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Split Runge-Kutta Method for Simultaneous Equations

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Consider two simultaneous first order differential equations $x'(t)=F(x,y,t)$, $y'(t)=G(x,y,t)$. Runge-Kutta type integration methods are developed which allow different integration steps to be used for these equations. These methods retain the desirable properties of Runge-Kutta methods, namely the self-starting property and ease of change of integration step. Two different approaches are considered and extensive experimental work is reported upon. Experiments are done both in situations where these methods are advantageous and where they are not. It is seen that these methods are more efficient than the normal Runge-Kutta methods if they are at all applicable and in ideal situations they give the same accuracy with 90 percent less computation. These methods are applicable to six degree of freedom missile simulations, for which they were developed.

1. Introduction

Consider two simultaneous first order differential equations:

$$\frac{dx}{dt}=F(x,y,t) \qquad x(t_0)=x_0 \tag{1.1}$$

$$\frac{dy}{dt}=G(x,y,t) \qquad y(t_0)=y_0 \tag{1.2}$$

where $y(t)$ does not depend strongly on $x(t)$ or varies much more rapidly than $x(t)$. In a normal numerical integration method for these equations, the integration step h must be chosen small enough to adequately integrate both (1.1) and (1.2). In this paper Runge-Kutta type methods are described which allow different integration steps to be used for these equations. These methods retain the desirable properties of Runge-Kutta methods, namely the self-starting property and the ability to change the integration step easily.[2]

The problem is defined in detail and two different approches to the development of the formulas are given in section 2. The analysis for third order integration formulas is given in sections 3, 4, 5, and 6. In section 7 the results are stated without derivation for fourth order integration.

Three systems of differential equations have been solved using these formulas with varying values of the parameters. The first equation is of the type suited for these formulas and they result in a considerable saving in computation. The results are discussed in detail in section 9. They point out that the second approach gives formulas which are considerably more accurate than the first approach—although one would not expect this beforehand. The second equation is of a type not suited for these formulas. The results are discussed in section 10. The third equation is of the type suited for these formulas except that a very high frequency low-amplitude oscillation has been added to $x(t)$. The experimental results are somewhat erratic.

In the final section a detailed discussion is given for the situations where these formulas are most useful and also comparison is made of their efficiency with that of the usual methods. It is seen that these formulas are more efficient than the normal Runge-Kutta methods if they are at all applicable and that in ideal situations they may give the same accuracy as normal Runge-Kutta with 90 percent less computation. One area of application is to six degree of freedom missile simulations, for which these formulas were originally derived.

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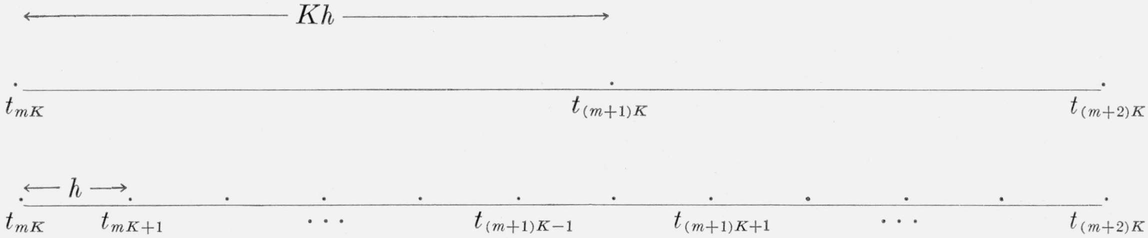
[1] This work was done while the author was at Autonetics, Inc. and at the National Bureau of Standards as an NRC-NBS Research Associate.
[2] For an account of the basic properties of Runge-Kutta see F. B. Hildebrand, Introduction to Numerical Analysis (McGraw-Hill Book Co., Inc., New York, N.Y., 1956).

2. Problem Definition

h denotes the integration step for (1.2), and Kh, where K is an integer, is the integration step for (1.1). Let t_n denote t_0+nh and let x_n and y_n denote the numerical solutions of (1.1) and (1.2), respectively at $t=t_n$. Round-off error is not considered in this paper.

Consider two t-axes,



the first for (1.1) and the second for (1.2). It is now desired to obtain Runge-Kutta type integration formulas that integrate (1.1) in steps of Kh and (1.2) in steps of h.

(1.1) can be integrated from t_{mK} to $t_{(m+1)K}$ by a normal Runge-Kutta method. For a third order method the pertinent equations are:

$$\left.\begin{aligned} k_0&=KhF(x_n,\ y_n,\ t_n)\\ k_1&=KhF(x_n+\gamma_1 k_0,\ y_n+\gamma_1 h_0,\ t_n+\gamma_1 Kh)\\ k_2&=KhF(x_n+\gamma_3 k_1+(\gamma_2-\gamma_3)k_0,\ y_n+\gamma_3 h_1+(\gamma_2-\gamma_3)h_0,\ t_n+\gamma_2 Kh)\\ h_0&=KhG(x_n,\ y_n,\ t_n)\\ h_1&=KhG(x_n+\gamma_1 k_0,\ y_n+\gamma_1 h_0,\ t_n+\gamma_1 Kh)\\ h_2&=KhG(x_n+\gamma_3 k_1+(\gamma_2-\gamma_3)k_0,\ y_n+\gamma_3 h_1+(\gamma_2-\gamma_3)h_0,\ t_n+\gamma_2 Kh)\\ x_{(m+1)K}&=x_{mK}+\beta_0 k_0+\beta_1 k_1+\beta_2 k_2. \end{aligned}\right\} \tag{2.1}$$

The integration parameters γ_1, γ_2, γ_3, β_0, β_1, and β_2 may be those of any third order Runge-Kutta method. Note that h_2 need not be computed for this integration.

The main difficulty in integrating (1.2) is to obtain values of $x(t)$ at the integration points between t_{mK} and $t_{(m+1)K}$. A natural way to obtain these values is to extrapolate $x(t)$ from t_{mK} The Runge-Kutta method is itself an extrapolation process and one extrapolation has been made in the integration of (1.1) from t_{mK} to $t_{(m+1)K}$. The values of k_0, k_1, and k_2 from (2.1) may also be used to extrapolate $x(t)$ from t_{mK} to the intermediate points. Let x_{mK+j} denote the extrapolated value of $x(t)$ at t_{mK+j}, $1\leq j\leq K-1$. Then, for the appropriate coefficients $\lambda_i(j)$,

$$x_{n+j}=\sum_{i=1}^{3}\lambda_i(j)k_{i-1}.$$

Other estimates of $x(t)$ are needed and they are obtained in the same manner.

For a third order method the equations for integrating (1.2) from t_{mK+j} to t_{mK+j+1} are:

$$\left.\begin{aligned} d_0(j)&=hG\,(x_{mK+j},y_{mK+j},t_{mK+j})\\ d_1(j)&=hG\left(x_{mK+j}+\sum_{i=4}^{6}\lambda_i(j)\,k_{i-4},y_{mK+j}+\mu_1 d_0,t_{mK+j}+\mu_1 h\right)\\ d_2(j)&=hG\left(x_{mK+j}+\sum_{i=7}^{9}\lambda_i(j)\,k_{i-7},y_{mK+j}+\mu_3 d_1+(\mu_2-\mu_3)\,d_0,t_{mK+j}+\mu_2 h\right)\\ y_{mK+j+1}&=y_{mK+j}+\alpha_0 d_0(j)+\alpha_1 d_1(j)+\alpha_2 d_2(j). \end{aligned}\right\} \tag{2.2}$$

Two basic approaches are considered for obtaining the integration parameters μ_1, μ_2, μ_3, α_0, α_1 and α_2 and the extrapolation parameters $\lambda_1(j), \ldots, \lambda_9(j)$. The first is to consider the truncation error of the resulting integration formula for $y(t)$. The parameters can be chosen so that the method is third order. This is done in sections 3, 4, and 5. The second approach is to determine the extrapolation parameters so as to make the extrapolation as accurate as possible and then to determine the integration parameters independently. This is done in section 6.

3. Preliminary Computations

For simplicity $\Sigma_{i=n}^{n+2}\lambda_i(j)k_{i-n}$ will be denoted by Σ_n. Likewise,

$$\Lambda_n(j): \qquad \lambda_n(j)+\lambda_{n+1}(j)+\lambda_{n+2}(j)$$

$$\Gamma_n(j): \qquad \gamma_1\lambda_{n+1}(j)+\gamma_2\lambda_{n+2}(j)$$

$$\Delta_n(j): \qquad \gamma_1^2\lambda_{n+1}(j)+\gamma_2^2\lambda_{n+2}(j)$$

$$\phi_n(j): \qquad \gamma_1\gamma_3\lambda_{n+2}(j)$$

These expressions will occur quite often. The argument j will be omitted if it leads to no confusion.

In order to consider the truncation error of the integration formula, various terms will be expanded in Taylor's series. There will be some derivatives evaluated at (x_{mK},y_{mK},t_{mK}) and some at $(x_{mK+j},y_{mK+j},t_{mK+j})$. Expansions will be obtained in this section that will allow a comparison of such terms. A Taylor's series expansion of Σ_n will also be obtained.

The notation: $F_p=\partial F/\partial p$, $G_p=\partial G/\partial p$, $F_{pq}=\partial^2F/\partial p\partial q$, $G_{pq}=\partial^2 G/\partial p\partial q$; $p,q=x,y,t$ will be used. The convention that F denotes $F(x_{mK},y_{mK},t_{mK})$ and that G denotes $G(x_{mK+j},y_{mK+j},t_{mK+j})$ will be adopted. The same convention will hold for derivatives of F and G. Furthermore F_1,G_1,F_2,G_2 will denote dF/dt, dG/dt,

$$F_{xx}F^2+2F_{xy}FG+F_{yy}G^2+2F_{xt}F+2F_{yt}G+F_{tt},$$

$$G_{xx}F^2+2G_{xy}FG+G_{yy}G^2+2G_{xt}F+2G_{yt}G+G_{tt},$$

respectively.

