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Electric Polarizability of a Short Right Circular Conducting Cylinder*

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A method similar to that employed by Smythe [1][2] for calculating the capacitance of a freely charged short right circular conducting cylinder is used to calculate the electric polarizability tensor in the principal axis system for such a cylinder. Calculations to an accuracy of approximately five significant figures are carried out for cylinders with radius to half-length ratios of ¼, ½, 1, 2, and 4. The results are applicable to the design of artificial dielectrics.

1. Introduction

A solid conducting object in the form of a short, that is noninfinite in length, right circular cylinder makes a suitable element for use in the construction of an artificial dielectric. For such an object, therefore, it becomes important to know the electric polarizability tensor α_{ij} which relates the induced dipole moment p_i to the inducing field E_j according to the equation

$$p_i = \alpha_{ij} E_j, \tag{1}$$

with the cylinder situated in free space. Once α_{ij} is known, the effective dielectric constant of a spatial array of identical and identically oriented cylinders can be calculated according to the methods of Kaprielian [2].

A centered coordinate system in which the z-axis coincides with the axis of rotational symmetry obviously constitutes a system of principal axes for a short right circular cylinder. The components α_{xx} and α_{yy} are clearly equal and are denoted α_{tt} where t stands for "transverse"; similarly, α_{zz} is denoted α_{ll} where l stands for "longitudinal." The cylinder cross section, with nomenclature, is illustrated in figure 1. This article describes a method which has been used successfully for calculating α_{ll} and α_{tt} for cylinders with radius to half-length (a/b) ratios of ¼, ½, 1, 2, and 4. Rationalized M.K.S. units are used throughout.

The problem which presents itself is that of finding, in the external space, a Laplacian function which has no tangential gradient at the surface of the cylinder and which reduces to the potential of a uniform field at infinity. With such a potential, there is associated physically a surface charge density on the cylinder proportional to the local normal gradient and a field-free condition in the interior. It is the presence of this density which gives rise to the induced dipole moment. Since there is no known coordinate system in which a cylinder of finite length with closed plane ends forms a coordinate surface and in which Laplace's equation is separable, the present problem must be regarded as intractable from the point of view of conventional methods and another method involving an arbitrarily good approximation to the surface charge density must be invoked.

As a mathematical abstraction, let the cylinder be regarded merely as a geometrical construct to which the surface charge distribution just mentioned is firmly affixed. If that portion of the potential which corresponds to the uniform (applied) field is now subtracted, the remaining potential will be that due to the charge alone and will be found to correspond to a localized external field with dipole-like appearance and a perfectly uniform *internal* field which is equal

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[2] Figures in brackets indicate the literature references at the end of this paper.

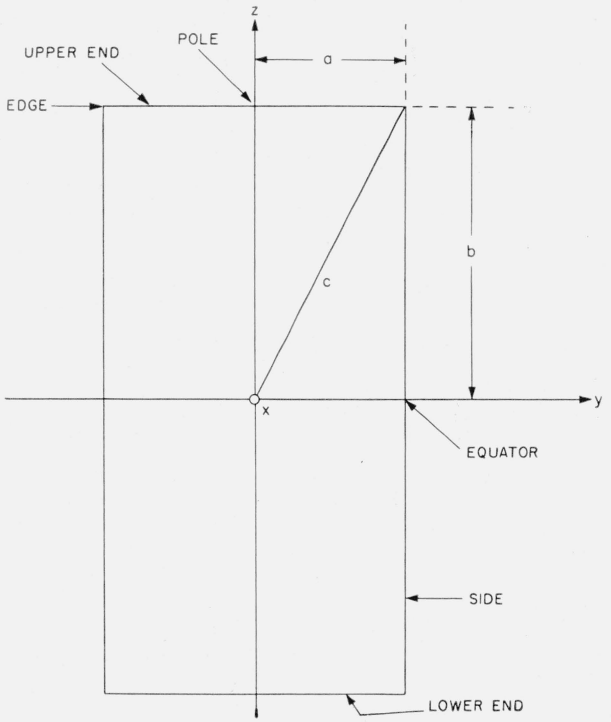

FIGURE 1. *Cross section of the cylinder with nomenclature.*

and opposite to the applied field. The problem may therefore be restated in terms of finding, on a finite cylindrical surface with closed plane ends, a charge distribution which, when acting alone, generates a uniform field in the interior. The technique employed in the present article makes use of this viewpoint and consists in setting up, on the surface, a charge density function which is completely determined by a finite number of parameters, then solving for the values of these parameters so as to obtain maximal uniformity of the charge field or, alternatively, maximal cancellation of the applied field, within. This method is similar to that employed by Smythe for calculating the capacitance of a freely charged cylinder [1]. The research reported here was performed under the direction of Prof. Smythe, to whom the author is indebted for many valuable discussions. The cooperation of the personnel of the California Institute of Technology Computing Center is also gratefully acknowledged.

