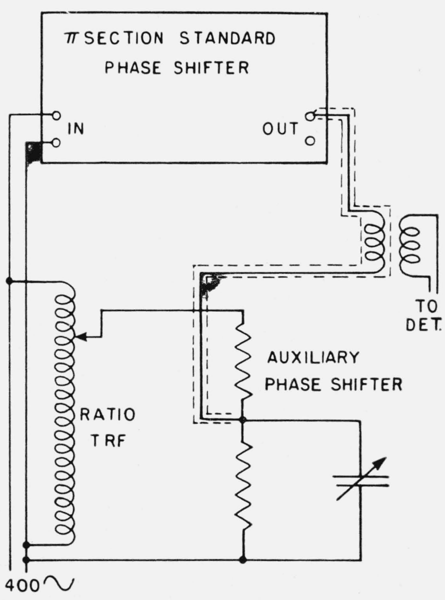

FIGURE 5. *Setup used in experimental calibration.*

0, i.e., maximum capacitance and the capacitor decade switch on step one, the auxiliary phase shifter was adjusted (together with the ratio transformer) until the detector showed a balance. Then, with the auxiliary phase shifter setting held constant and the decade capacitor switch set back to zero the detector was balanced by adjusting the fine control capacitor and ratio transformer. Since the auxiliary phase shifter remains unchanged for these two balances the phase shift put in by the first step on the decade capacitor must equal that read on the fine control dial after the second balance. The change in phase shift when the decade capacitor switch is changed from 2 to 3, 3 to 4, etc., can be measured in the same manner. The actual values as measured are given in table 3.

TABLE 3. *Phase shift for each step of capacitor decade switch—as determined by comparison with fine control dial*

Decade capacitor step	Phase shift of each step	Σ	Phase shift corrected for 180° check	Σ
	deg			
1	0. 4468		0. 4470	
2	. 4429	0. 8897	. 4431	0. 8901
3	. 4454	1. 3351	. 4456	1. 3357
4	. 4340	1. 7691	. 4342	1. 7699
5	. 4436	2. 2127	. 4438	2. 2137
6	. 4440	2. 6567	. 4442	2. 6579
7	. 4344	3. 0911	. 4346	3. 0925
8	. 4447	3. 5358	. 4449	3. 5374
9	. 4406	3. 9764	. 4408	3. 9782
10	. 4428	4. 4192	. 4430	4. 4212

To measure the phase shift of each small π-section, the switching arrangement was modified slightly from that shown in figure 1, so that any one of the three sections could be connected or disconnected individually. Then by using the method just described for the capacitor decade steps, the phase shift of each small π-section was measured in terms of the decade capacitor switch and the fine control dial. Another modification of the switching arrangement made it possible to connect any individual large π-section in by itself, and the phase shift of each of these sections was measured in terms of the three small sections plus the setting on the capacitor decade and the fine control dial. Before measuring each π-section its impedance was checked and adjusted to be equal to R_0 and C_0 in parallel by the method described in section 4.2.

The experimental calibration values obtained by the method just described and those computed for the fine control dial (given in table 2) were actually carried out to 0.0001°. The estimated error for each step of the capacitor decade is 0.0005°. If this error is not completely random it would be accumulative and might add up to an appreciable value for the large π-sections. Consequently the values assigned to the 14.6° π-sections based entirely on the computed phase shift for the fine control capacitor were found to differ from computed values based on measured values of inductance of each section by as much as 0.02°. Variations in these differences upon repeated calibrations were usually found to be less than 0.005°. In order to correct for these fairly large errors in the 14.6° π-sections introduced by very small errors in the calibration of the capacitor decade, an independent check of the 180° point was required.

