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A Reduction Formula for Partitioned Matrices

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A theorem of L. Goddard and H. Schneider, concerning square matrices A and B, of orders n and m, respectively, which satisfy an equation $AX=XB$ for some $n\times m$ matrix X, is generalized here for rectangular matrices A and B, with dimensions $n_1\times n_2$, $m_1\times m_2$, which satisfy $AX_2=X_1B$, where X_i has dimensions $n_i\times m_i$ for $i=1,2$. This result is used to find reduction formulas for partitioned matrices with submatrices, A_{ij}, having dimensions $n_i\times n_j$, and satisfying equations $A_{ij}X_j=X_iB_{ij}$. The reduction formulas given here are also generalizations of a theorem by J. Williamson concerning partitioned matrices whose submatrices are all square and satisfy $AX=XB$, where B is triangular and X is square.

1. Introduction

Given a matrix $A=(a_{ij})$ of order n, if there exists a nonsingular matrix P such that

$$\tilde{A}=P^{-1}AP=\begin{pmatrix} B & O \\ * & C \end{pmatrix} \tag{1}$$

where B is a square matrix of order r, and O is an $r\times(n-r)$ matrix of zeros, A is called a *reducible* matrix and the formula (1) is a *reduction formula*. The characteristic equation of A is then factorable:

$$|A-\lambda I_n|=|B-\lambda I_r||C-\lambda I_{n-r}|$$

where I_n represents the identity matrix of order n.

The results in this paper concern a reduction formula of the type (1) for partitioned matrices, and give a connection between, and an extension of, results by L. Goddard and H. Schneider,[1] and J. Williamson.[2] The theorem of Goddard and Schneider is generalized in theorem 1, and this is applied in theorem 2 to give a general reduction formula for partitioned matrices,

$$A=\begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1t} \\ A_{21} & A_{22} & \ldots & A_{2t} \\ \\ A_{t1} & A_{t2} & \ldots & A_{tt} \end{bmatrix} \tag{2}$$

where the submatrices A_{ij} have dimensions $n_i\times n_j$, $i,j=1, \ldots, t$.

Theorem 2 contains the results of Goddard and Schneider and those of Williamson as special cases.

[1] L. S. Goddard and H. Schneider, Matrices with a nonzero commutator, Proc. Camb. Phil. Soc. **51**, 551 (1955).

[2] J. Williamson, The latent roots of a matrix of special type, Bul. Am. Math. Soc. **37**, 585 (1931).

2. Results of Goddard and Schneider

Goddard and Schneider showed that if two square matrices A and B of orders n and m, respectively, are related by the equation,

$$AX=XB, \tag{3}$$

where X is an $n\times m$ matrix of rank r, then there exist matrices P and Q, depending on X, such that

$$P^{-1}AP=\begin{pmatrix} E & * \\ O & C \end{pmatrix}, \qquad Q^{-1}BQ=\begin{pmatrix} E & O \\ * & D \end{pmatrix}, \tag{4}$$

where E is square, of order r, so that A and B have r roots in common.

More generally, Goddard and Schneider proved, for any $m\times n$ matrix K and any polynomial $f(x, y)$,

$$\begin{aligned} P^{-1}f(A,XK)P&=\begin{pmatrix} f(E,G) & * \\ O & f(C,0) \end{pmatrix} \\ Q^{-1}f(B,KX)Q&=\begin{pmatrix} f(E,G) & O \\ * & f(D,0) \end{pmatrix} \end{aligned} \tag{5}$$

where G depends on K. So all such functions of A and B are reducible if $r<\min(m,n)$, and each corresponding pair (for which f and K are the same) has r roots in common.

If $r=m<n$, A and $f(A,XK)$ are reducible and (1) holds for A and B as defined in (3) and C as defined in (4).

If $r=m=n$, for the matrices in (3), we have $X^{-1}AX=B$, so if B is reduced, A and $f(A, XK)$ are reducible. Naturally, similar results would hold for B if $r=n\leqq m$.

Thus, although the results of Goddard and Schneider concern a pair of matrices, the cases given above can be considered as leading to a reduction formula for one or both of the matrices.