A second article, now in preparation, applies this method to the problem of determining the magnetic polarizability tensor under the assumption of negligible field penetration. If both tensors are known and if the wavelength is long compared with element spacing, it becomes possible to calculate the wave propagation constant of artifical media whose elements are cylinders of the type considered here.

2. Functions for Expressing the Charge Densities

A system of generalized orthogonal polynomials and weighting functions has been constructed for the purpose of expressing the charge densities on the side and ends of the cylinder. These polynomials involve an argument u, which represents either z/b or ρ/a as required, and a weighting function $(1-u^2)^\nu$. The parameter ν is given the value of minus one-third in order that the charge densities may be asymptotic to a constant times $l^{-1/3}$ as $l \to 0$, where l is the distance from the edge of the cylinder. The product of the polynomial of order m and the weighting function is denoted by the symbol $\bar{\psi}_m$ and is defined as follows:

$$\bar{\psi}_m(\gamma,\zeta,\nu,u)=\frac{(-1)^m 2^{-\nu}\Gamma(m+\gamma+\zeta)}{\Gamma(\gamma+\zeta)\Gamma(m+1+\nu)}\,(1-u^2)^\nu u^\zeta {}_2F_1(-m,m+\nu+\gamma+\zeta;\gamma+\zeta;u^2). \qquad (2)$$

If the parameter γ is given the value of one-half, this function becomes suitable for use on the side of the cylinder; if unity, for use on the end. The parameter ζ controls the parity of the function and is zero for even parity, unity for odd parity. Another function, involving the polynomial alone, is given by

$$\psi_m(\gamma,\zeta,\nu,u)=\frac{(-1)^m 2^{1+\nu}(2m+\nu+\gamma+\zeta)\Gamma(m+\nu+\gamma+\zeta)}{\Gamma(\gamma+\zeta)\Gamma(m+1)}\, u^{\zeta}{}_2F_1(-m,m+\nu+\gamma+\zeta;\gamma+\zeta;u^2). \tag{3}$$

The multipliers in (2) and (3) are designed to simplify both the orthogonality relation and certain integral transforms (see app. A) which play an important part in the analysis. The former is simply

$$\int_0^1 \bar{\psi}_m \psi_n u^{2\gamma-1} du=\delta_{mn}. \tag{4}$$

The polynomials themselves are adaptations of Gegenbauer and Jacobi polynomials; expressions for $\bar{\psi}_m$ and ψ_m in terms of Jacobi polynomials of argument $(2u^2-1)$ are possible.

3. The Longitudinal Problem

In the longitudinal problem, a uniform electric field of magnitude E is applied in the positive z-direction. The assumed charge density on the side depends upon the coefficients r_b and r_m, $0\leq m\leq N_s-1$, as follows:

$$\frac{\sigma_s}{\epsilon_0 E}=R(z)=\left\{\begin{array}{ll} r_b\frac{z}{b}+\sum_{m=0}^{N_s-1} r_m\bar{\psi}_m\left(\frac{1}{2},1,-\frac{1}{3},\frac{z}{b}\right), & |z|\leq b. \\ 0, & |z|>b. \end{array}\right\} \tag{5}$$

The linear term,

$$r_b\frac{z}{b}=r_b\bar{\psi}_0\left(\frac{1}{2},1,0,\frac{z}{b}\right), \tag{6}$$

is called the "basic" term and is included in order that (5) may more easily approximate the true distribution especially when a/b is small.

The potential generated in the interior by the charge density (5) acting alone is easily expressed as an integral transform:

$$\frac{V_s(\rho,z)}{cE}=\frac{a}{c}\int_0^\infty\left[\frac{2}{\pi}\int_0^b R(z')\sin kz'dz'\right]K_0(ka)I_0(k\rho)\sin kz\, dk. \tag{7}$$

The Fourier sine transform of the charge density, enclosed in square brackets above, may be found with the aid of appendix A. The function $I_0(k\rho)\sin kz$, which is regular in a neighborhood of the origin, may be represented as a spherical harmonic expansion about the origin. Thus (7) becomes:

$$\frac{V_s(r,\theta)}{cE}=\sum_{p=0}^{\infty}\frac{ab}{c}\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty\left[r_b\frac{J_{3/2}(kb)}{(kb)^{1/2}}+\sum_{m=0}^{N_s-1}r_m\frac{(-1)^m J_{2m+3/2+\nu}(kb)}{(kb)^{1/2+\nu}}\right]K_0(ka)(kc)^{2p+1}dk$$
$$\frac{(-1)^p}{(2p+1)!}\left(\frac{r}{c}\right)^{2p+1}P_{2p+1}(\cos\theta). \tag{8}$$

This expansion converges within that sphere which has its center at the origin and which passes through the nearest point occupied by charge, in other words, for all $r<a$.