Then it is seen that

$$\left.\begin{aligned}
&F(x_{mK+j},y_{mK+j},t_{mK+j})\\
&\quad=F+F_x(x_{mK+j}-x_{mK})+F_y(y_{mK+j}-y_{mK})+F_tjh+\tfrac{1}{2}[F_{xx}(x_{mK+j}-x_{mK})^2\\
&\qquad+2F_{xy}(x_{mK+j}-x_{mK})(y_{mK+j}-y_{mK})+F_{yy}(y_{mK+j}-y_{mK})^2+F_{tt}(jh)^2\\
&\qquad+2F_{xt}(x_{mK+j}-x_{mK})(jh)+2F_{yt}(y_{mK+j}-y_{mK})(jh)]+\ldots\\
&G(x_{mK},y_{mK},t_{mK})=G-G_x(x_{mK+j}-x_{mK})-G_y(y_{mK+j}-y_{mK})-G_t(jh)+\ldots\\
&\Sigma_n=\Lambda_nKhF+(Kh)^2\Gamma_n[F_xF+F_yG(x_{Km},y_{mK},t_{mK})+F_t]+\frac{(Kh)^3}{2}(\Delta_n)[F_{xx}F^2\\
&\qquad+2F_{xy}FG(x_{mK},y_{mK},t_{mK})+F_{yy}G^2(x_{mK},y_{mK},t_{mK})+2F_{xt}F+2F_{yt}G(x_{mK},y_{mK},t_{mK})+F_{tt}]\\
&\qquad+(Kh)^3\phi_n[F_x(F_xF+F_yG(x_{mK},y_{mK},t_{mK})+F_t)+F_y(G_x(x_{mK},y_{mK},t_{mK})F\\
&\qquad+G_y(x_{mK},y_{mK},t_{mK})G(x_{mK},y_{mK},t_{mK})+G_t(x_{mK},y_{mK},t_{mK}))]+\ldots\\
&\hspace{22em}(n=1,4,7)
\end{aligned}\right\}\qquad(3.1)$$

Let $y_n(t)$ denote the solution of (1.2) which assumes the value y_n at t_n. Then

$$y_{mK+j}-y_{mK}=y_{mK+j}-y_{mK+j}(t_{mK})+y_{mK+j}(t_{mK})-y_{mK}.$$

Since this will be a third order integration method,

$$y_{mK+j}(t_{mK})-y_{mK}=ah^4+bh^5+\ .\ .\ .\ .$$

Now

$$y_{mK+j}-y_{mK+j}(t_n)=Gjh+\frac{(jh)^2}{2}\,[G_xF(x_{mK+j},\,y_{mK+j},\,t_{mK+j})+G_yG+G_t]+\ .\ .\ .$$

and hence

$$y_{mK+j}-y_{mK}=Gjh+\frac{(jh)^2}{2}\,[G_xF(x_{mK+j},\,y_{mK+j},\,t_{mK+j})+G_yG+G_t]+\ .\ .\ . \tag{3.2}$$

With equation (3.2) and by repeated substitution, eqs (3.1) assume the form:

$$F(x_{mK+j},\,y_{mK+j},\,t_{mK+j})=F+\Lambda_1KhF_xF+jh(F_yG+F_t)$$

$$+\frac{h^2}{2}\,[F_{xx}F^2K^2\Lambda_1^2+2F_{xy}FGK\Lambda_1j+F_{tt}j^2+F_{yy}G^2j^2+2F_{xt}K\Lambda_1j+2F_{yt}Gj^2]+\ .\ .\ . \tag{3.3}$$

$$G(x_{mK},y_{mK},t_{mK})=G-G_xFK\Lambda_1h-jh(G_yG+G_t)+\ .\ .\ . \tag{3.4}$$

$$\Sigma_n=FK\Lambda_nh+(Kh)^2\Gamma_nF_1+\frac{(Kh)^3}{2}\,\Delta_nF_2+(Kh)^3\phi_n(F_xF_1+F_yG_1)$$

$$-(Kh)^2\Gamma_n[G_xFK\Lambda_1h+G_yGjh+G_tjh]+\ .\ .\ . \qquad (n=1,\ 4,\ 7). \tag{3.5}$$

4. The First Approach

The difference $y_{mK+j}(t_{mK+j+1})-y_{mK+j+1}$ will be expanded in a Taylor's series and the coefficients of all terms of third order or less in h will be equated to zero. The resulting equations will be used to determine the integration and extrapolation parameters. Now

$$y_{mK+j}(t_{mK+j+1})-y_{mK+j+1}=y_{mK+j}(t_{mK+j+1})-y_{mK+j}-\alpha_0d_0-\alpha_1d_1-\alpha_2d_2.$$

With eqs (3.3), (3.4), and (3.5) it is seen that:

$$\left.\begin{aligned}
y_{mK+j}(t_{mK+j+1})=&\,y_{mK+j}+Gh+\frac{1}{2}h^2[G_xF(x_{mK+j},y_{mK+j},t_{mK+j})+G_yG+G_t]\\
&+\frac{1}{6}h^3[G_{tt}+2G_{ty}G+G_{yy}G^2+G_{xx}F^2(x_{mK+j},y_{mK+j},t_{mK+j})\\
&+2G_{xt}F(x_{mK+j},y_{mK+j},t_{mK+j})+2G_{xy}GF(x_{mK+j},y_{mK+j},t_{mK+j})]\\
&+\frac{1}{6}h^3[G_y(G_xF(x_{mK+j},y_{mK+j},t_{mK+j})+G_yG+G_t)\\
&+G_x\{F_x(x_{mK+j},y_{mK+j},t_{mK+j})F(x_{mK+j},y_{mK+j},t_{mK+j})\\
&+GF_y(x_{mK+j},y_{mK+j},t_{mK+j})+F_t(x_{mK+j},y_{mK+j},t_{mK+j})\}]+\ .\ .\ .\\
=&\,y_{mK+j}+Gh+\frac{1}{2}h^2G_1+\frac{1}{6}h^3G_2\\
&+\frac{1}{6}h^3(G_yG_1+G_xF_1)+\frac{1}{2}h^3G_x(FF_xK\Lambda_1+F_yGj+F_tj)+.\ .\ .
\end{aligned}\right\} \tag{4.1}$$

$$d_0(j)=hG, \tag{4.2}$$

$$
\left.\begin{aligned}
d_1(j)&=hG+h(G_x\Sigma_4+G_yG\mu_1h+G_t\mu_1h)\\
&\quad+\tfrac{1}{2}h[G_{xx}\Sigma_4^2+2G_{xy}G\Sigma_4\mu_1h+2G_{xt}\Sigma_4\mu_1h+G_{yy}G^2(\mu_1h)^2+2G_{yt}G(\mu_1h)^2+G_{tt}(\mu_1h)^2]\\
&=hG+h^2(G_xFK\Lambda_4+G_yG\mu_1+G_t\mu_1)\\
&\quad+\tfrac{1}{2}h^3[G_{xx}F^2K^2\Lambda_4^2+2G_{xy}GFK\Lambda_4\mu_1+2G_{xt}FK\Lambda_4\mu_1+G_{yy}G^2\mu_1^2\\
&\qquad+2G_{yt}G\mu_1^2+G_{tt}\mu_1^2]+h^3K^2\Gamma_4G_xF_1+\ \ldots
\end{aligned}\right\}\tag{4.3}
$$

$$
\begin{aligned}
d_2(j)&=hG+h(G_x\Sigma_7+G_yG\mu_2h+G_t\mu_2h)\\
&\quad+\tfrac{1}{2}h[G_{xx}\Sigma_7^2+2G_{xy}G\Sigma_7\mu_2h+2G_{xt}\Sigma_7\mu_2h+G_{yy}G^2(\mu_2h)^2+2G_{yt}(\mu_2h)^2+G_{tt}(\mu_2h)^2]\\
&\quad+\mu_3h^2G_y(G_x\Sigma_4+G_yG\mu_1h+G_t\mu_1h)+\ \ldots\\
&=hG+h^2[G_xFK\Lambda_7+G_yG\mu_2+G_t\mu_2]\\
&\quad+\tfrac{1}{2}h^3[G_{xx}F^2K^2\Lambda_7^2+2G_{xy}GFK\Lambda_7\mu_2+2G_{xt}FK\Lambda_7\mu_2+G_{yy}G^2\mu_2^2+2G_{yt}G\mu_2^2+G_{tt}\mu_2^2]\\
&\quad+\mu_3h^3G_y(G_xFK\Lambda_4+G_yG\mu_1+G_t\mu_1)+h^3K^2\Gamma_7G_xF_1+\ \ldots
\end{aligned}\tag{4.4}
$$

The following system of equations results when the coefficients of the various terms are set equal to zero:

$$
\text{(4.5)}\qquad \alpha_0+\alpha_1+\alpha_2=1 \qquad\qquad \alpha_1\mu_1^2+\alpha_2\mu_2^2=\frac{1}{3}
$$

$$
\alpha_1\mu_1+\alpha_2\mu_2=\frac{1}{2} \qquad\qquad \alpha_2\mu_1\mu_3=\frac{1}{6}
$$

$$
\text{(4.6)}\qquad \alpha_1\Lambda_4K+\alpha_2\Lambda_7K=\frac{1}{2} \qquad\qquad \alpha_2\mu_3K\Lambda_4=\frac{1}{6}\qquad\text{(4.9)}
$$

$$
\text{(4.7)}\qquad \alpha_1\Lambda_4^2K^2+\alpha_2\Lambda_7^2K^2=\frac{1}{3} \qquad\qquad \alpha_1K^2\Gamma_4+\alpha_2K^2\Gamma_7=\frac{1}{6}+\frac{j}{2}\qquad\text{(4.10)}
$$

$$
\text{(4.8)}\qquad \alpha_1\mu_1K\Lambda_4+\alpha_2\mu_2K\Lambda_7=\frac{1}{3} \qquad\qquad \alpha_1K^2\Gamma_4+\alpha_2K^2\Gamma_7=\frac{1}{6}+\frac{K\Lambda_1}{2}\qquad\text{(4.11)}
$$

Equations (4.5) are the usual equations for the integration parameters of a third order Runge-Kutta method. From (4.9) it follows that $K\Lambda_4=\mu_2$ and hence by (4.6) $K\Lambda_7=\mu_2$. From (4.10) and (4.11) it is seen that $K\Lambda_1=j$. Equations (4.6) through (4.11) may then be replaced by

$$
\Lambda_1=\frac{j}{k}\qquad \Lambda_7=\frac{\mu_2}{K}\qquad \Lambda_4=\frac{\mu_1}{K}\qquad \alpha_1\Gamma_4+\alpha_2\Gamma_7=\frac{1}{6K^2}\,(1+3j)\cdot \tag{4.12}
$$

No attempt will be made to discuss all possible solutions of (4.5) and (4.12). However, a few obvious facts will be pointed out. From (4.5) it is seen that $\mu_1\neq 0$, $\mu_3\neq 0$, $\alpha_2\neq 0$. Likewise $\gamma_1\neq 0$. It is clear that Λ_1, Λ_4, Λ_7, Γ_4 and Γ_7 are independent. Since the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0\\
0 & 1 & 0 & 0\\
0 & 0 & \alpha_1 & \alpha_2
\end{pmatrix}
$$

is of rank three any set of third order Runge-Kutta integration parameters may be used.