3. Results of Williamson

J. Williamson[2] deals with partitioned matrices as in (2) in which all submatrices, or *blocks*, are square. He showed that if a partitioned matrix A, of order nt, has blocks of order n which can be simultaneously reduced to triangular form, then A is reducible to a *block-triangular* matrix, i.e., a partitioned matrix (2) in which $A_{ij}=0$, $j>i$.

Williamson's results could be expressed as follows: Given a partitioned matrix $A=(A_{ij})$ of order nt, with blocks of order n, if there exists a nonsingular, $n\times n$ matrix X such that

$$X^{-1}A_{ij}X=B_{ij}$$

or

$$A_{ij}X=XB_{ij}, \quad (i,j=1, \ldots, t) \tag{6}$$

where B_{ij} is triangular with elements $\lambda^{(ij)}, \ldots, n^{(ij)}$ on the diagonal, then A is similar to a block-triangular matrix with blocks

$$\tilde{A}_{kk}=(\lambda_k^{(ii)})$$

on the diagonal.

This theorem is then generalized by Williamson to show that, given any partitioned matrix $A=(A_{ij})$ satisfying (6) with B_{ij} triangular, the partitioned matrix which has blocks

$$G_{ij}=f_{ij}(A_{ij})$$

where $f_{ij}(A)$ is a rational function of A with nonsingular denominator, has as roots the roots of the n matrices of order t,

$$\tilde{G}_{kk}=f_{ij}(\lambda_k^{(ii)} \qquad (k=1, \ldots, n; \quad i,j=1, \ldots, t).$$

4. Connection Between Theorems of Goddard and Schneider and of Williamson

It can be seen, from eqs (3) and (6), that although the theorem by Williamson and that of Goddard and Schneider are quite different, there is a connection between them.

In theorem 1 we generalize the results of Goddard and Schneider to rectangular matrices A and B of dimensions $n_1\times n_2$ and $m_1\times m_2$, respectively, satisfying

$$AX_2=X_1B, \tag{7}$$

where X_i is $n_i\times m_i$ of rank r_i, $i=1$ or 2.

We then apply this result in theorem 2 to square partitioned matrices A and B, with rectangular blocks $n_i\times n_j$ and $m_i\times m_j$, respectively, in which the following relations hold between the blocks,

$$A_{ij}X_j=X_iB_{ij}, \quad (i,j=1, \ldots, t) \tag{8}$$

where X_i is $n_i\times m_i$ of rank r_i.

[2] J. Williamson The latent roots of a matrix of special type, Bul. Am. Math. Soc. **37**, 585 (1931).

Then the theorem of Goddard and Schneider can be obtained as a special case of theorem 2 when we set $t=1$ in (8). If we have $r_i=m_i=n_i=n$, $i=1, \ldots, t$, and B_{ij} triangular, $i,j=1, \ldots, t$, we have the theorem of Williamson.

5. Theorem of Goddard and Schneider for Rectangular Matrices

The proof of theorem 1 is essentially the same as that of Goddard and Schneider, the only difference being the addition of subscripts to correspond with the two different matrices, X_1 and X_2.

THEOREM 1. *If A and B are rectangular matrices satisfying (7), there exist matrices P_1, P_2, Q_1 and Q_2, such that*

$$P_1^{-1}AP_2=\begin{pmatrix} E & * \\ O & C \end{pmatrix}, \qquad Q_1^{-1}BQ_2=\begin{pmatrix} E & O \\ * & D \end{pmatrix}, \tag{9}$$

where E is an $r_1\times r_2$ matrix, and C and D are rectangular matrices having dimensions $(n_1-r_1)\times(n_2-r_2)$ and $(m_1-r_1)\times(m_2-r_2)$, respectively. Moreover, if $f(x, y)$ is a polynomial in two noncommutative indeterminates (which is linear and homogeneous if A and B are not square, or if $r_1\neq r_2$) and K is an arbitrary $m_1\times n_2$ matrix,

$$P_1^{-1}f(A,X_1K)P_2=\begin{pmatrix} f(E,G) & * \\ O & f(C,0) \end{pmatrix}, \qquad Q_1^{-1}f(B,KX_2)Q_2=\begin{pmatrix} f(E,G) & O \\ * & f(D,0) \end{pmatrix}, \tag{10}$$

where G depends upon K.