Consider next the ends of the cylinder. The assumed charge density on the upper end depends upon the coefficients t_b and t_m, $0\leq m\leq N_e-1$, as follows:

$$\frac{\sigma_e}{\epsilon_0 E}=T(\rho)=\left\{\begin{array}{ll} t_b+\sum_{m=0}^{N_e-1} t_m\bar{\psi}_m\left(1,0,-\frac{1}{3},\frac{\rho}{a}\right), & \rho\leq a. \\ 0, & \rho>a. \end{array}\right\} \tag{9}$$

The constant term,

$$t_b = t_b \overline{\psi}_0\left(1, 0, 0, \frac{\rho}{a}\right), \tag{10}$$

is now the "basic" term. Again, the potential generated in the interior of the cylinder by both the charge density (9) and an equal and opposite density situated on the lower end, is easily expressed as an integral transform:

$$\frac{V_e(\rho,z)}{cE} = \frac{a}{c}\int_0^\infty \left[\frac{1}{a}\int_0^a T(\rho')J_0(k\rho')\rho' d\rho'\right] e^{-kb} J_0(k\rho)\sinh kz\, dk. \tag{11}$$

This, in turn, may also be given as an expansion in spherical harmonics which converges for all $r<b$:

$$\frac{V_e(r,\theta)}{cE} = \sum_{p=0}^{\infty}\frac{a^2}{c}\int_0^\infty \left[t_b\frac{J_1(ka)}{ka} + \sum_{m=0}^{N_e-1} t_m \frac{(-1)^m J_{2m+1+\nu}(ka)}{(ka)^{1+\nu}}\right] e^{-kb}(kc)^{2p+1} dk \frac{1}{(2p+1)!}\left(\frac{r}{c}\right)^{2p+1} P_{2p+1}(\cos\theta). \tag{12}$$

Evidently the sum of (8) and (12) constitutes an expansion of the interior potential due to the total charge distribution. Although this expansion converges only within the largest sphere which can be inscribed in the cylinder, it can be continued [3, p. 196] to any interior point of the cylinder. This expansion of the interior potential may be expressed in the following form:

$$\frac{V(r,\theta)}{cE} = \sum_{p=0}^{\infty}\left[r_b X_p^{sb} + \sum_{m=0}^{N_s-1} r_m X_p^{sm} + t_b X_p^{eb} + \sum_{m=0}^{N_e-1} t_m X_p^{em}\right]\left(\frac{r}{c}\right)^{2p+1} P_{2p+1}(\cos\theta), \tag{13}$$

where the X's are obtained from (8) and (12) by integrating [4, p. 137] over k. These X's become the matrix elements in a system of simultaneous linear equations which may be solved for the r_b, r_m, t_b, and t_m such that the coefficient of $(r/c)^{2p+1}P_{2p+1}(\cos\theta)$ will be unity for $p=0$ and zero for as many $p>0$ as possible. A solution of this type approaches exactness as the number of simultaneous equations approaches infinity; however, good accuracy was obtained with only eighteen equations. Among these is included an additional relationship known as the "edge condition,"

$$\sum_{m=0}^{N_s-1} r_m - \left(\frac{a}{b}\right)^{1/3}\sum_{m=0}^{N_e-1} t_m = 0, \tag{14}$$

which insures that the side and end densities match one another as they both tend to infinity at the edge. The expressions for the matrix elements are as follows:

$$X_p^{sm} = \frac{(-1)^{p+m}\, 2^{2p-\nu}\,\Gamma\left(m+p+\frac{3}{2}\right)\Gamma\left(m+p+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(2p+2)\,\Gamma\left(2m+\frac{5}{2}+\nu\right)}\left(\frac{a}{c}\right)\left(\frac{b}{c}\right)^{2m+2} \cdot {}_2F_1\left(m+p+\frac{3}{2},\ m-p+1+\nu;\ 2m+\frac{5}{2}+\nu;\ \frac{b^2}{c^2}\right); \tag{15}$$

$$X_p^{em} = \frac{(-1)^m 2^{2p-\nu}\,\Gamma\left(m+p+\frac{3}{2}\right)\Gamma(m+p+1)}{\Gamma\left(\frac{1}{2}\right)\Gamma(2p+2)\,\Gamma\left(2m+2+\nu\right)}\left(\frac{b}{c}\right)\left(\frac{a}{c}\right)^{2m+2} \cdot {}_2F_1\left(m+p+\frac{3}{2},\ m-p+1+\nu;\ 2m+2+\nu;\ \frac{a^2}{c^2}\right). \tag{16}$$

Separate expressions for X_p^{sb} and X_p^{eb} are not necessary since these elements may be obtained merely by setting $\nu=0$ in (15) and (16).