This was done by connecting the primary of a uniformly wound toroidal transformer in parallel with the input to the standard phase shifter. The secondary of the toroidal transformer was used to get an exact 180° phase shift by (1) reversing polarity or (2) using its two coils as a center tap winding with the center grounded. The phase shifter zero was set with the secondary polarity giving zero phase shift, and then with the opposite polarity the 180° point was measured on the phase shifter. The magnitude was balanced by connecting a ratio transformer either at the primary or at the secondary of the toroidal transformer. The 180° point was measured using five different arrangements for the windings of the toroidal transformer and with the two locations of ratio transformer for each of the five arrangements. Since the results obtained on all measurements were the same to within 0.01° it was concluded that the phase shift being measured was exactly 180°. Using the preliminary experimental calibration, based on computed values of phase shift for the fine control capacitor, the measured value came out to be 179.92°. Thus a correction factor of 180/179.92 or (1+0.000417) was applied to the preliminary calibration values to get the correct values as listed in table 1 for the 14.6 and 4.3 degree π-sections and in columns 4 and 5 of table 3 for the capacitor decade steps. As indicated in table 1 these measured values agree quite well with the values computed from measured values of inductance for each π-section. Although the change in the capacitor decade calibration introduced by the 180° check is quite small, as seen from table 3, it should be included for accurate measurements.

7. Requirements for Use of Phase Standards

The master phase angle π-section standard described in this paper is a continuously variable phase shifter whose absolute calibration has been accurately determined. It is primarily intended to be used to establish corrections for other continuously variable phase shifters and phasemeters used as standards. Herein, a phasemeter is defined as an instrument capable of measuring the phase shift between two alternating voltages of the same frequency. Throughout this kind of calibration testing the two factors most likely to affect accuracy are (1) the impedance of the instrument under test and (2) waveform distortion in supply voltage. These two points are of sufficient importance to be considered separately

7.1 Compensation for Impedance of Instrument Under Test

The connections used for calibrating a phasemeter are shown in the block diagram in figure 6. The

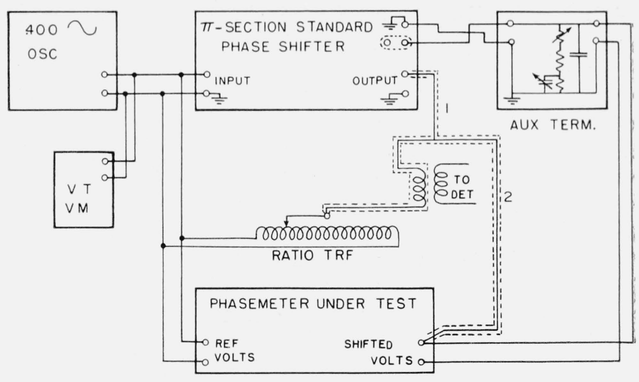

Figure 6. *Setup used for calibrating a phasemeter.*

"input" to the π-section phase standard is supplied by a 2-w low distortion oscillator with a regulated output of 5 v. This value of voltage was chosen because it puts 0.25 v across each 14.6° π-section, at which value the change of inductance with voltage is minimum for the iron core inductors. It was found that a 5-percent change in voltage at this level was required to produce a detectable change in inductance. Actually the voltage is maintained constant to within 2 percent. The reference or unshifted voltage input terminals of the meter under test are connected in parallel with the standard "input." At this point the value of impedance of the instrument under test is not critical because it merely acts as an additional load on the oscillator, and its only possible effect would be a slight change in waveform of the supply voltage. This effect would be entirely negligible for most instruments used as reference standards. If a value of voltage other than that required for the π-section standard (5 v) is required by the meter under test, a transformer would normally be used. In such a case the phase shift introduced by the transformer must be known so that a correction for it can be applied.

If the impedance looking into the phasemeter through the shifted voltage binding posts were infinitely high, the output of the π-section standard could be connected directly to these "shifted voltage" binding posts without affecting the terminating impedance of the π-section line. Under this condition the "auxiliary termination" would not be needed. However, since the performance of the π-section line is sensitive to small changes in its termination, Z_0, some compensation will normally be required for the impedance at the "shifted voltage" binding posts. This compensation is provided for by two insulated binding posts at the output end of the π-section phase standard, (see figs. 6 and 1), one being connected to the output end of the π-section line and the other to R_0 and C_0 (in parallel) which form the normal line termination. When these two binding posts are connected by a copper link the π-section line is terminated inside the π-section standard case and the normal output terminals can be used. By removing the link the internal line termination is disconnected and the "auxiliary termination" may be connected as shown in figure 6. This auxiliary termination consists of a stable adjustable R–C network which is connected in parallel with the leads from the π-section standard and arranged so that with the "shifted volts" leads also connected in parallel, the total impedance thus formed can be adjusted to be equal to Z_0 (the π-section impedance).