One could at this point make an analysis of various types of simple solutions of (4.12). However, in the next section it will be seen that there are some other considerations. Two sets of parameters are given for comparison purposes.

$$\left.\begin{array}{lll} \alpha_0,\ \alpha_1,\ \alpha_2,\ \mu_1,\ \mu_2,\ \mu_3 & \multicolumn{2}{l}{\text{(arbitrary solutions of (4.5))}} \\ \lambda_1=\dfrac{j}{K} & \lambda_4=\dfrac{\mu_1}{K} & \lambda_7=\dfrac{\mu_2}{K}-\lambda_8 \\ \lambda_8=\dfrac{1+3j}{6\alpha_2\gamma_1 K^2} & \multicolumn{2}{l}{\lambda_2=\lambda_3=\lambda_5=\lambda_6=\lambda_9=0} \end{array}\right\} \tag{4.13}$$

$$\left.\begin{array}{lll} \alpha_0=0 & \alpha_1=\dfrac{3}{4} & \alpha_2=\dfrac{1}{4} \\ \mu_1=\dfrac{2}{3} & \mu_2=0 & \mu_3=1 \\ \lambda_1=\dfrac{j}{K} & \lambda_4=\dfrac{2}{3K}-\lambda_5 & \lambda_5=\dfrac{2}{9\gamma_1 K^2}(1+3j) \\ \multicolumn{3}{c}{\lambda_2=\lambda_3=\lambda_6=\lambda_7=\lambda_8=\lambda_9=0.} \end{array}\right\} \tag{4.14}$$

5. The Truncation Error

In this section the truncated terms of the various expansions will be considered in some detail. A term shall be said to be $T(j^\alpha K^\beta h^\gamma)$ if it is of the form $j^\alpha K^\beta h^\gamma f$ where f is a function independent of j, K and h. It would be desirable for all of the fourth order terms truncated in the integration of (1.2) to be $T(h^4)$. The procedure would be pointless if some of the truncated terms are $T(K^4h^4)$. It will be seen that there are terms which are $T(Kh^4)$ and $T(j^2K^{-1}h^4)$ which cannot be simultaneously eliminated by any choice of extrapolation parameters.

All of the truncated terms from $y_{mK+j}(t_{mK+j+1})-y_{mK+j+1}$ which are not $T(h^4)$ are listed below. It is assumed that $\Lambda_1=j/K$, $\Lambda_4=\mu_1/K$, $\Lambda_7=\mu_2/K$. The terms from $y_{mK+j}(t_{mK+j+1})$, $d_1(j)$ and $d_2(j)$ are grouped in that order.

$$\frac{3+8j}{24}(G_{xx}F+G_{xy}G+G_{xt})F_1+\frac{(1+8j+12K^2\Gamma_1)}{24}G_xF_xF_1$$
$$+\frac{(1+4j)}{24}G_xG_yF_1+\frac{(1+4j+6j^2)}{24}G_xF_2+\frac{(1+4j-6j^2)}{24}G_xF_yG_1;$$

$$\mu_1K^2\Gamma_4(G_{xx}F+G_{xy}G+G_{xt})F_1+K^3\phi_4G_xF_xF_1+\tfrac{1}{2}K^3\Delta_4G_xF_2+K^2(K\phi_4-j\Gamma_4)G_xF_yG_1;$$

$$\mu_2K^2\Gamma_7(G_{xx}F+G_{xy}G+G_{xt})F_1+K^3\phi_7G_xF_xF_1+\tfrac{1}{2}K^3\Delta_7G_xF_2+K^2(K\phi_7-j\Gamma_7)G_xF_yG_1+\mu_3K^2\Gamma_4G_xG_yF_1.$$

The original statement was that $y(t)$ did not depend strongly on $x(t)$ or varied more rapidly than $x(t)$. This statement may be replaced by the following explicit assumptions on F and G:

Assumption 1: The following inequalities are valid:

$$|F_p|\le|G_p|/K, \qquad p=x,y,1,2.$$

This assumption implies that $y(t)$ does vary more rapidly than $x(t)$.

Assumption 2: G_x is $T(K^{-1})$.

It is seen from (4.12) that Γ_4 and Γ_7 are $T(jK^{-2})$.

With assumption 1 and the preceding remark it is seen that there are two groups of terms which are not $T(h^4)$. They are

$$\frac{(1+4j+6j^2)}{24}G_xF_2, \qquad \frac{1}{2}K^3\Delta_4G_xF_2, \qquad \frac{1}{2}K^3\Delta_7G_xF_2$$

and

$$\frac{(1+4j-6j^2)}{24}G_xF_yG_1, \qquad K^2(K\phi_4-j\Gamma_4)G_xF_yG_1, \qquad K^2(K\phi_7-j\Gamma_7)G_xF_yG_1.$$

These terms would be eliminated if the following equations held:

$$\frac{1+4j+6j^2}{12K^3}=\alpha_1\Delta_4+\alpha_2\Delta_7$$

$$\frac{1+4j-6j^2}{24K^2}=\alpha_1(K\phi_4-j\Gamma_4)+\alpha_2(K\phi_7-\Gamma_7).$$

These two equations may be transformed into

$$\frac{1+4j+6j^2}{12K^3}=\alpha_1\Delta_4+\alpha_2\Delta_7. \tag{5.1}$$

$$\frac{1+8j+6j^2}{24K^3}=\alpha_1\phi_4+\alpha_2\phi_7. \tag{5.2}$$

Unfortunately, it is not possible to satisfy (4.12), (5.1) and (5.2) simultaneously. The equations

$$\alpha_1\Gamma_4+\alpha_2\Gamma_7=\frac{1+3j}{6K^2}$$

$$\alpha_1\Delta_4+\alpha_2\Delta_7=\frac{1+4j+6j^2}{12K^3}$$

$$\alpha_1\phi_4+\alpha_2\phi_7=\frac{1+8j+6j^2}{24K^3}$$

are incompatible. It is possible to satisfy (4.12) and either (5.1) or (5.2). Since the terms involved by (5.1) and (5.2) are $T(j^2K^{-1}h^4)$ and $T(Kh^4)$, respectively, the smallest truncation error results when (4.12) and (5.2) are satisfied.

If (5.2) or (5.1) are to be satisfied, then some of the simplicity of the extrapolation coefficients is lost. Two sets of parameters are given below; the first satisfies (5.1) and the second satisfies (5.2).

$$\left.\begin{array}{l}\mu_1,\ \mu_2,\ \gamma_1,\ \gamma_2,\ \gamma_3 \text{ (arbitrary Runge-Kutta integration parameters)}\\[1ex] \lambda_1=\dfrac{j}{K} \qquad \lambda_4=\dfrac{\mu_1}{K}-\lambda_5-\lambda_6 \qquad \lambda_5=\dfrac{1+3j}{6\alpha_1\gamma_1K^2}-\dfrac{\lambda_6\gamma_2}{1}\\[2ex] \lambda_6=\dfrac{1+4j+6j^2-2\gamma_1K(1+3j)}{12K\alpha_1\gamma_2(\gamma_2-\gamma_1)} \qquad \lambda_7=\dfrac{\mu_2}{K} \qquad \lambda_2=\lambda_3=\lambda_8=\lambda_9=0\end{array}\right\} \tag{5.3}$$

$$\left.\begin{array}{l}\mu_1,\ \mu_2,\ \gamma_1,\ \gamma_2,\ \gamma_3 \text{ (arbitrary Runge-Kutta integration parameters)}\\[1ex] \lambda_1=\dfrac{j}{K} \qquad \lambda_4=\dfrac{\mu_1}{K}-\lambda_5-\lambda_6 \qquad \lambda_5=\dfrac{1+3j}{3\alpha_1\gamma_1K^2}-\dfrac{\lambda_6\gamma_2}{\gamma_1}\\[2ex] \lambda_6=\dfrac{1+8j+6j^2}{24K^3\alpha_1\gamma_1\gamma_3} \qquad \lambda_7=\dfrac{\mu_2}{K} \qquad \lambda_2=\lambda_3=\lambda_8=\lambda_9=0\end{array}\right\} \tag{5.4}$$

These equations are somewhat more complicated than those given in section 4.

With assumption 2 it is seen that there are three groups of terms which are not $T(h^4)$. They are

$$\frac{(1+8j+12K^2\Gamma_1)}{24}G_xF_xF_1, \qquad K^3\phi_4G_xF_xF_1, \qquad K^3\phi_7G_xF_xF_1$$

and the two groups found with assumption 1. These terms would be eliminated if the following equations held.

$$\frac{1+4j+6j^2}{24K^3}=\alpha_1\Delta_4+\alpha_2\Delta_7$$

$$\frac{1+8j+6j^2}{24K^3}=\frac{1+8j+12K^2\Gamma_1}{24K^3}=\alpha_1\phi_4+\alpha_2\phi_7.$$

Hence if Γ_1 is set equal $j^2/2K^2$ the same equations are found as with assumption 1.

6. The Second Approach

In this section the extrapolation parameters will be determined so as to make the truncation error of the extrapolation as small as possible. This procedure is similar to the analysis for the Runge-Kutta integration of one equation.