The dipole moment in the longitudinal problem has only a z-component and is found by integrating $z\sigma$ over the entire surface of the cylinder. The orthogonality properties of the $\bar{\psi}_m$ and ψ_m greatly simplify this integration, which results in the following formula for the longitudinal polarizability:

$$\frac{\alpha_{ee}}{v_0\epsilon_0}=\frac{2b}{a}\left[\frac{1}{3}r_b+\frac{\Gamma(3/2)}{2^{2/3}\Gamma(13/6)}r_0\right]+2\left[\frac{1}{2}t_b+\frac{1}{2^{2/3}\Gamma(5/3)}t_0\right], \tag{17}$$

where v_0 is the geometrical volume of the cylinder.

4. The Transverse Problem

In the transverse problem, a uniform electric field of magnitude E is applied in the positive x-direction. The assumed charge density on the side depends upon the coefficients s_b and s_m, $0\leq m\leq N_s-1$, as follows:

$$\frac{\sigma_s}{\epsilon_0 E}=[\cos\phi]S(z)=[\cos\phi]\left\{\begin{array}{ll} s_b+\sum_{m=0}^{N_s-1}s_m\bar{\psi}_m\left(\frac{1}{2},\,0,\,-\frac{1}{3},\,\frac{z}{b}\right), & |z|\leq b. \\ 0\,, & |z|>b. \end{array}\right\} \tag{18}$$

The potential of this density in the interior of the cylinder is given by the integral transform

$$\frac{V_s(\rho,\phi,z)}{cE}=\frac{a}{c}\int_0^\infty\left[\frac{2}{\pi}\int_0^b S(z')\cos kz'\,dz'\right]K_1(ka)I_1(k\rho)\cos kz\,dk\cos\phi. \tag{19}$$

Its spherical harmonic representation, convergent for $r<a$, becomes

$$\frac{V_s(r,\theta,\phi)}{cE}=\sum_{p=0}^\infty\frac{ab}{c}\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty\left[s_b\frac{J_{1/2}(kb)}{(kb)^{1/2}}+\sum_{m=0}^{N_s-1}s_m\frac{(-1)^mJ_{2m+1/2+\nu}(kb)}{(kb)^{1/2+\nu}}\right]$$
$$\cdot K_1(ka)(kc)^{2p+1}dk\,\frac{(-1)^p}{(2p+2)!}\left(\frac{r}{c}\right)^{2p+1}P^1_{2p+1}(\cos\theta)\cos\phi. \tag{20}$$

The assumed charge density on the upper end involves the coefficients w_b and w_m, $0\leq m\leq N_e-1$, and is given by:

$$\frac{\sigma_e}{\epsilon_0 E}=[\cos\phi]W(\rho)=[\cos\phi]\left\{\begin{array}{ll} w_b\frac{\rho}{a}+\sum_{m=0}^{N_e-1}w_m\bar{\psi}_m\left(1,\,1,\,-\frac{1}{3},\,\frac{\rho}{a}\right), & \rho\leq a. \\ 0\,, & \rho>a. \end{array}\right\} \tag{21}$$

The potential generated in the interior of the cylinder by both this charge density and an equal density on the lower end becomes

$$\frac{V_e(\rho,\phi,z)}{cE}=\frac{a}{c}\int_0^\infty\left[\frac{1}{a}\int_0^a W(\rho')J_1(k\rho')\rho'\,d\rho'\right]e^{-kb}J_1(k\rho)\cosh kz\,dk\cos\phi. \tag{22}$$

Again, the spherical harmonic expansion, convergent for $r<b$, is given by

$$\frac{V_e(r,\theta,\phi)}{cE}=\sum_{p=0}^\infty\frac{a^2}{c}\int_0^\infty\left[w_b\frac{J_2(ka)}{ka}+\sum_{m=0}^{N_e-1}w_m\frac{(-1)^mJ_{2m+2+\nu}(ka)}{(ka)^{1+\nu}}\right]$$
$$\cdot e^{-kb}(kc)^{2p+1}dk\,\frac{1}{(2p+2)!}\left(\frac{r}{c}\right)^{2p+1}P^1_{2p+1}(\cos\theta)\cos\phi. \tag{23}$$