In order to make this adjustment a ratio transformer and a sensitive detector with a shielded input transformer are connected as shown in figure 6. First the leads from the π-section standard to the auxiliary termination are disconnected, the link replaced on the π-section standard and the switch put in "check" position (see fig. 1). Under these conditions the π-section line termination is the same as it was when the capacitor legs were adjusted, i.e., it consists of C_0, R_0, and the compensating capacitor C_s in parallel. With the detector transformer lead in position "1" and all π-section standard dials set on "zero," the detector is balanced by adjusting the ratio transformer and the "zero set capacitor." Next the line termination C_0, R_0, C_s is replaced by the "auxiliary termination." This is done by removing the link from the π-section standard binding posts, putting the switch in the "test" position, and connecting the auxiliary termination to the π-section standard and the phasemeter under test as shown in figure 6. The detector is then balanced by adjusting the impedance of the auxiliary termination. To correct for any phase shift introduced by the "auxiliary termination" and its leads, the detector lead position is changed from "1" to "2" and the zero set capacitor is adjusted to balance the detector. Any correction for the phase shift in the ratio transformer can be made during this balance by actually setting this phase shift on the π-section standard fine control dial before making the final balance. The phasemeter under test can now be calibrated for any phase shift angle by first disconnecting all leads from the phase shifters to the detector transformer, then setting the required angle on the π-section phase standard and reading the corresponding value on the phasemeter.

If a voltage higher than the 1 v available from the π-section standard output must be applied to the phasemeter under test, an amplifier can be used between the auxiliary termination and the "shifted volts" binding posts. The only requirements are (1) the amplifier must have low distortion and (2) the three balances as described in the previous paragraph must be made with the amplifier and its leads in position and the amplifier gain set to the same voltage as to be used during the calibration. The phase shift introduced by the amplifier may be fairly large, thus requiring considerable change in the zero set capacitor during the final detector balance before starting actual calibration.

For calibrating another continously variable phase shifter the procedure would be essentially the same as that just described for a phasemeter except that an external detector must be used to indicate when the phase shift put in by the phase shifter being tested is equal to that of the standard. For a given setting of the phase shifter being tested the detector would be balanced by adjusting the phase shift put in by the π-section standard and the magnitude of the output voltage from the phase shifter being tested by using a ratio transformer.

7.2. Effect of Waveform Distortion

At the very beginning of this paper phase angle was defined to be the quantity used as a measure of the time phase relation between two voltages having pure sine waveform. So far the effects of distortion have not been mentioned. However, since it is impossible to obtain pure sine waves in practice, such effects must be considered. For a distorted waveform, phase angle is herein defined as the time phase relation between the fundamental sine-wave components of the two voltages. Thus, there is no change in definition but a definite basis is established for a discussion of the effects of unwanted harmonics. If all the circuit elements were linear (i.e., no iron cores or lossy dielectrics were used) and if measuring and detecting instruments were only sensitive to the fundamental, then the use of a distorted waveform for making the measurements would make no difference. Unfortunately, these postulations cannot be met in practice even though very selective amplifier detectors are available.

Since the π-section standard as described in this paper is made up of iron core inductors the voltage drop across each inductor is always kept low enough so that flux density in the core is well below saturation. Yet it is possible that distortion could be introduced by the inductors even though the supply voltage has a pure sine wave form. To investigate this possibility, first an oscillator of very low distortion was used to supply the phase shifter "input," and distortion was measured at the "output" posts for various phase angle settings, i.e., for different numbers of π-sections in the "line." The waveform with phase shifter set on "0" i.e., no iron core inductors connected contained 0.02 percent 2d harmonic, 0.01 percent 3d harmonic, and 0.01 percent of higher harmonics. As inductors were connected all harmonics remained at about the same value except the 3d which gradually increased to about 0.04 percent with all twelve 14.6° π-sections in circuit. Thus for a nearly pure sine-wave input the inductors in the π-sections slightly increase the 3d harmonic distortion.