The extrapolation of $x(t)$ to any value t_{mK} to any value $t_{mK}+\tau$ is given by

$$x_{mK}+\lambda_0(\tau)k_0+\lambda_1(\tau)k_1+\lambda_2(\tau)k_2$$

where k_0, k_1 and k_2 are from (2.1). $\lambda_0(\tau)$, $\lambda_1(\tau)$ and $\lambda_2(\tau)$ are determined by equating the coefficients of h in the expansion of the error equal to zero. The resulting equations are

$$\lambda_0(\tau)+\lambda_1(\tau)+\lambda_2(\tau)=\tau/Kh \tag{6.1}$$

$$\lambda_1(\tau)\gamma_1+\lambda_2(\tau)\gamma_2=\frac{1}{2}\tau^2/(Kh)^2 \tag{6.2}$$

$$\lambda_1(\tau)\gamma_1^2+\lambda_2(\tau)\gamma_2^2=\frac{1}{3}\tau^3/(Kh)^3 \tag{6.3}$$

$$\lambda_2(\tau)\gamma_1\gamma_3=\frac{1}{6}\tau^3/(Kh)^3. \tag{6.4}$$

Equation (6.1) results from equating the coefficients of h to zero, (6.2) results from equating the coefficients of h^2 to zero, and (6.3) and (6.4) result from equating the coefficients of h^3 to zero. In general only three of these equations can be satisfied One would naturally choose a solution which satisfies both (6.1) and (6.2).

These equations may be written for the extrapolation parameters. The resulting equations are:

$$\Lambda_1=\frac{j}{K} \qquad \Lambda_4=\frac{\mu_1}{K} \qquad \Lambda_7=\frac{\mu_2}{K} \tag{6.5}$$

$$\Gamma_1=\frac{j^2}{2K^2} \qquad \Gamma_4=\frac{(j+\mu_1)^2}{2K^2}-\Gamma_1 \qquad \Gamma_7=\frac{(j+\mu_2)^2}{2K^2}-\Gamma_1 \tag{6.6}$$

$$\Delta_1=\frac{j^3}{3K^3} \qquad \Delta_4=\frac{(j+\mu_1)^3}{3K^3}-\Delta_1 \qquad \Delta_7=\frac{(j+\mu_2)^3}{3K^3}-\Delta_1 \tag{6.7}$$

$$\phi_1=\frac{j^3}{6K^3} \qquad \phi_4=\frac{(j+\mu_1)^3}{6K^3}-\phi_1 \qquad \phi_7=\frac{(j+\mu_2)^3}{6K^3}-\phi_1. \tag{6.8}$$

Equations (6.6) and 6.5) imply that (4.12) is satisfied. The converse is not true. It is easily shown that (6.7) implies

$$\alpha_1\Delta_4+\alpha_2\Delta_7=\frac{6j^2+4j+4(\alpha_1\mu_1^3+\alpha_2\mu_2^3)}{12K^3}$$

and that (6.8) implies

$$\alpha_1\phi_4+\alpha_2\phi_7=\frac{6j^2+4j+4(\alpha_1\mu_1^3+\alpha_2\mu_2^3)}{24K^3}.$$

Therefore (6.7) implies that the G_xF_2 term is $T(j^2K^{-1}h^4)$ and (6.8) implies that the $G_xF_yG_1$ term is $T(Kh^4)$. Hence (6.7) and (6.8) have the same effect on the truncation error as (5.1) and (5.2) although (6.7) and (6.8) do not actually imply (5.1) and (5.2), respectively.

It is rather surprising that $\Gamma_1=j^2/2K^2$ does not appear among the equations derived in sections 4 and 5 with assumption 1. It is certainly plausible to include this equation in any set of equations taken to determine the extrapolation parameters.

Two sets of parameters are given, the first satisfies (6.5), (6.6), and (6.7) and the second satisfies (6.5), (6.6), and (6.8).

$$\left.\begin{array}{lll}
\multicolumn{3}{l}{\mu_1,\ \mu_2,\ \gamma_1,\ \gamma_2,\ \gamma_3 \text{ (arbitrary Runge-Kutta integration parameters)}}\\[2mm]
\lambda_1=\dfrac{j}{K}-\lambda_2-\lambda_3, & \lambda_2=\dfrac{2j^3-3\gamma_2Kj^2}{6K^3\gamma_1(\gamma_1-\gamma_2)} & \lambda_3=\dfrac{2j^3-3Kj^2\gamma_1}{6K^3\gamma_2(\gamma_2-\gamma_1)}\\[2mm]
\lambda_4=\dfrac{\mu_1}{K}-\lambda_5-\lambda_6, & \lambda_5=\dfrac{2(j+\mu_1)^3-3\gamma_2K(j+\mu_1)^2}{6K^3\gamma_1(\gamma_1-\gamma_2)}-\lambda_2, & \lambda_6=\dfrac{2(j+\mu_1)^3-3K(j+\mu_1)^2\gamma_1}{6K^3\gamma_2(\gamma_2-\gamma_1)}\\[2mm]
\lambda_7=\dfrac{\mu_2}{K}-\lambda_8-\lambda_9, & \lambda_8=\dfrac{2(j+\mu_2)^3-3\gamma_2K(j+\mu_2)^2}{6K^3\gamma_1(\gamma_1-\gamma_2)}-\lambda_2, & \lambda_9=\dfrac{2(j+\mu_2)^3-3K(j+\mu_2)^2\gamma_1}{6K^3\gamma_2(\gamma_2-\gamma_1)}
\end{array}\right\} \quad (6.9)$$

$$\left.\begin{array}{lll}
\multicolumn{3}{l}{\mu_1,\ \mu_2,\ \gamma_1,\ \gamma_2,\ \gamma_3 \text{ (arbitrary Runge-Kutta integration parameters)}}\\[2mm]
\lambda_1=\dfrac{j}{K}-\lambda_2-\lambda_3, & \lambda_2=\dfrac{j^2}{2K^2\gamma_1}-\dfrac{\gamma_2j^3}{6K^3\gamma_1^2\gamma_3}, & \lambda_3=\dfrac{j^3}{6K^3\gamma_1\gamma_3}\\[2mm]
\lambda_4=\dfrac{\mu_1}{K}-\lambda_5-\lambda_6, & \lambda_5=\dfrac{(j+\mu_1)^2}{2K^2\gamma_1}-\dfrac{\gamma_2(j+\mu_1)^3}{6K^3\gamma_1^2\gamma_3}-\lambda_2, & \lambda_6=\dfrac{(j+\mu_1)^3}{6K^3\gamma_1\gamma_3}-\lambda_3\\[2mm]
\lambda_7=\dfrac{\mu_2}{K}-\lambda_8-\lambda_9, & \lambda_8=\dfrac{(j+\mu_2)^2}{2K^2\gamma_1}-\dfrac{\gamma_2(j+\mu_2)^3}{6K^3\gamma_1^2\gamma_3}-\lambda_2, & \lambda_9=\dfrac{(j+\mu_2)^3}{6K^3\gamma_1\gamma_3}-\lambda_3
\end{array}\right\} \quad (6.10)$$

If only (6.5) and (6.6) are to be satisfied there is considerable simplification in the extrapolation coefficients. The following set satisfies (6.5) and (6.6):

$$\left.\begin{array}{lll}
\lambda_1=\dfrac{j}{K}-\lambda_2 & \lambda_2=\dfrac{j^2}{2K^2\gamma_1} & \lambda_3=0\\[2mm]
\lambda_4=\dfrac{\mu_1}{K}-\lambda_5 & \lambda_5=\dfrac{(j+\mu_1)^2}{2K^2\gamma_1}-\lambda_2 & \lambda_6=0\\[2mm]
\lambda_7=\dfrac{\mu_2}{K}-\lambda_8 & \lambda_8=\dfrac{(j+\mu_2)^2}{2K^2\gamma_1}-\lambda_2 & \lambda_9=0
\end{array}\right\} \quad (6.11)$$

7. Fourth Order Formulas

An analysis similar to that of sections 4, 5, and 6 may be made for a fourth order integration method. Only the results of such an analysis are given here. The equations for a fourth order integration method are:

$$k_0 = KhF(x_{mK},\ y_{mK},\ t_{mK})$$

$$k_1 = KhF(x_{mK}+\gamma_1 k_0,\ y_{mK}+\gamma_1 h_0,\ t_{mK}+\gamma_1 Kh)$$

$$k_2 = KhF(x_{mK}+\gamma_3 k_1+(\gamma_2-\gamma_3)k_0,\ y_{mK}+\gamma_3 h_1+(\gamma_2-\gamma_3)h_0,\ t_{mK}+\gamma_2 Kh)$$

$$k_3 = KhF(x_{mK}+\gamma_6 k_2+\gamma_5 k_1+(\gamma_4-\gamma_5-\gamma_6)k_0,\ y_{mK}+\gamma_6 h_2+\gamma_5 h_1+(\gamma_4-\gamma_5-\gamma_6)h_0,\ t_{mK}+\gamma_4 Kh)$$

$$h_0 = KhG(x_{mK},\ y_{mK},\ t_{mK})$$

$$h_1 = KhG(x_{mK}+\gamma_1 k_0,\ y_{mK}+\gamma_1 h_0,\ t_{mK}+\gamma_1 Kh)$$

$$h_2 = KhG(x_{mK}+\gamma_3 k_1+(\gamma_2-\gamma_3)k_0,\ y_{mK}+\gamma_3 h_1+(\gamma_2-\gamma_3)h_0,\ t_{mK}+\gamma_2 Kh)$$

$$h_3 = KhG(x_{mK}+\gamma_6 k_2+\gamma_5 k_1+(\gamma_4-\gamma_5-\gamma_6)k_0,\ y_{mK}+\gamma_6 h_2+\gamma_5 h_1+(\gamma_4-\gamma_5-\gamma_6)h_0,\ t_{mK}+\gamma_4 Kh)$$

$$x_{(m+1)K} = x_{mK}+\beta_0 k_0+\beta_1 k_1+\beta_2 k_2+\beta_3 k_3$$

$$x_{mK+j} = \sum_{i=1}^{4} \lambda_i(j) k_{i-1} + x_{mK}$$

$$d_0(j) = hG(x_{mK+j}, y_{mK+j}, t_{mK+j})$$

$$d_1(j) = hG(x_{mK+j}+\sum_{i=5}^{8} \lambda_i k_{i-5}, y_{mK+j}+\mu_1 d_0, t_{mK+j}+\mu_1 h)$$