PROOF. For $i=1$ or 2, we have X_i of rank r_i, so there exist corresponding nonsingular matrices, P_i and Q_i, of order n_i and m_i, respectively, such that

$$Y_i=P_i^{-1}X_iQ_i=\begin{pmatrix} I_{r_i} & O \\ O & O \end{pmatrix} \tag{11}$$

is an $n_i\times m_i$ matrix with an identity matrix of order r_i in the upper left corner.

Let

$$\tilde{A}=P_1^{-1}AP_2, \qquad \tilde{B}=Q_1^{-1}BQ_2, \qquad \tilde{K}=Q_1^{-1}KP_2 \tag{12}$$

and partition each matrix so that there is an $r_1\times r_2$ matrix in the upper left corner, i.e.,

$$\tilde{A}=\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix},$$

and B and K are partitioned similarly.

Then, from (11) and (12),

$$P_1^{-1}AX_2Q_2=\tilde{A}Y_2=\begin{pmatrix}\tilde{A}_{11} & O\\ \tilde{A}_{21} & O\end{pmatrix},$$
$$P_1^{-1}X_1BQ_2=Y_1\tilde{B}=\begin{pmatrix}\tilde{B}_{11} & \tilde{B}_{12}\\ O & O\end{pmatrix}. \tag{13}$$

Since, by (7), $\tilde{A}Y_2=Y_1\tilde{B}$, (13) implies

$$\tilde{A}_{11}=\tilde{B}_{11},\quad \tilde{A}_{21}=\tilde{B}_{12}=0. \tag{14}$$

Thus, if we let $\tilde{A}_{11}=E$, $\tilde{A}_{22}=C$, $\tilde{B}_{22}=D$, from (12) and (14) we have (9). Also

$$P_1^{-1}X_1KP_2=Y_1\tilde{K}=\begin{pmatrix}\tilde{K}_{11} & \tilde{K}_{12}\\ O & O\end{pmatrix}$$
$$Q_1^{-1}(KX_2)Q_2=\tilde{K}Y_2=\begin{pmatrix}\tilde{K}_{11} & O\\ \tilde{K}_{21} & O\end{pmatrix} \tag{15}$$

so if we let $\tilde{K}_{11}=G$, we obtain (10).

6. A General Reduction Formula

In this section we apply theorem 1 to partitioned matrices with rectangular blocks satisfying (8). We will in general obtain formulas of the type (4) which can be considered as reduction formulas for A or B or both.

Specific applications of this theorem in the reduction of certain special partitioned matrices have been given previously by the author.[3]

Before applying theorem 1 to theorem 2 it will be necessary to prove the following.

LEMMA. Given a partitioned matrix of order N, with $n_i\times n_j$ blocks A_{ij}, if

$$A_{ij}=\begin{pmatrix}B_{ij} & O\\ * & C_{ij}\end{pmatrix} \tag{16}$$

where all matrices B_{ij} are square, of order r, and the matrices C_{ij} are $(n_i-r)\times(n_j-r)$, then A is reducible as in (1) where P is a permutation matrix, and $B=(B_{ij})$, $C=(C_{ij})$.

PROOF. The proof consists merely in defining the permutation matrix P, which is equivalent to writing the order in which the rows and columns of A should be arranged.

Let $\sum_{i=1}^{k}n_i=N_k$. Then $N_1=n_1$, $N_t=N$. If we arrange the rows and columns of A in the following order:

$$\begin{array}{llll}
1, & 2, & \ldots, & r;\\
N_1+1, & N_1+2, & \ldots, & N_1+r;\\
 & \ldots & & \\
N_{t-1}+1, & N_{t-1}+2, & \ldots, & N_{t-1}+r;\\
r+1, & r+2, & \ldots, & N_1;\\
N_1+r+1, & N_1+r+2, & \ldots, & N_2;\\
 & \ldots & & \\
N_{t-1}+r+1, & N_{t-1}+r+2, & \ldots, & N_t;
\end{array} \tag{17}$$

we have a new matrix A in which the matrices B_{ij} are together in the upper left corner, and the matrices C_{ij} are together in the lower right corner, so $\tilde{A}$ will have the form (1).