Similar tests were made using an oscillator with somewhat greater distortion, i.e., about 0.03 percent 2d, 0.5 percent 3d, and 0.16 percent 4th. For this input voltage the effect of adding π-section inductors on the distortion was too small to be significant (less than about 0.02 percent). In order to check for possible effects of this distortion on measurements involving only the fundamental component, the 180° point was checked as described in section 6 using each of the oscillators whose distortions have just been given. The results using the oscillator with 0.5 percent 3d harmonic agreed with those using the nearly pure sine wave oscillator to within about 0.005°. Thus it was concluded that for accuracies within 0.01° the total wave form distortion should be less than 0.5 percent. It should be borne in mind that in making these tests a highly selective, feed-back tuned, amplifier was used in the detector circuit.

Waveform may also have an effect on or may be affected by the phase shifter or phasemeter being calibrated. For testing another phase shifter of the same design as the π-section standard the restrictions on distortion should be the same as those given above. For other types of phase shifters or phasemeters the effects of wave-form distortion should be determined by repeat tests using two oscillators of significantly different waveform. From a limited number of such tests it is believed that in general for all oscillators with less than 0.5 percent total distortion the phase-angle test results would be the same to within about 0.01°.

8. Summary and Conclusions

A continuously variable, 0° to 180° phase shifter for 400 cps has been described in detail. By using a special procedure for matching the characteristic impedance of the L–C π-sections used in this phase shifter, a method for experimentally measuring the phase shift of each section has been devised. This phase shift has also been computed from measured values of inductance for each π-section. The results obtained by these two methods agree to within 0.01°. These results indicate that this phase shifter can be used as a master standard in calibrating other phase shift devices.

Similar phase angle standards for higher audio-frequencies could be designed, but on the basis of the experience gained it is thought that the upper limit in frequency would be about 20 kc. At higher frequencies, stray capacitance introduced by connecting and switching leads might prove troublesome. It might be feasible to use the same master standard over a frequency range of about 2 to 1 by some readjustments on the π-sections and termination for each frequency.

Appendix 1. Computation of Characteristic Impedance of π-Section—Losses Considered

A single terminated π-section with losses in both inductor and capacitor considered, is shown schematically in figure 7. To keep the derivation general, the two capacitor legs are not assumed to be equal. R'' represents the total losses in $C''/2$ expressed as a parallel resistance and likewise R' for $C'/2$. The total losses in the inductor are represented by r, as a series resistance. Five circuit equations may be written by inspection of figure 7:

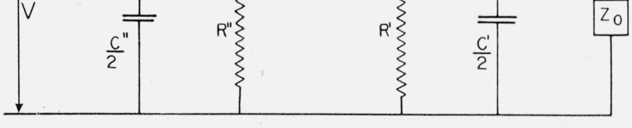

FIGURE 7. *A terminated π-section with losses considered.*

$$i_1a-i_2a=V,$$

$$-i_1a+i_2b-i_3R''=0,$$

$$-i_2R''+i_3d-i_4R'=0,$$

$$-i_3R'+i_4f-i_5g=0,$$

$$-i_4g+i_5(Z_0+g)=0,$$

where

$$a=\frac{1}{j\omega\frac{C''}{2}},\qquad b=R''+\frac{1}{j\omega\frac{C''}{2}},\qquad d=R''+r+R'+j\omega L,$$

$$f=R'+\frac{1}{j\omega\frac{C'}{2}},\qquad \text{and}\qquad g=\frac{1}{j\omega\frac{C'}{2}}.$$

Using these equations and putting $1/Z_0=(1/R_0)+j\omega C_0$, the following expression for input admittance Y_{in} is obtained:

$$Y_{in}=j\omega\frac{C''}{2}+\frac{\left[1+j\omega R_0\left(C_0+\frac{C'}{2}\right)\right]\left[1+\frac{r}{R''}+j\omega\frac{L}{R''}\right]+R_0\left[\frac{1}{R'}+\frac{r}{R'R''}+\frac{1}{R''}+\frac{j\omega L}{R'R''}\right]}{\left[1+j\omega R_0\left(C_0+\frac{C'}{2}\right)\right](r+j\omega L)+R_0\left[1+\frac{r}{R'}+\frac{j\omega L}{R'}\right]}. \tag{5}$$