Once more, the sum of (20) and (23) constitutes an expansion of the interior potential due to the total charge distribution, which expansion converges within the largest sphere that can be inscribed in the cylinder. The expansion may be expressed as follows:

$$\frac{V(r,\theta,\phi)}{cE}=\sum_{p=0}^{\infty}\left[s_bY_p^{sb}+\sum_{m=0}^{N_s-1}s_mY_p^{sm}+w_bY_p^{eb}+\sum_{m=0}^{N_e-1}w_mY_p^{em}\right]\left(\frac{r}{c}\right)^{2p+1}P_{2p+1}^1(\cos\theta)\cos\phi, \tag{24}$$

where the Y's are again obtained by integrating over k. It is desired that s_b, s_m, w_b, and w_m be such that the coefficient of $(r/c)^{2p+1}P_{2p+1}^1(\cos\theta)\cos\phi$ be unity for $p=0$ and zero for as many $p>0$ as possible, and a system of simultaneous linear equations must be solved. The edge condition now becomes

$$\sum_{m=0}^{N_s-1}s_m-\left(\frac{a}{b}\right)^{1/3}\sum_{m=0}^{N_e-1}w_m=0, \tag{25}$$

and the expressions for the matrix elements are:

$$Y_p^{sm}=\frac{(-1)^{p+m}2^{2p-\nu}\Gamma\left(m+p+\frac{3}{2}\right)\Gamma\left(m+p+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(2p+3)\Gamma\left(2m+\frac{3}{2}+\nu\right)}\left(\frac{a^2}{bc}\right)\left(\frac{b}{c}\right)^{2m+2}$$
$$\cdot{}_2F_1\left(m+p+\frac{3}{2},\ m-p+1+\nu;2m+\frac{3}{2}+\nu;\frac{b^2}{c^2}\right); \tag{26}$$

$$Y_p^{em}=\frac{(-1)^m2^{2p-\nu}\Gamma\left(m+p+\frac{3}{2}\right)\Gamma(m+p+2)}{\Gamma\left(\frac{1}{2}\right)\Gamma(2p+3)\Gamma(2m+3+\nu)}\left(\frac{a}{c}\right)\left(\frac{a}{c}\right)^{2m+2}{}_2F_1\left(m+p+\frac{3}{2},\ m-p+1+\nu;2m+3+\nu;\frac{a^2}{c^2}\right). \tag{27}$$

Note that Y_p^{sb} and Y_p^{eb} are obtained by setting $\nu=0$ in the above.

The dipole moment now has only an x component and is found by integrating $x\sigma$ over the entire surface of the cylinder. The expression for the transverse polarizability is:

$$\frac{\alpha_{tt}}{v_0\epsilon_0}=\left[s_b+\frac{\Gamma(1/2)}{2^{2/3}\Gamma(7/6)}s_0\right]+\frac{a}{b}\left[\frac{1}{4}w_b+\frac{1}{2^{2/3}\Gamma(8/3)}w_0\right]. \tag{28}$$

5. Results

The two distinct tensor components of the electric polarizability are given in table 1.

TABLE 1

a/b	$\alpha_{ll}/v_0\epsilon_0$	$\alpha_{tt}/v_0\epsilon_0$
0	∞	2.0000
¼	15.071	2.3151
½	7.0966	2.6115
1	3.8614	3.1707
2	2.4325	4.2173
4	1.7507	6.1814
∞	1.0000	∞

$v_0=2\pi a^2b$

These results may be compared with the polarizability of a conducting sphere which is equal to $3v_0\epsilon_0$ in rationalized M.K.S. units. Actual values of the charge coefficients are included in appendix B. An estimate of the accuracy of the results was obtained by solving the problem repeatedly with larger and larger numbers of equations and observing the limit

toward which the calculated value of the polarizability tended. It was found that, for a maximum number of equations equal to 18, the polarizability could be found to 5 significant figures with some doubt about the least significant figure. As might be expected, the accuracy is best for a/b equal to unity, in which case the least significant figure is in error by at most one unit.

As an additional check upon the calculations, the potential, V, at the pole and the electric field, E_z, at the equator due to the charge alone in the longitudinal problem were calculated and compared with the corresponding values associated with the applied field. The small errors involved can, by appealing to the appropriate gradients, be translated into local discrepancies, $\Delta\rho$ at the equator and Δz at the pole, between the actual surface and an imaginary surface which is an equipotential under the joint influence of charge and applied field. A corresponding check, in which the potential, V, at the equator and the electric field, E_ρ, at the pole due to the charge alone were compared with the corresponding values associated with the applied field, was carried out for the transverse problem. The results of both checks are given in appendix B and are seen to be very satisfactory.