THEOREM 2. *Suppose we have partitioned matrices, A and B, with rectangular blocks, A_{ij} and B_{ij}, satisfying (8), where $r_i=r$, $i=1, \ldots, t$. Then, A and B have tr roots in common.*

Moreover, if $G=(G_{ij})$ and $H=(H_{ij})$ have rectangular blocks,

$$G_{ij}=f_{ij}(A_{ij},X_iK_{ij}),\qquad H_{ij}=f_{ij}(B_{ij},K_{ij}X_j) \tag{18}$$

where the matrices K_{ij} are arbitrary $m_i\times n_j$ matrices and the polynomials $f_{ij}(x,y)$ are as defined in theorem 1, then all pairs of matrices defined in (18) have tr roots in common.

PROOF. Let P_i and Q_i be matrices satisfying (11) for $i,j=1, \ldots, t$ and let P and Q be the direct sums of the matrices P_i and Q_i respectively, i.e.,

$$P=P_1\dot{+}P_2\dot{+}\cdots\dot{+}P_t=\sum_{i=1}^{t}\dot{}P_i,\ Q=\sum_{i=1}^{t}\dot{}Q_i.$$

Then we have, using block multiplication of matrices,

$$P^{-1}AP=\tilde{A}=(\tilde{A}_{ij}),\quad Q^{-1}BQ=\tilde{B}=(\tilde{B}_{ij}),$$

where

$$\tilde{A}_{ij}=P_i^{-1}A_{ij}P_j,\quad \tilde{B}_{ij}=Q_i^{-1}B_{ij}Q_j.$$

Thus, from (9),

$$\tilde{A}_{ij}=\begin{pmatrix}E_{ij} & *\\ O & C_{ij}\end{pmatrix},\qquad \tilde{B}_{ij}=\begin{pmatrix}E_{ij} & O\\ * & D_{ij}\end{pmatrix}.$$

[3] E. Haynsworth, Applications of a theorem on partitioned matrices, J. Research NBS **63B**, 73 (1959).

So, by the lemma, $\tilde{A}$ and $\tilde{B}$ are similar to the matrices $\hat{A}$ and $\hat{B}$,

$$\hat{A}=\begin{pmatrix} E & * \\ O & C \end{pmatrix}, \qquad \hat{B}=\begin{pmatrix} E & O \\ * & D \end{pmatrix},$$

where $E=(E_{ij})$, $C=(C_{ij})$, $D=(D_{ij})$. Thus the roots of E are roots of both matrices.

Also, using (15) for the matrices K_{ij}, P_i, P_j, Q_i, Q_j, we can in the same way obtain the more general result concerning the matrices G and H.

COROLLARY 1. *If a matrix A of order nt can be partitioned into square blocks A_{ij}, of order n, having r linearly independent characteristic vectors x_h in common, corresponding to the roots, $\lambda_h^{(ij)}$, $h=1, \ldots, r$; $i, j=1, \ldots, t$, then tr of the roots of A are roots of the matrices $(\lambda_h^{(ij)})$.*

PROOF: Let

$$X_i=X=(x_1,x_2, \ldots, x_r)$$

be the $n\times r$ matrix whose columns are the given vectors and

$$B_{ij}=\text{diag }(\lambda_1^{(ij)}, \lambda_2^{(ij)}, \ldots, \lambda_r^{(ij)}).$$

Then (8) holds for the blocks A_{ij}, and tr of the roots of A are roots of $B=(B_{ij})$. If we now rearrange the rows and columns of B in the order,

$$\begin{array}{lllll} 1, & r+1, & 2r+1, & \ldots, & (t-1)\,r+1; \\ 2, & r+2, & 2r+2, & \ldots, & (t-1)\,r+2; \\ & r, & 2r, & \ldots, & tr; \end{array}$$

B will be in block-diagonal form with the matrices $(\lambda_h^{(ij)})$ on the diagonal.

This corollary is applied by the author to certain special partitioned matrices (see footnote 3).

COROLLARY 2. *If a matrix A, of order nt, can be partitioned into square submatrices of order n which are mutually commutative and have roots, $\lambda_h^{(ij)}$, $(h=1, \ldots, n)$, the roots of A are the roots of the n matrices, $(\lambda_1^{(ij)})$, $(\lambda_2^{(ij)})$, $\ldots$, $(\lambda_n^{(ij)})$.*

This corollary would follow also from Williamson's theorem since any set of mutually commutative matrices can be simultaneously triangularized.

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