To obtain a relation which requires the characteristic impedance of the π-section to be $R_0/(1+j\omega C_0R_0)$ equate the above value of Y_{in} to $(1/R_0)+j\omega C_0$. This gives a complex equation from which two equations involving circuit parameters only can be formed. By equating the real parts of the complex equation the following is obtained:

$$\omega^2R_0rC_0^2-\left[\omega^2R_0r\left(\frac{C''}{2}-\frac{C'}{2}\right)-2\omega^2L\right]C_0+\omega^2L\left(\frac{C'}{2}-\frac{C''}{2}\right)-\omega^2R_0r\frac{C'C''}{4}-\frac{r}{R_0}+m=0, \tag{6}$$

where

$$m=\frac{R_0}{R'}\left[1+\frac{R'}{R''}-\frac{r}{R_0}+\frac{rR'}{R_0R''}+\frac{r}{R''}-\frac{\omega^2LR'}{R''}\left(C_0+\frac{C'}{2}\right)+\omega^2L\left(C_0-\frac{C''}{2}\right)\right].$$

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By equating the imaginary parts of the complex equation the following is obtained:

$$\omega r\left(2C_0+\frac{C'}{2}-\frac{C''}{2}\right)$$
$$+\frac{\omega L}{R_0}\left[1-\omega^2R_0^2\left(C_0^2+\frac{C_0C'}{2}-\frac{C_0C''}{2}-\frac{C'C''}{4}\right)\right]$$
$$+\omega R_0\left(-\frac{C''}{2}-\frac{C'}{2}\right)+\frac{n}{R_0}=0, \tag{7}$$

where

$$n/R_0=\omega L\left(\frac{1}{R'}-\frac{1}{R''}-\frac{R_0}{R'R''}\right)$$
$$+\omega rR_0\left[\frac{C_0-\frac{C''}{2}}{R'}-\frac{C_0+\frac{C'}{2}}{R''}\right].$$

For $C'/2=C''/2=C/2$ these two equations can be used to solve for $C/2$ and C_0 when L, R_0, r, R', R'' are known. Equation (6) is used to solve for C_0. Putting $C/2=(L/2R_0^2)\,(1+\delta)$ where $\delta<1$, this equation becomes:

$$\omega^2R_0rC_0^2+2\omega^2LC_0-\frac{r\omega^2L^2}{4R_0^3}\,(1+\delta)^2-\frac{r}{R_0}+m=0. \tag{8}$$

Solving it for C_0 gives

$$C_0=\frac{r}{2\omega^2LR_0}\,(1+\alpha), \tag{9}$$

where

$$1+\alpha=\left[1+\frac{\omega^2L^2}{4R_0^2}(1+\delta)^2-\frac{R_0}{r}m\right]\left(1-\frac{\Delta}{4}+\frac{\Delta^2}{8}-\frac{5\Delta^3}{64}+\cdots\right),$$
$$\Delta=\frac{r^2}{\omega^2L^2}\left[1+\frac{\omega^2L^2}{4R_0^2}\,(1+\delta)^2-\frac{R_0}{r}\,m\right].$$

An estimated value is used first for δ. The exact value is determined later from eq (10) and a recomputation made if necessary. Using this value for C_0 and $C'/2=C''/2=C/2$, eq (7) becomes

$$\omega^2L\left(\frac{C}{2}\right)^2-2\frac{C}{2}+\frac{L}{R_0^2}\left[1+\frac{3r^2}{4\omega^2L^2}\left(1+\frac{2\alpha}{3}-\frac{\alpha^2}{3}\right)+\frac{n}{\omega L}\right]=0.$$

Solving for $C/2$ gives

$$\frac{C}{2}=\frac{L}{2R_0^2}\left[1+\gamma+\frac{\omega^2L^2}{4R_0^2}(1+\gamma)^2+\frac{\omega^4L^4}{8R_0^4}(1+\gamma)^3+\ldots\right] \tag{10}$$

where

$$\gamma=\frac{3r^2}{4\omega^2L^2}\left(1+\frac{2\alpha}{3}-\frac{\alpha^2}{3}\right)+\frac{n}{\omega L}.$$