6. Appendix A: Hankel Transforms of the $\bar{\psi}_m$ Functions

It will be shown that all of the integral transforms of the $\bar{\psi}_m$ functions used in this article are special cases of the general transform:

$$\int_0^1 u^\sigma (1-u^2)^\nu P_m^{(\nu,\sigma)}(2u^2-1) J_\sigma(tu)\, u\, du = \frac{(-1)^m \Gamma(m+1+\nu) 2^\nu J_{2m+1+\sigma+\nu}(t)}{\Gamma(m+1) t^{1+\nu}};$$

$$\text{Re } \sigma > -1;\ \text{Re } \nu > -1;\ t \text{ real and } > 0, \qquad \text{(A1)}$$

which will now be proved. $P_m^{(\nu,\sigma)}(2u^2-1)$ is a Jacoby polynomial [5, p. 168]. Denote the integral in eq (A1) by $f(t)$. Then

$$f(t) = \sum_{r=0}^{\infty} \frac{(-1)^r (t/2)^{2r+\sigma}}{\Gamma(r+1)\Gamma(r+\sigma+1)} \int_0^1 u^{2r+2\sigma} (1-u^2)^\nu P_m^{(\nu,\sigma)}(2u^2-1)\, u\, du. \qquad \text{(A2)}$$

Letting $2u^2-1=x$ in [4, p. 284],

$$\int_0^1 u^{2l} (1-u^2)^\nu P_m^{(\nu,\sigma)}(2u^2-1)\, u\, du = \frac{\Gamma(l+1)\Gamma(m+1+\nu)\Gamma(l-\sigma+1)}{2\Gamma(m+1)\Gamma(l-\sigma-m+1)\Gamma(\nu+l+m+2)};\ \text{Re}\, l > -1;\ \text{Re}\, \nu > -1. \qquad \text{(A3)}$$

Let $l=r+\sigma$; Re $\sigma > -1$:

$$f(t) = \frac{\Gamma(m+1+\nu)}{2\Gamma(m+1)} \sum_{r=0}^{\infty} \frac{(-1)^r (t/2)^{2r+\sigma}}{\Gamma(r-m+1)\Gamma(\nu+r+\sigma+m+2)};\ \text{Re } \sigma > -1;\ \text{Re } \nu > -1. \qquad \text{(A4)}$$

Make a change in the index of summation; let $r=m+k$:

$$f(t) = \frac{(-1)^m \Gamma(m+1+\nu)}{2\Gamma(m+1)} \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{2k+2m+\sigma}}{\Gamma(1+k)\Gamma(2m+2+\sigma+\nu+k)}. \qquad \text{(A5)}$$

This is:

$$f(t) = \frac{(-1)^m \Gamma(m+1+\nu)}{2\Gamma(m+1)} \left(\frac{t}{2}\right)^{-1-\nu} J_{2m+1+\sigma+\nu}(t), \qquad \text{(A6)}$$

and the transform of eq (A1) is true.

From [5, p. 170], one finds that:

$$P_m^{(\nu,\sigma)}(2u^2-1) = \frac{(-1)^m \Gamma(m+1+\sigma)}{\Gamma(m+1)\Gamma(\sigma+1)}\, {}_2F_1(-m, m+\nu+\sigma+1; \sigma+1; u^2). \qquad \text{(A7)}$$

If σ is set equal to $\gamma+\zeta-1$, the transform becomes

$$\int_0^1 \bar{\psi}_m(\gamma,\zeta,\nu,u) J_{\gamma+\zeta-1}(tu)(tu)^\gamma du=\frac{(-1)^m J_{2m+\gamma+\zeta+\nu}(t)}{(t)^{1-\gamma+\nu}}. \tag{A8}$$

Now if the parameters γ and ζ are allowed to take all values relevant to the present problem, one obtains:

Side, even:

$$\int_0^1 \bar{\psi}_m\left(\frac{1}{2},0,\nu,u\right)\cos tu\, du=\left(\frac{\pi}{2}\right)^{1/2}\frac{(-1)^m J_{2m+1/2+\nu}(t)}{t^{1/2+\nu}}; \tag{A9}$$

Side, odd:

$$\int_0^1 \bar{\psi}_m\left(\frac{1}{2},1,\nu,u\right)\sin tu\, du=\left(\frac{\pi}{2}\right)^{1/2}\frac{(-1)^m J_{2m+3/2+\nu}(t)}{t^{1/2+\nu}}; \tag{A10}$$