$$d_2(j) = hG(x_{mK+j}+\sum_{i=9}^{12} \lambda_i k_{i-9}, y_{mK+j}+\mu_3 d_1+(\mu_2-\mu_3)d_0, t_{mK+j}+\mu_2 h)$$

$$d_3(j) = hG(x_{mK+j}+\sum_{i=13}^{16} \lambda_i k_{i-13}, y_{mK+j}+\mu_6 d_2+\mu_5 d_1+(\mu_4-\mu_5-\mu_6)d_0, t_{mK+j}+\mu_4 h)$$

$$y_{mK+j+1} = y_{mK+j}+\alpha_0 d_0+\alpha_1 d_1+\alpha_2 d_2+\alpha_3 d_3$$

The integration parameters must satisfy:

$$\left.\begin{array}{ll} \alpha_0+\alpha_1+\alpha_2+\alpha_3=1 & \alpha_1\mu_1^3+\alpha_2\mu_2^3+\alpha_3\mu_4^3=\frac{1}{4} \\ \alpha_1\mu_1+\alpha_2\mu_2+\alpha_3\mu_4=\frac{1}{2} & \alpha_2\mu_1\mu_2\mu_3+\alpha_3\mu_2\mu_4\mu_6+\alpha_3\mu_1\mu_4\mu_5=\frac{1}{8} \\ \alpha_1\mu_1^2+\alpha_2\mu_2^2+\alpha_3\mu_4^2=\frac{1}{3} & \alpha_2\mu_1^2\mu_3+\alpha_3\mu_2^2\mu_6+\alpha_3\mu_1^2\mu_5=\frac{1}{12} \\ \alpha_2\mu_1\mu_3+\alpha_3\mu_2\mu_6+\alpha_3\mu_1\mu_5=\frac{1}{6} & \alpha_3\mu_1\mu_3\mu_6=\frac{1}{24} \end{array}\right\} \quad (7.1)$$

The extrapolation parameters must satisfy the following equations in order for the integration method to be fourth order:

$$\left.\begin{array}{ll}
\Lambda_1=\dfrac{j}{K} & \alpha_1\Gamma_5+\alpha_2\Gamma_9+\alpha_3\Gamma_{13}=\dfrac{1+3j}{6K^2} \\[2ex]
\Lambda_5=\dfrac{\mu_1}{K} & \alpha_1\Delta_5+\alpha_2\Delta_9+\alpha_3\Delta_{13}=\dfrac{1+4j+6j^2}{12K^3} \\[2ex]
\Lambda_9=\dfrac{\mu_2}{K} & \alpha_1\mu_1\Gamma_5+\alpha_2\mu_2\Gamma_9+\alpha_3\mu_4\Gamma_{13}=\dfrac{3+8j}{24K^2} \\[2ex]
\Lambda_{13}=\dfrac{\mu_2}{K} & \alpha_1\phi_5+\alpha_2\phi_9+\alpha_3\phi_{13}=\dfrac{1+8j+6j^2}{24K^3} \\[2ex]
\Gamma_1=\dfrac{j^2}{2K^2} & \alpha_2\mu_3\Gamma_5+\alpha_3\mu_5\Gamma_5+\alpha_3\mu_6\Gamma_9=\dfrac{1+4j}{24K^2}
\end{array}\right\} \qquad (7.2)$$

where

$$\Lambda_n=\lambda_n+\lambda_{n+1}+\lambda_{n+2}+\lambda_{n+3} \qquad \Delta_n=\gamma_1^2\lambda_{n+1}+\gamma_2^2\lambda_{n+2}+\gamma_4^2\lambda_{n+3}$$

$$\Gamma_n=\gamma_1\lambda_{n+1}+\gamma_2\lambda_{n+2}+\gamma_4\lambda_{n+3} \qquad \phi_n=\gamma_1\gamma_3\lambda_{n+2}+(\gamma_1\gamma_5+\gamma_2\gamma_5)\lambda_{n+3}$$

The truncated terms have not been examined as in section 5 due to the extremely tedious nature of such an examination.

The equations analogous to (6.5), (6.6), (6.7), and (6.8) are:

$$\Lambda_1=\frac{j}{K} \qquad \Lambda_5=\frac{\mu_1}{K} \qquad \Lambda_9=\frac{\mu_2}{K} \qquad \Lambda_{13}=\frac{\mu_4}{K} \qquad (7.3)$$

$$\Gamma_1=\frac{j^2}{2K^2} \qquad \Gamma_5=\frac{(j+\mu_1)^2}{2K^2}-\Gamma_1 \qquad \Gamma_9=\frac{(j+\mu_2)^2}{2K^2}-\Gamma_1 \qquad \Gamma_{13}=\frac{(j+\mu_4)^2}{2K^2}-\Gamma_1 \qquad (7.4)$$

$$\Delta_1=\frac{j^3}{3K^3} \qquad \Delta_5=\frac{(j+\mu_1)^3}{3K^3}-\Delta_1 \qquad \Delta_9=\frac{(j+\mu_2)^3}{3K^3}-\Delta_1 \qquad \Delta_{13}=\frac{(j+\mu_4)^3}{3K^3}-\Delta_1 \qquad (7.5)$$

$$\phi_1=\frac{j^3}{6K^3} \qquad \phi_5=\frac{(j+\mu_1)^3}{6K^3}-\phi_1 \qquad \phi_9=\frac{(j+\mu_2)^3}{6K^3}-\phi_1 \qquad \phi_{13}=\frac{(j+\mu_4)^3}{6K^3}-\phi_1. \qquad (7.6)$$

The relation between the λ_j and the Λ_n, Γ_n, Δ_n and ϕ_n is

$$\left|\begin{array}{c}\Lambda_n\\ \Gamma_n\\ \Delta_n\\ \phi_n\end{array}\right| = \left|\begin{array}{cccc}1 & 1 & 1 & 1\\ 0 & \gamma_1 & \gamma_2 & \gamma_4\\ 0 & \gamma_1^2 & \gamma_2^2 & \gamma_4^2\\ 0 & 0 & \gamma_1\gamma_3 & (\gamma_1\gamma_5+\gamma_2\gamma_6)\end{array}\right| \left|\begin{array}{c}\lambda_n\\ \lambda_{n+1}\\ \lambda_{n+2}\\ \lambda_{n+3}\end{array}\right|$$

This relationship can be inverted and

$$\left|\begin{array}{c}\lambda_n\\ \lambda_{n+1}\\ \lambda_{n+2}\\ \lambda_{n+3}\end{array}\right| = A \left|\begin{array}{c}\Lambda_n\\ \Gamma_n\\ \Delta_n\\ \phi_n\end{array}\right|$$

where A is the matrix with elements a_{ij}/ρ,

$$\rho=\gamma_1[(\gamma_2-\gamma_1)\gamma_2(\gamma_1\gamma_5+\gamma_2\gamma_6)+\gamma_1\gamma_3\gamma_4(\gamma_1-\gamma_4)]$$

$$a_{11}=\rho \quad a_{12}=(\gamma_1^2-\gamma_2^2)(\gamma_1\gamma_5+\gamma_2\gamma_6)-\gamma_1\gamma_3(\gamma_1-\gamma_4) \quad a_{13}=(\gamma_2-\gamma_1)(\gamma_1\gamma_5+\gamma_2\gamma_6)+\gamma_1\gamma_3(\gamma_1-\gamma_4)$$

$$a_{14}=(\gamma_1-\gamma_2)(\gamma_2-\gamma_4)(\gamma_1-\gamma_4)$$

$$a_{21}=0 \quad a_{22}=\gamma_2^2(\gamma_1\gamma_5+\gamma_2\gamma_6)-\gamma_1\gamma_3\gamma_4^2 \quad a_{23}=\gamma_1\gamma_3\gamma_4-\gamma_2(\gamma_1\gamma_5+\gamma_2\gamma_6) \quad a_{24}=\gamma_2\gamma_4^2-\gamma_2^2\gamma_4$$

$$a_{31}=0 \quad a_{32}=-\gamma_1^2(\gamma_1\gamma_5+\gamma_2\gamma_6) \quad a_{33}=\gamma_1(\gamma_1\gamma_5+\gamma_2\gamma_6) \quad a_{34}=\gamma_1\gamma_4(\gamma_1-\gamma_4)$$

$$a_{41}=0 \quad a_{42}=\gamma_1^3\gamma_3 \quad a_{43}=-\gamma_1^2\gamma_3 \quad a_{44}=\gamma_1\gamma_2(\gamma_2-\gamma_1).$$

If Runge-Kutta-Gill integration parameters are used, then

$$A=\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0.58578646 & 2.8284272 & -6.8284272 \\ 0 & 3.414214 & -6.8284272 & 6.8284272 \\ 0 & -1 & 2 & 0 \end{bmatrix}$$

8. Experiments

Experiments have been made with these formulas on three pairs of differential equations. They are

$$\left.\begin{aligned} \frac{dx}{dt}&=x/2 & x(0)&=1 \\ \frac{dy}{dt}&=x\cos(25t) & y(0)&=1/1250.5 \end{aligned}\right\} \tag{8.1}$$

$$\left.\begin{aligned} \frac{dx}{dt}&=y^6 & x(0)&=a^6/13 \\ \frac{dy}{dt}&=\frac{13x}{a^5(t+1)^{12}} & y(0)&=a \end{aligned}\right\} \tag{8.2}$$

$$\left.\begin{aligned} \frac{dx}{dt}&=.1+(1.5\times10^{-4})\cos(150t) & x(0)&=0 \\ \frac{dy}{dt}&=x\cos(25t) & y(0)&=.00016. \end{aligned}\right\} \tag{8.3}$$

Equation (8.1) is a typical example of a system suited for split Runge-Kutta. Equation (8.2) is an example of a system not suited for split Runge-Kutta. Yet even in this situation there may be some advantage to using it if the solution for $y(t)$ is needed with much more accuracy than that for $x(t)$. In (8.3), $y(t)$ is much more rapidly varying than $x(t)$, yet the small oscillation in $x(t)$ gives it a comparatively large second derivative. The experiments for each of these equations are discussed in considerably more detail in sections 9, 10, and 11.

No experiments were made with parameters derived from assumption 2 in section 5.