Now going back to eq (6) and putting $C'/2=C/2\,(1+\lambda')$ and $C''/2=C/2\,(1=\lambda'')$, where $C/2$ is the value from eq (10), it follows that

$$\lambda'-\lambda''+\frac{C_0R_0r}{L}\,(\lambda'-\lambda'')-\frac{rR_0C}{2L}\,(\lambda'+\lambda''+\lambda'\lambda'')$$
$$=\frac{rR_0}{L}\frac{C}{2}-\frac{rR_0C_0^22}{LC}-\frac{2C_02}{C}+\frac{r2}{\omega^2LR_0C}-\frac{m2}{\omega^2LC} \tag{11}$$

Likewise eq (7) gives

$$\lambda'+\lambda''-\frac{r}{R_0}(\lambda'-\lambda'')+\omega^2LC_0\,(\lambda'-\lambda'')$$
$$-\omega^2L\frac{C}{2}\,(\lambda'+\lambda''+\lambda'\lambda'')=\frac{2rC_02}{R_0C}+\frac{L2}{R_0^2C}$$
$$-\frac{\omega^2LC_0^22}{C}+\frac{\omega^2LC}{2}-2+\frac{2n}{\omega R_0^2C}. \tag{12}$$

Equations (11) and (12) can be used to compute actual changes in $C'/2$ and $C''/2$ that are made experimentally when adjusting the characteristic impedance of a given π-section to exactly match the terminating impedance (R_0 and C_0 in parallel). First the values for $C/2$ and C_0 are computed using eqs (10) and (9) by assuming average values for L, r, R' and R''. A value for R_0 must also be chosen. Then if these values of R_0 and C_0 are kept the same the effect of small changes in r, R', and/or R'' on values of $C''/2$ and $C'/2$ can be computed using eqs (11) and (12). The actual value of $C'/2$ can then be used to accurately compute the phase shift for each individual π-section as indicated in appendix 2.

The above procedure may be best illustrated by taking actual values for the 14.6° π-section:

$$\mathrm{L}=100\text{ mh},\qquad r=2.8\ \Omega,\qquad R'=R''=1\times10^7\Omega,$$

$$R_0=1{,}000\ \ \Omega,\qquad \omega=2513.27,\qquad \omega^2=6.31653\times10^6.$$

Putting these values in eqs (9) and (10) gives

$$C_0=0.002099\times10^{-6},$$

$$C/2=0.050819_1\times10^{-6}.$$

When these values are put in eqs (11) and (12) they give $\lambda'=\lambda''=0$ for the values of r, R', and R'' given above. For small changes in r, R', and R'' these equations give the corresponding values for λ' and λ'' required to keep the characteristic impedance the same.

First keep all values the same as above except change r to 3.1 ohms. Putting values in eq (11) gives

$$0.99849\ \lambda'-1.00164\ \lambda''=0.00952. \tag{13}$$

Putting values in eq (12) gives

$$0.96613\ \lambda'+0.96967\ \lambda''=0.000065. \tag{14}$$

Solving eqs (13) and (14) for λ' and λ'' gives

$$\lambda'=0.00480\qquad\text{and}\qquad\lambda''=-0.00472.$$

Next keep all values the same as assumed above except put $R'=10^8$. Then eq (11) becomes:

$$0.99864\ \lambda'-1.00148\ \lambda''=0.00275. \tag{15}$$

Equation (12) becomes

$$0.96643\ \lambda'+0.96937\ \lambda''=-0.000137 \qquad (16)$$

Solvings eqs (15) and (16) for λ' and λ'' gives:

$$\lambda'=0.00131 \qquad \text{and} \qquad \lambda''=-0.00144$$

If the same computation is made for R'' changed to 10^8 the values of λ' and λ'' will be very nearly the same because m is the same and the effect of the change in n is quite small.

Now keep all values as originally assumed except $R'=R''=10^8$. Then eq (11) becomes

$$0.99864\ \lambda'-1.00148\ \lambda''=0.00544. \qquad (17)$$

Equation (12) becomes

$$0.96643\ \lambda'+0.96937\ \lambda''=0.000040. \qquad (18)$$

Solving (17) and (18) for λ' and λ'' gives

$$\lambda'=0.00274 \qquad \text{and} \qquad \lambda''=-0.00270.$$

These values of λ' and λ'' were worked out so that they could be used to compute phase shift under these particular assumed conditions, as will be done in appendix 2.