End, even:

$$\int_0^1 \bar{\psi}_m(1,0,\nu,u) J_0(tu) u du=\frac{(-1)^m J_{2m+1+\nu}(t)}{t^{1+\nu}}; \tag{A11}$$

End, odd:

$$\int_0^1 \bar{\psi}_m(1,1,\nu,u) J_1(tu) u du=\frac{(-1)^m J_{2m+2+\nu}(t)}{t^{1+\nu}};\ \mathrm{Re}\ \nu>-1. \tag{A12}$$

Because of the special values taken by γ and ζ, the singularity at the origin which might have occurred in $(tu)^\gamma J_{\gamma+\zeta-1}(tu)$ does not appear. Both sides of eqs (A9) through (A12) become entire functions of t and no restrictions need be placed upon t.

7. Appendix B: Values of the Charge Coefficients

The values of the charge coefficients and the results of the checking procedure at pole and equator are given in tables 2 through 6. The notation "$A(p)$" means "A times 10^p"

TABLE 2

$a/b=\frac{1}{4}$		
	Longitudinal	Transverse
m	r_m	s_m
Basic	+0.12905187 (+0)	+0.40387637 (+0)
0	+0.23603290 (+1)	+0.14690787 (+1)
1	−0.11393541 (+0)	−0.43239914 (+0)
2	−0.19185823 (−1)	−0.65706908 (−1)
3	−0.52174681 (−2)	−0.17221441 (−1)
4	−0.16484165 (−2)	−0.54219231 (−2)
5	−0.53665621 (−3)	−0.17807221 (−2)
6	−0.16813125 (−3)	−0.56407302 (−3)
7	−0.48283874 (−4)	−0.16353609 (−3)
8	−0.12195426 (−4)	−0.41591873 (−4)
9	−0.25981907 (−5)	−0.88990999 (−5)
10	−0.44409406 (−6)	−0.15241129 (−5)
11	−0.56727317 (−7)	−0.19470442 (−6)
12	−0.47918712 (−8)	−0.16423381 (−7)
13	−0.20011235 (−9)	−0.68402739 (−9)
Equator check	$E_z/E=-1.00000002$	$V/aE\cos\phi=+1.00000002$
$\Delta\rho/a$	Negligible	Negligible
m	t_m	w_m
Basic	+0.15200248 (−1)	−0.17172000 (+0)
0	+0.35624820 (+1)	+0.14693027 (+1)
1	−0.39128623 (−1)	+0.32011553 (−1)
Pole check	$V/bE=+0.99999875$	$E\rho/E\cos\phi=-0.99902328$
$\Delta z/b$	−0.00000037	−0.0022077

TABLE 3

$a/b=\frac{1}{2}$		
	Longitudinal	Transverse
m	r_m	s_m
Basic	−0.19225150 (−4)	+0.38965487 (+0)
0	+0.18367917 (+1)	+0.16055212 (+1)
1	−0.48838691 (−1)	−0.35509127 (+0)
2	−0.54930270 (−2)	−0.43967119 (−1)
3	−0.10582670 (−2)	−0.10119125 (−1)
4	−0.23834773 (−3)	−0.28016839 (−2)
5	−0.54147255 (−4)	−0.78539136 (−3)
6	−0.11308514 (−4)	−0.20187101 (−3)
7	−0.20056279 (−5)	−0.43956298 (−4)
8	−0.27611488 (−6)	−0.74296836 (−5)
9	−0.25757920 (−7)	−0.85391338 (−6)
10	−0.12042175 (−8)	−0.49561337 (−7)
Equator check	$E_z/E=-1.00000003$	$V/aE\cos\phi=+1.00000009$
$\Delta\rho/a$	Negligible	Negligible
m	t_m	w_m
Basic	+0.46938430 (−1)	+0.50835705 (−2)
0	+0.23357609 (+1)	+0.13797193 (+1)
1	−0.82663119 (−1)	+0.88406931 (−1)
2	−0.76227915 (−2)	+0.25064651 (−1)
3	−0.12692924 (−2)	+0.77264312 (−2)
4	−0.16584913 (−3)	+0.15415744 (−2)
Pole check	$V/bE=+0.99999997$	$E\rho/E\cos\phi=-1.00002004$
$\Delta z/b$	Negligible	+0.00000890