9. First Example—The Typical Case

The solution of (8.1) is

$$x(t)=e^{t/2} \tag{9.1}$$

$$y(t)=(\tfrac{1}{2}\cos 25t+25\sin 25t)e^{t/2}/625.25. \tag{9.2}$$

There are three basic variables in the solution of (8.1). They are (i) the method of solution, (ii) K, (iii) h. The methods of solution are numbered 1 through 6 corresponding to equations (4.13), (5.3), (5.4), (6.11), (6.9), and (6.10), respectively. The same integration parameters were used throughout. They were:

$$\alpha_0=\beta_0=2/9,\quad \alpha_1=\beta_1=1/3,\quad \alpha_2=\beta_2=4/9,\quad \gamma_1=\mu_1=1/2,\quad \gamma_2=\mu_2=3/4,\quad \gamma_3=\mu_3=3/4.$$

In figure 1 the errors in the y integration are studied as a function of K for various methods of solution. The integration interval h is fixed at .01 and the error is the maximum error for $0\leq t\leq 1$. In general this occurs close to $t=1$. The value of the error of the x integration at $t=1$ is also given. Here the variation of K is equivalent to varying the integration step. This error behaves like $K^{2.7}$ in the range of values considered.

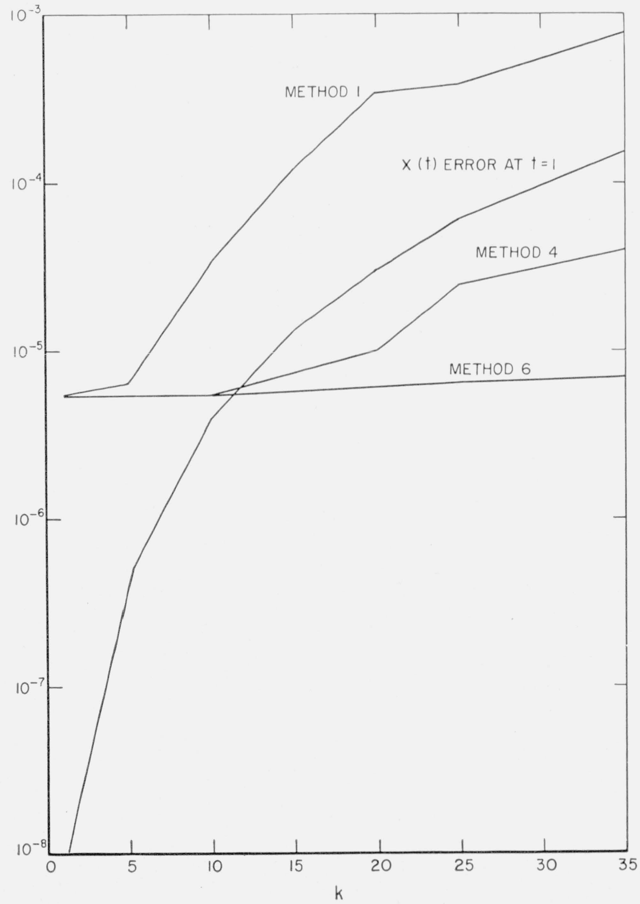


Figure 1. *Maximum integration errors as a function of* K *and the extrapolation method.*

There is very little difference in the results for the first three methods and only one curve is given. These methods proved to be inferior by a large factor compared with the second three methods.

There is little difference in the results for the fourth and fifth methods. Only the results for the fourth method are plotted. This lack of difference can be explained. Method 4 corresponds to satisfying eqs (6.1) and (6.2), and method 5 corresponds to satisfying (6.1), (6.2), and (6.3). Now (6.3) arises from setting the coefficient of F_2 equal to zero and for this system $F=ax$ and $F_2=0$. Hence there is no improvement when method 5 is used rather than method 4.

On the other hand method 6, which satisfies (6.1), (6.2), and (6.4) is a definite improvement over method 4. Equation (6.4) arises from setting the coefficient of $F_xF_1+F_yG_1=e^{t/2}/8$ equal to zero and hence an improvement would be expected.

There is another phenomenon apparent in figure 1 and a very significant one. That is the fact that the error does not decrease with K after a certain point. Indeed, the same accuracy for $y(t)$ may be obtained with $Kh=0.35$ as with $Kh=0.01$ if method 6 is used. This phenomenon is explained by the fact that there are two factors that contribute to the error in the $y(t)$ integration. One of them is the error in extrapolation of $x(t)$ and the other is the error inherent in the actual integration of $y(t)$. For some values of K the extrapolation errors are dominant and for other values the integration errors are dominant. When the extrapolation errors are no longer significant then improvement can be made only by decreasing h.

In figure 2 the errors are studied as a function of h with K fixed. This was done only for

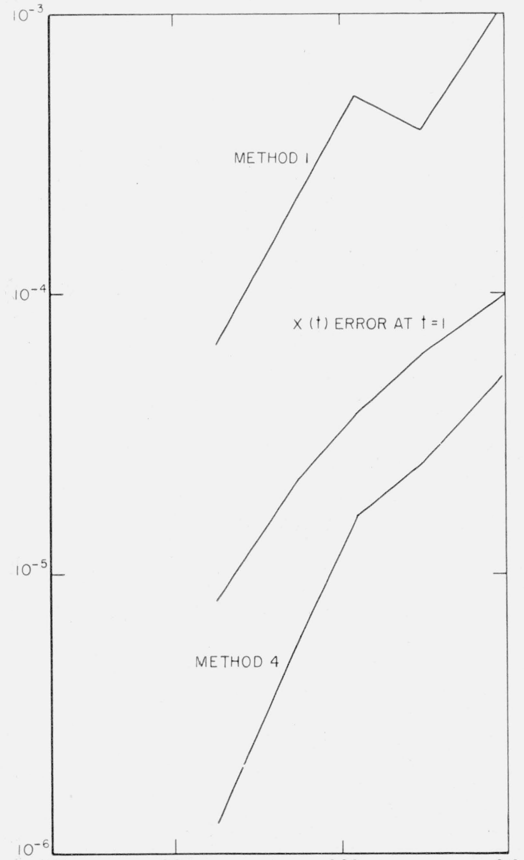


FIGURE 2. *Maximum integration errors as a function of* h *and the extrapolation method.*
K is fixed at 25.

the first and fourth methods. Figure 2 is self-explanatory except to point out that things are not always as one thinks they should be in numerical analysis. There is no particular explanation for the fact that the error does not decrease monotonically with h, it just happens that way.

Figures 3 through 7 are plots of error curves for various situations. Figure 3 is for method 1, $K=25$ and $h=.01$. Figure 4 is for method 4, $K=25$ and $h=.01$. Figure 5 is for method 4, $K=10$, $h=.01$. It is identical with the curves with $K=5$, 1. Figure 6 is for method 6, $K=25$, $h=.01$ and it is identical with the curves with $K=10$, 5, 1 and very similar to the curve with $K=35$. Figure 7 is for method 1, $K=25$, $h=.01/\sqrt[4]{2}\approx .0084$.

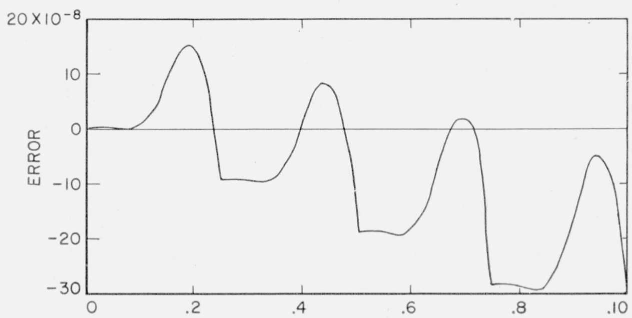


FIGURE 3. *Error curve for method 1 with* K=*25 and* h=*0.01.*

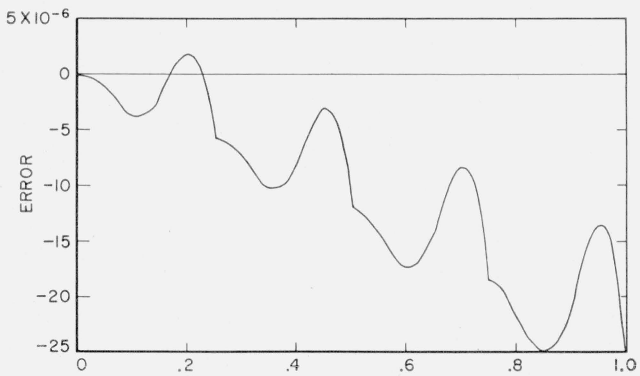


FIGURE 4. *Error curve for method 4 with* K=*25 and* h=*0.01.*

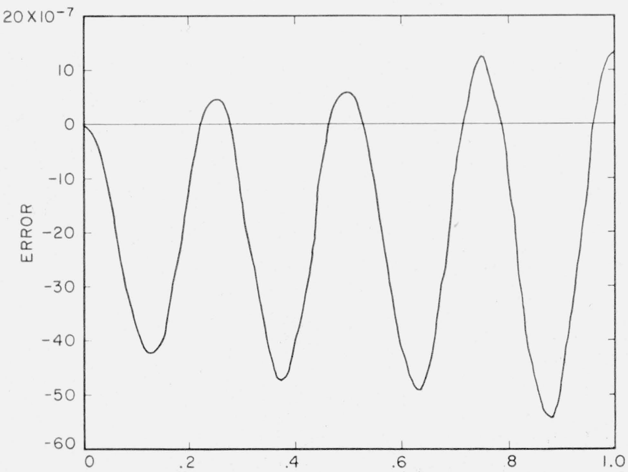


FIGURE 5. *Error curve for method 4 with* K=*10 and* h=*0.01.*
Identical curves are obtained for smaller values of K.