Appendix 2. Computation of Phase Shift of a π-Section From Value of Inductance

For computing phase shift the first capacitor leg of the π-section need not be considered because it is in parallel with the supply voltage or "V_{in}." Thus the circuit diagram is as shown in figure 8.

The purpose of this derivation is to obtain an expression for the phase shift between V_{in} and V_{out} when the complete π-section (see fig. 7) has been adjusted experimentally so that its impedance as seen from the input end is equal to Z_0.

Three circuit equations may be written by inspection of figure 8:

$$i_1(R'+r+j\omega L)-i_2R'=V_{\text{in}},$$

$$-i_1R'+i_2\left(R'+\frac{1}{j\omega\frac{C'}{2}}\right)-i_3\frac{1}{j\omega\frac{C'}{2}}=0,$$

$$-i_2\frac{1}{j\omega\frac{C'}{2}}+i_3\left(\frac{1}{j\omega\frac{C'}{2}}+Z_0\right)=0.$$

For $Z_0=R_0/1+j\omega C_0R_0$, these equations give the relation

$$\frac{V_{\text{out}}}{V_{\text{in}}}=\frac{1}{1+\frac{r}{R_0}+\frac{r}{R'}-\frac{\omega LA}{R_0}+\frac{jrA}{R_0}+\frac{j\omega L}{R'}+j\frac{\omega L}{R_0}},$$

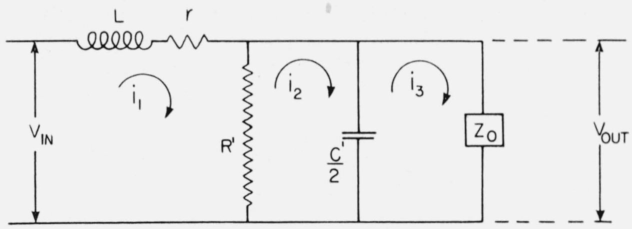

FIGURE 8. *A terminated π-section without first capacitor leg.*

where

$$A=\omega R_0\left(C_0+\frac{C'}{2}\right).$$

The phase shift, θ, is then the angle of the vector in the denominator, thus,

$$\tan\theta=\frac{\omega L/R_0+\omega L/R'+r\omega\left(C_0+\frac{C'}{2}\right)}{1+r/R_0+r/R'-\omega^2L\left(C_0+\frac{C'}{2}\right)}. \qquad (19)$$

This equation can be used to compute $\tan\theta$ for various values of L, r, $C'/2$, R', C_0, and R_0. First using average values ($L=100$ mh, $r=2.8\Omega$, $R'=R''=10^7\Omega$, and $R_0=1{,}000\Omega$) from eqs. (9) and (10), $C/2=C'/2=0.0508191\ \mu$f and $C_0=0.002099\ \mu$f. These values give

$$\tan\theta=0.25968.$$

For the next set of values assumed in appendix 1, i.e., $L=100$ mh, $R'=R''=10^7\Omega$, $r=3.1\ \Omega$, $R_0=1{,}000\ \Omega$, $C_0=0.002099\ \mu$f, $C'/2=0.051063\ \mu$f (from $\lambda'=0.00480$).

$$\tan\theta=0.25968.$$

For $L=100$ mh, $r=2.8\ \Omega$, $R_0=1{,}000\ \Omega$, $C_0=0.002099\ \mu$f, $R''=10^7\Omega$, $R'=10^8\Omega$, $C'/2=0.05089\ \mu$f (from $\lambda'=0.00131$),

$$\tan\theta=0.25967.$$

For the same values as above, except $R''=10^8\Omega$ and $R'=10^7\Omega$, $C'/2$ still equals $0.05089\ \mu$f and

$$\tan\theta=0.25969.$$

Now for all values the same, except $R'=R''=10^8\Omega$, giving $C'/2=0.05095\ \mu$f (from $\lambda'=0.00274$) and $C_0+C'/2=0.05305\ \mu$f