TABLE 4

$a/b=1$

	Longitudinal	Transverse
m	r_m	s_m
Basic	−0.37616817 (−1)	+0.25340314 (+0)
0	+0.14921036 (+1)	+0.19371617 (+1)
1	−0.83747159 (−2)	−0.32219422 (+0)
2	+0.23944987 (−2)	−0.38977600 (−1)
3	+0.12471808 (−2)	−0.93990090 (−2)
4	+0.48160777 (−3)	−0.25945051 (−2)
5	+0.14698364 (−3)	−0.64974220 (−3)
6	+0.31457070 (−4)	−0.12162489 (−3)
7	+0.34949152 (−5)	−0.12244613 (−4)
Equator check	$E_z/E=-1.00000001$	$V/aE\cos\phi=+0.99999999$
$\Delta\rho/a$	Negligible	Negligible
m	t_m	w_m
Basic	+0.87113326 (−1)	−0.13425511 (+0)
0	+0.16193448 (+1)	+0.14792105 (+1)
1	−0.11224086 (+0)	+0.58149961 (−1)
2	−0.14340444 (−1)	+0.17095607 (−1)
3	−0.34812949 (−2)	+0.59774893 (−2)
4	−0.95870761 (−3)	+0.20469643 (−2)
5	−0.23986408 (−3)	+0.59421158 (−3)
6	−0.44962281 (−4)	+0.12429137 (−3)
7	−0.45407822 (−5)	+0.13669068 (−4)
Pole check	$V/bE=+1.00000001$	$E\rho/E\cos\phi=-0.99999996$
$\Delta z/b$	Negligible	Negligible

TABLE 5

$a/b=2$

	Longitudinal	Transverse
m	r_m	s_m
Basic	−0.65161515 (−2)	+0.16191166 (+0)
0	+0.12482169 (+1)	+0.23744087 (+1)
1	+0.24818534 (−1)	−0.28135556 (+0)
2	+0.91060187 (−2)	−0.33694020 (−1)
3	+0.28707901 (−2)	−0.72222609 (−2)
4	+0.56748900 (−3)	−0.11581174 (−2)
Equator check	$E_z/E=-1.00000271$	$V/aE\cos\phi=+0.99999988$
$\Delta\rho/a$	+0.00000121	−0.00000005
m	t_m	w_m
Basic	+0.16025169 (+0)	−0.22721442 (+0)
0	+0.11683321 (+1)	+0.15658591 (+1)
1	−0.12492981 (+0)	+0.42693596 (−1)
2	−0.17150631 (−1)	+0.12814464 (−1)
3	−0.42141901 (−2)	+0.43243348 (−2)
4	−0.12098644 (−2)	+0.14993639 (−2)
5	−0.34737952 (−3)	+0.49000507 (−3)
6	−0.90947390 (−4)	+0.14126899 (−3)
7	−0.20111927 (−4)	+0.33697692 (−4)
8	−0.34459642 (−5)	+0.61418057 (−5)
9	−0.40094338 (−6)	+0.75263582 (−6)
10	−0.23533528 (−7)	+0.46182535 (−7)
Pole check	$V/bE=+1.00000002$	$E\rho/E\cos\phi=-1.00000003$
$\Delta z/b$	Negligible	Negligible

TABLE 6

$a/b=4$

	Longitudinal	Transverse
m	r_m	s_m
Basic	−0.91407486 (−1)	+0.36981774 (+0)
0	+.11655470 (+1)	+.27748758 (+1)
1	+.92149445 (−2)	−.13470990 (+0)
Equator check	$E_z/E=-0.99979406$	$V/aE\cos\phi=+1.00001613$
$\Delta\rho/a$	−0.00005287	+0.00000530
m	t_m	w_m
Basic	+0.16831628 (+0)	−0.10545599 (+0)
0	+0.92928755 (+0)	+0.15388757 (+1)
1	−0.15311162 (+0)	+0.82957625 (−1)
2	−0.25352751 (−1)	+0.25448502 (−1)
3	−0.71931540 (−2)	+0.96857682 (−2)
4	−0.23904452 (−2)	+0.39007948 (−2)
5	−0.81543454 (−3)	+0.15309282 (−2)
6	−0.26584526 (−3)	+0.55635828 (−3)
7	−0.78893956 (−4)	+0.18020312 (−3)
8	−0.20466149 (−4)	+0.50262768 (−4)
9	−0.44553523 (−5)	+0.11635018 (−4)
10	−0.77488570 (−6)	+0.21335661 (−5)
11	−0.10037598 (−6)	+0.28945170 (−6)
12	−0.85747540 (−8)	+0.25757752 (−7)
13	−0.36132625 (−9)	+0.11256670 (−8)
Pole check	$V/bE=+1.00000001$	$E\rho/E\cos\phi=-1.00000001$
$\Delta z/b$	Negligible	Negligible

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