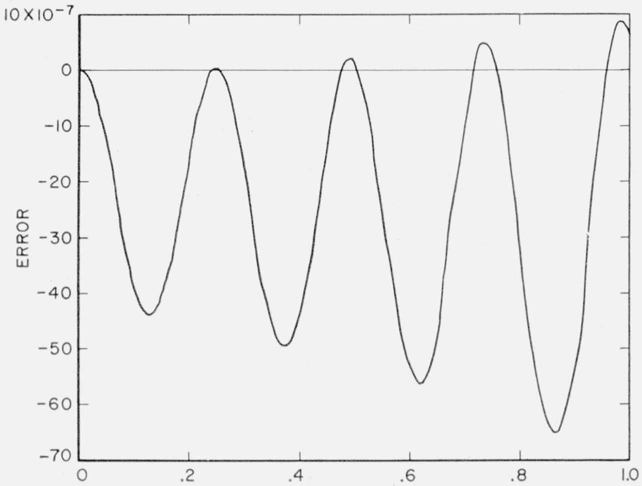


FIGURE 6. *Error curve for method 6 with* K=*25 and* h=*0.01.*
Identical curves are obtained for smaller values of K and for $K=35$ the error curve is very similar.

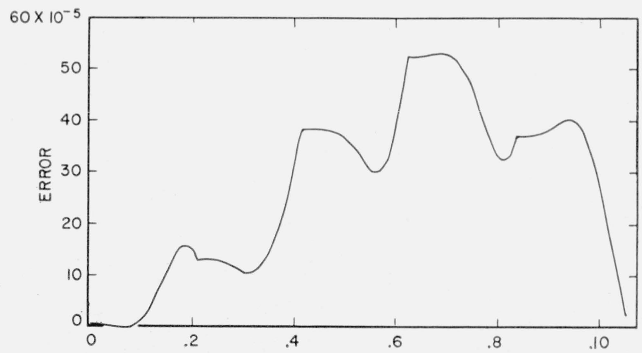


FIGURE 7. *Error curve for method 1 with* K=*25 and* h$\approx$*0.0084.*
It is seen that the relatively larger error here is due to the chance shape of the curve and not to any basic lower accuracy.

10. Second Example—An Unprofitable Case

The solution of (8.2) is

$$x(t)=a^6(t+1)^{13}/13 \tag{10.1}$$

$$y(t)=a(t+1)^2. \tag{10.2}$$

All of the results discussed below are for $a=.1$.

In figure 8 the errors are studied as a function of K for various methods of solution.

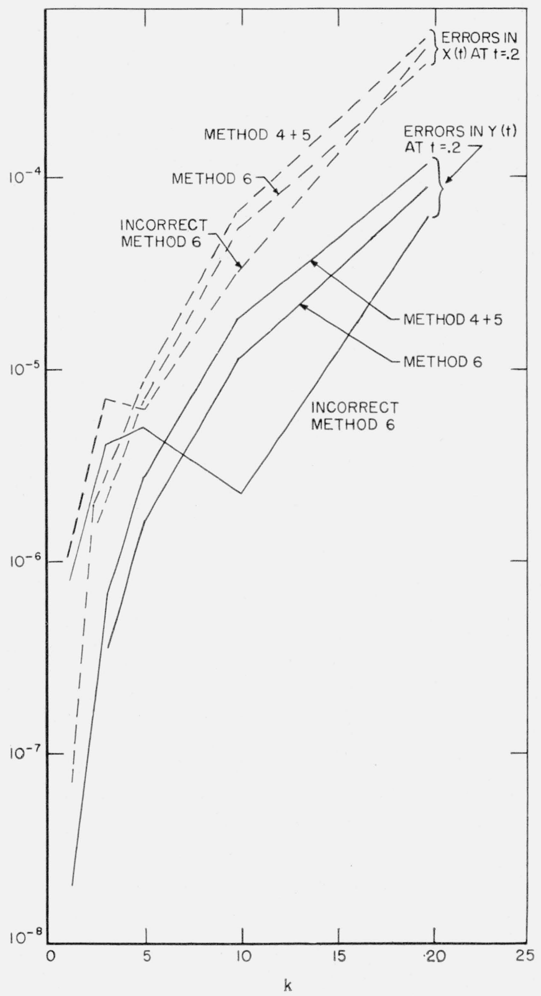


FIGURE 8. *Integration error at* t=*0.2 as a function of* K *and extrapolation method.*

It is seen that the $y(t)$ error depends directly on the $x(t)$ error. The maximum error accurs at $t=0.2$.

Methods 1, 2, and 3 are not used. Methods 4 and 5 give very similar results while method 6 does somethat better than either of them. Also included is an "incorrect method 6" which gives the best results of all for larger K's. In this method λ_8 was computed from

$$\lambda_8=\frac{(j+\mu_2)^2}{2K^2\gamma_1}-\frac{\gamma_2(j+\mu_1)^3}{6K^3\gamma_1^2\gamma_3}$$

instead of its true value given in (6.10).

The fact that method 6 is better than methods 4 and 5 cannot be explained as previously. Indeed, by that argument method 5 should be the best and method 4 the worst with method 6 in between. Recall that method 5 satisfies (6.3) which is obtained from setting the coefficient of F_2 equal to zero, likewise method 6 is obtained from setting the coefficient of $F_xF_1+F_yG_1$

equal to zero. Now in this case $F_2=120a^6\ (t+1)^{10}$ and $F_xF_1+F_yG_1=6a^6\ (t+1)^{10}$ and hence method 5 should be the best. This points out that the truncation error is not always a good measure of the integration error.

It is seen that the integration error does not level off as K decreases. This is due to the fact that the major portion of the error in the $y(t)$ integration is due to the errors in the integration of $x(t)$, as differentiated from the errors in the extrapolation of $x(t)$. This is borne out by figures 9 and 13. In figure 13 the $y(t)$ integration error is plotted for the system of equations

$$\frac{dx}{dt}=a^6(t+1)^{12}$$

$$\frac{dy}{dt}=13x/a^5(t+1)^{12}$$

which have the same solution as (8.2). Method 6, $K=20$ and $h=.002$ is used. This leads to much more accurate values of $x(t)$ which in turn lead to a much more accurate $y(t)$ integration even though the extrapolation accuracy has not improved.

In figure 9 the error curves for methods 4, 5, and 6 are compared with $K=20$, $h=0.002$ and for $0\leq t\leq .2$.

The error curve for method 4 with $K=1$, $h=0.002$ is given in figure 10.

In figure 11 two error curves are given which have the same interval of integration for the $x(t)$ equation. Both use method 6, one with $K=1$ and $h=.02$ and one with $K=10$ and

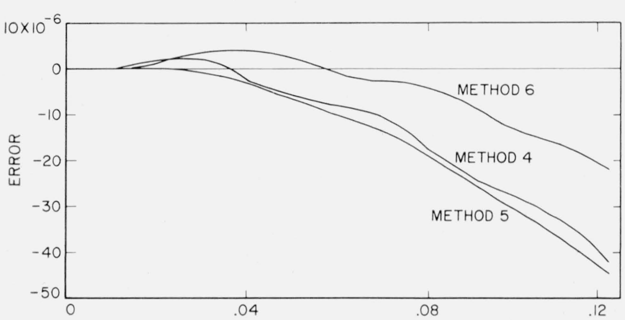


FIGURE 9. *Error curves for methods 4, 5 and 6 with* K=20 *and* h=0.002.

The improvement of method 6 seems to lie in the fact that the error remained positive much longer than for the other two methods.

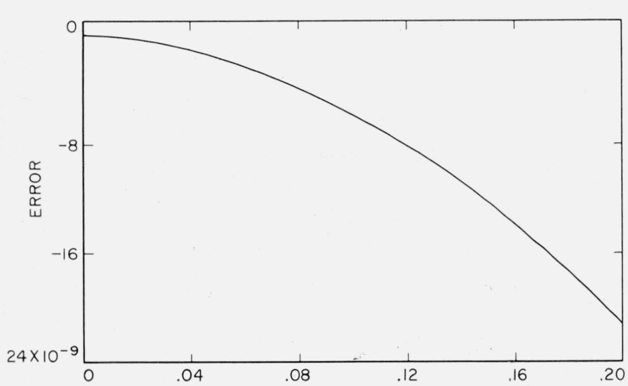


FIGURE 10. *Error curve for method 4 with* K=1 *and* h=0.002.

This corresponds to the normal Runge-Kutta integration procedure.

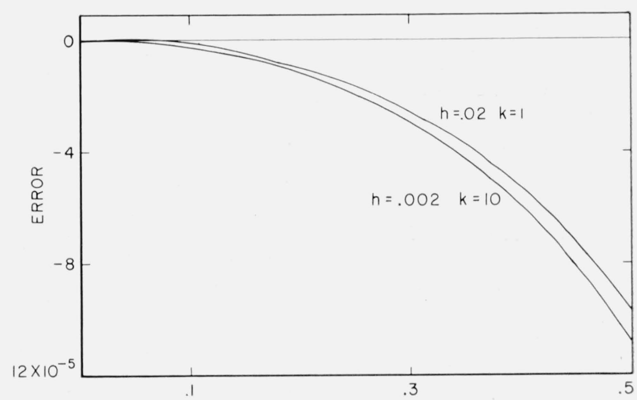


FIGURE 11. *Error curves for method 6 with* K *and* h *taken to be (10, 0.002) and (1, 0.02).*

These curves emphasize the fact that the accuracy is determined by *Kh* and not *h* alone.

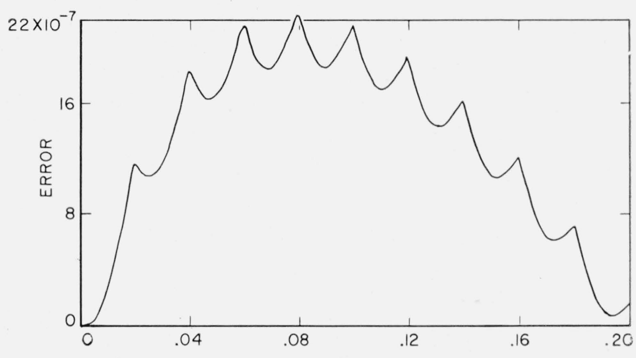


FIGURE 12. *The unusual error curve for the incorrect method 6 with* K=10 *and* h=0.002.

The improved answers are apparently due to the fact that the error remains positive for a long period.