$$\tan\theta=0.25968.$$

The changes in r, $R'+R''$ as assumed above are the maximum to be encountered among the twelve 14.6° sections, as found from actual measurements. Since $\tan\theta$ is very nearly the same for all of these assumed values, the average values based on the actual measurements may be used, i.e., $r=2.8$ ohm, $R'=R''=10^7$ ohm. For $R_0=1{,}000$ ohms these values give $C/2=0.0508191$ μf and $C_0=0.002099$ μf. When these values are put in eq (19) it reduces to a constant times L and for $f=400$ cps this gives

$$\tan\theta=2.59678\ L.$$

Thus for the actual inductors used, if the terminating impedance is held constant ($R_0=1{,}000$ ohms and $C_0=0.002099$ μf) and if the trimmers on each capacitance leg are adjusted so that the impedance looking into the line equals the terminating impedance (by the method explained in section 4.2), the phase shift introduced by each section is equal to a constant times its inductance.

Appendix 3. Computation of Phase Shift of the R–C Network

Figure 9 is a simplified circuit diagram of the R–C network. R_1 is a precision resistor accurately adjusted to 4,000 ohms. C_1 is a precision air capacitor with a capacitance remove dial variable from 1,050 to 50 pf. The change in C_1 can be read from the dial setting to within 1 pf so that it can be used as a fine control of phase shift. R_0 is the resistance component of the π-line input impedance (1,000 ohms). C_2 includes the capacitance component C_0 of the π-line together with the decade capacitor (ten steps of 0.44° each), the zero setting capacitor and any lead capacitance. The purpose of the following derivation is to get an expression for phase shift between voltages E and e in terms of the circuit constants.

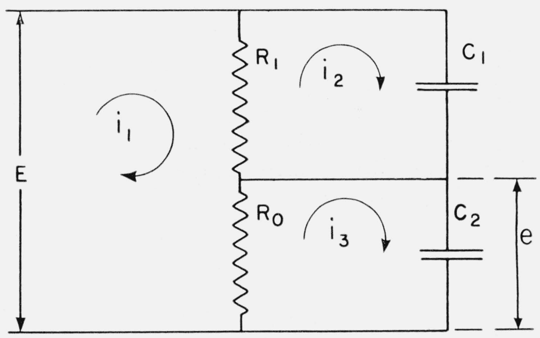

FIGURE 9. *Schematic diagram of the RC circuit in the phase standard.*

The following circuit equation may be written by inspection of figure 9:

$$i_1(R_1+R_0)-i_2R_1-i_3R_0=E,$$

$$-i_1R_1+i_2\left(R_1+\frac{1}{j\omega C_1}\right)=0,$$

$$-i_1R_0+i_3\left(R_0+\frac{1}{j\omega C_2}\right)=0.$$

These equations yield the following relation:

$$\frac{e}{E}=\frac{R_0}{R_1+R_0}\,\frac{1+j\omega C_1R_1}{1+\dfrac{j\omega R_1R_0}{R_1+R_0}(C_1+C_2)}. \qquad (20)$$

By clearing the j term from denominator the following expression for $\tan\theta$ is obtained where θ is the angle by which e leads E:

$$\tan\theta=\frac{\omega R_1\left(C_1\dfrac{R_1}{R_1+R_0}-C_2\dfrac{R_0}{R_1+R_0}\right)}{1+\dfrac{\omega^2R_1^2R_0}{R_1+R_0}C_1(C_1+C_2)}. \qquad (21)$$

For

$R_1=4{,}000$ ohms, $R_0=1{,}000$ ohms, and $\begin{cases}\omega=2513.27\\ \omega^2=6.3165_3\times10^6,\end{cases}$

the expression becomes

$$\tan\theta=\frac{2.01062\times10^6(4C_1-C_2)}{1+20.213\times10^{12}C_1(C_1+C_2)}.$$

Using this expression, phase shifts, θ, were computed for three values of C_1 (200, 600, and 1,200 pf) and for values of C_2 from 0.004 μf to 0.043 μf in steps of 0.0039 μf. The results which were carried out to 0.0001° in order to show small changes are presented in table 2.

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