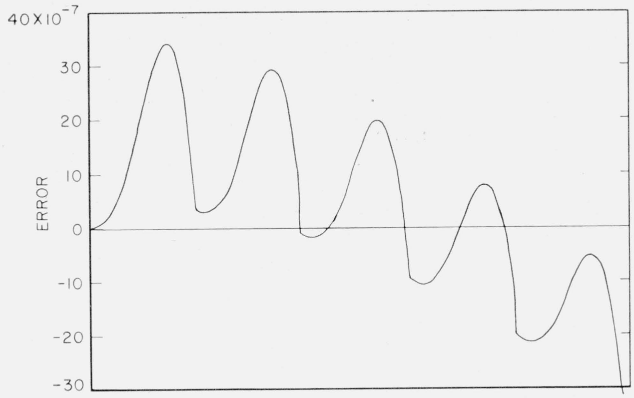


FIGURE 13. *Error curve of the modified equation for method 6 with* K=*20 and* h=*0.002.*
The modification removed the dependence of the $x(t)$ equation on $y(t)$. The maximum error is decreased by a factor of 10.

$h=.002$. The close similarity of the two curves again emphasize that the accuracy of the $x(t)$ integration determines the accuracy of the $y(t)$ integration.

The unusual error curve for the "incorrect method 6" is given in figure 12 with $K=10$ and $h=0.002$.

Some studies were also made with $a=2$ but the overall behavior was the same.

11. Third Example—Noise

The solution to (8.3) is

$$x(t)=t/10+10^{-6} \sin 150$$

$$ty(t)=4\times 10^{-8}[\sin 25t-6 \sin^3 25t+(48/5) \sin^5 25t-(32/7) \sin^7 25t]+0.004\,[t \sin 25t+.04 \cos 25t].$$

This system is very similar to (8.1) with the addition of a rapid oscillation on $x(t)$. However the experimental results bear no resemblance to those for (8.1). In fact they seem to be completely chaotic. Although the errors in the $y(t)$ integration are all relatively small and fairly uniform in size, there appears to be no correlation between the maximum error and K or method. An examination of the error curves reveals the same lack of trend. Even the errors in the $x(t)$ integration do not behave rationally as a function of K, which is equivalent to the integration interval. Indeed the largest error occurs for $K=4$.

In figure 14 the $y(t)$ integration errors are plotted as functions of K and method. The $x(t)$ integration errors, which are independent of the extrapolation method, are also plotted as a function of K.

Figures 15 to 18 are error curves for various situations.

12. Numerical and Efficiency Considerations

The ingredients of an ideal situation for these formulas are as follows: (1) $F(x,y,t)$ is complicated and difficult to evaluate, (2) the solution for $y(t)$ is relatively insensitive to errors in $x(t)$; i.e., the main error source is the inherent inaccuracy of the $y(t)$ integration. The first ingredient magnifies the efficiency and the second is necessary to attain high accuracy. The second ingredient may be replaced by "the accuracy desired for $y(t)$ is lower than that required for $x(t)$." It should be remembered that numerical integration is a delicate business in general and these formulas are particularly so and hence each case should be examined on its own merits.

A comparison will now be made between the work required for split Runge-Kutta and the normal Runge-Kutta methods.

Unless the parameter K is changed very often the extrapolation parameters should be computed only once. For the integration of (1.1) and (1.2) from t_0 to t_0+Kh the following tables give the required number of operations for normal Runge-Kutta and split Runge-Kutta

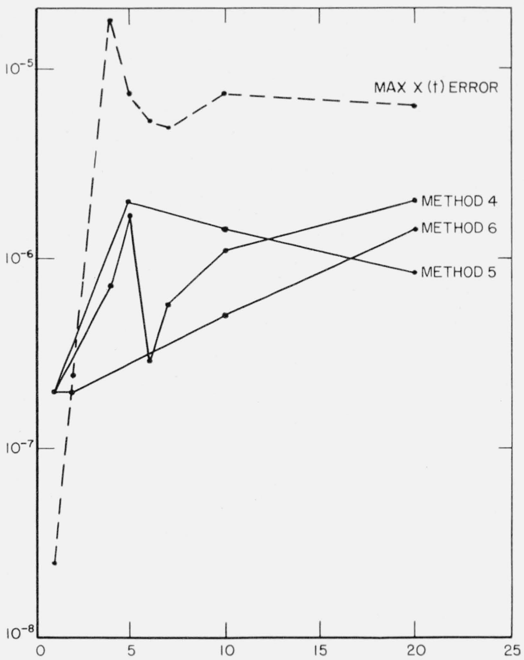


FIGURE 14. *Maximum integration errors as a function of* K *and extrapolation method.*

The lines only join the data points and are not intended to represent the actual behavior of the curves between data points.

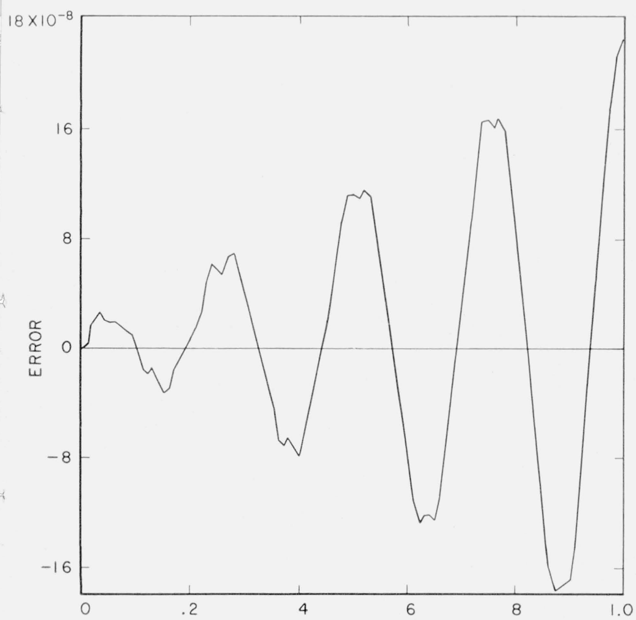


FIGURE 15. *Error curve for method 6 with* K=*2 and* h=*0.01.*

This curve was reproduced for methods 5 and 6 with $K=1$ and $h=0.01$. Apparently the majpr portion of the error oscillates with a period of about 0.23. Upon this then there is superimposed an error with much smaller magnitude and a much shorter period. Compare the period of the error with figure 5.

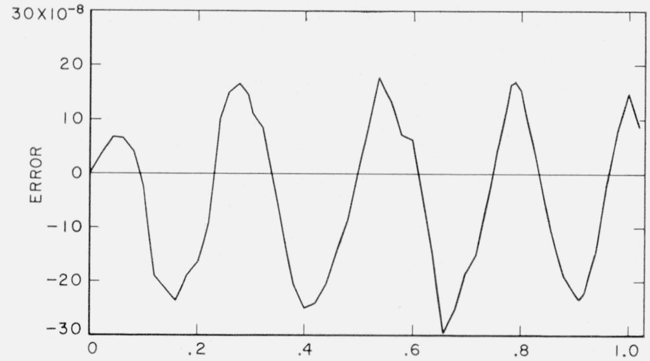


FIGURE 16. *Error curve for method 4 with* K=*6 and* h=*0.01.*

The error oscillates with approximately constant amplitude and a period similar to that of figures 5, 15, and 17.

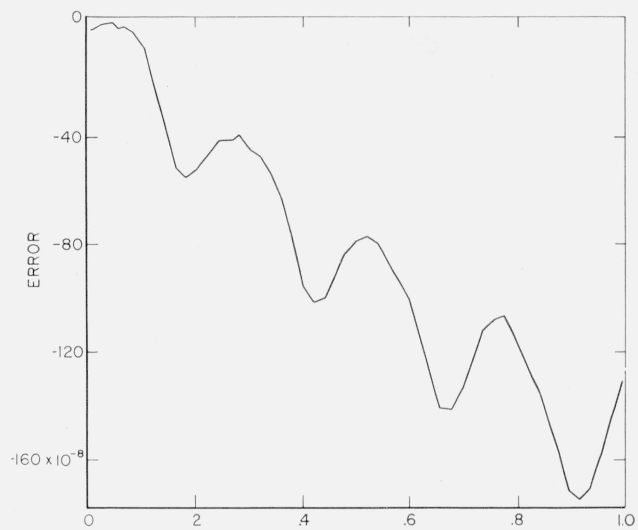


FIGURE 17. *Error curve for method 4 with* K=*5 and* h=*0.01.*

The error oscillates as in figure 16 except it is now oscillating about the line; error$=(1.2t+0.2)10^{-6}$, instead of zero.

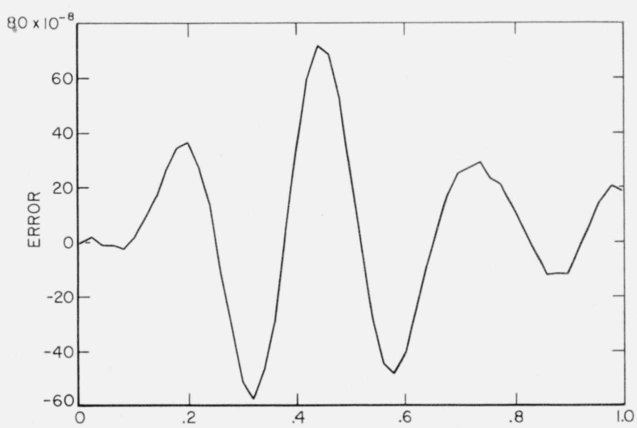


FIGURE 18. *Error curve for method 4 with* K=*4 and* h=*0.01.*

The same oscillation period is present and the oscillation is about zero, but the error curve appears to be "amplitude modulated."

methods. Table 1 is for third order integration and table 2 is for fourth order integration. From these tables it is seen that if a large K may be used then there is a large saving in computation even if F is extremely simple. If F is complicated then the efficiency is even greater.

TABLE 1

	Add.	Mult.	F evaluation	G evaluation
Normal Runge-Kutta	$22K$	$28K$	$3K$	$3K$
Split Runge-Kutta method 4	$14(K+1)$	$17K+19$	3	$3K+2$
Split Runge-Kutta methods 5, 6	$16K+14$	$19(K+1)$	3	$3K+2$

TABLE 2

	Add.	Mult.	F evaluation	G evaluation
Normal Runge-Kutta	$38K$	$46K$	$4K$	$4K$
Split Runge-Kutta	$25K+27$	$29K+34$	4	$4K+3$

(Paper 64B3–32)