JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 64B, No. 2, April–June 1960

Criteria for the Existence and Equioscillation of Best Tchebycheff Approximations

John R. Rice [1]

(January 19, 1960)

Let $F(a,x)=\Sigma a_i\varphi_i(x)$ where $\varphi_i(x)$ are continuous in [0,1] and the a_i are real parameters. The following theorem answers the principal questions of a general nature in the theory of Tchebycheff approximations: THEOREM. Let $\{\varphi_i(x)\}$ form a Tchebycheff set and let $f(x)$ be an arbitrary function continuous on [0,1]. Then (A) $f(x)$ possesses a best approximation, (B) a necessary and sufficient condition that $F(a^*,x)$ be a best approximation to $f(x)$ is that $\max|F(a^*,x)-f(x)|$ alternates at least n times in [0,1], (C) the best approximation to $f(x)$ is unique. A. Haar [Math. Ann. **78,** 43–56 (1928)] posed and answered the following question: What conditions on F are necessary and sufficient for theorem C to be valid? The condition he found is that $\{\varphi_i(x)\}$ must form a Tchebycheff set. This paper poses and answers the following three questions: (1) What conditions on F are necessary and sufficient for theorem A to be valid? (2) What conditions on F are necessary and sufficient for theorem B to be valid? (3) What conditions on F are necessary and sufficient for both theorems A and B to be valid?

This paper does not tacitly assume that the a_i may assume all values.

1. Let T be a set of n functions $\varphi_1(x)$, $\varphi_2(x)$, . . ., $\varphi_n(x)$ continuous on [0,1] and let $F(a,x)=\Sigma_{i=1}^n a_i\varphi_i(x)$ be called a *T-polynomial.* $F(a,x)$ is said to be a *nontrivial T-polynomial* if $\Sigma_{i=1}^n|a_i|>0$. It is assumed that there is no nontrivial T-polynomial which is identically zero. The *parameter set* P of F is the domain of the a_i. T is said to be a *Tchebycheff set* if every nontrivial T-polynomial has at most $n-1$ zeros. $\max|F(a,x)-f(x)|$ is said to *alternate n times* if there are $n+1$ points $0\leq x_1<x_2< \ . \ . \ . <x_{n+1}\leq 1$ such that

$$F(a,x_j)-f(x_j)=-[F(a,x_{j+1})-f(x_{j+1})] =\pm\max|F(a,x)-f(x)|.$$

All maximums and minimums are taken over $x\in[0,1]$ unless otherwise stated, and all summations are from 1 to n unless otherwise stated. Braces, { }, denote sets and sequences, and $\{x| \ . \ . \ . \}$ is read as "the set of x such that . . .".

The following theorem answers the principal questions of a general nature in the theory of Tchebycheff approximations.

THEOREM: *Let T be a set of n functions and let f(x) be an arbitrary function continuous on [0,1]. Then*

A. f(x) possesses a best approximation,

B. a necessary and sufficient condition that $F(a^*,x)$ be a best approximation to f(x) is that $\max|F(a^*,x)-f(x)|$ alternates at least n times.

C. the best approximation to f(x) is unique.

A. Haar [2] has posed and answered the following question: What conditions on F are necessary and sufficient in order that theorem C be valid for every continuous $f(x)$? The condition he found is that T must be a Tchebycheff set.

Three similar questions are posed in this paper. They are:

(1) What conditions on F are necessary and sufficient for theorem A to be valid for every continuous $f(x)$?

(2) What conditions on F are necessary and sufficient for both theorems A and B to be valid for every continuous $f(x)$?

(3) What conditions on F are necessary and sufficient for theorem B to be valid for every continuous $f(x)$?

These questions are answered by theorems 1, 2, and 3, respectively.

In this study it has not been assumed that $P=E_n$, and indeed the topological nature of P plays a central role in the analysis.

2. The following definition will be useful:

DEFINITION: *F has property Z if $a\neq a^*$ implies that $F(a,x)-F(a^*,x)$ has at most $n-1$ zeros in [0,1].*

Note that this does not imply that T is a Tchebycheff set for the parameters are restricted to P.

The first theorem gives an answer to the first question posed.

THEOREM 1: *Theorem A is valid if and only if P is closed.*

PROOF: Let $d=\inf_{a\in P}\max|F(a,x)-f(x)|$, then it can be shown [3] that there is an $M<\infty$ such that $\Sigma|a_i|>M$ implies $\max|F(a,x)-f(x)|>d+1$. $f(x)$ has a best approximation if there is an $a^*\in P$ such that

$$\max|F(a^*,x)-f(x)|=\inf\max|F(a,x)-f(x)|.$$

Let

$$P_M=\{a|a\in P,\sum|a_i|\leq M\}.$$

[1] Part of this work was done while the author was an NRC–NBS Postdoctoral Research Associate.

[2] A. Haar, Die Minkowskische Geometrie und die Annäherung an stetige Functionnen, Math. Ann. **78,** 43–56 (1928).

[3] N. I. Achieser, Theory of approximation, p. 10 (Ungar Publishing Co., New York, N.Y., 1956).

542538—60——2

Clearly a may be restricted to values from P_M. If P is closed, then $\inf_{a \in P_M} \max|F(a,x)-f(x)|$ is attained and $f(x)$ possesses a best approximation.

Assume that P is not closed and let a' be a boundary point of P not contained in P. Consider the function $\Sigma a'_i\varphi_i(x)$. If theorem A is valid, then it has a best approximation, say $F(a^*,x)$. Now it is clear that

$$\inf_{a \in P} \max |F(a,x)-\sum a'_i\varphi_i(x)|=0,$$

hence

$$\sum (a'_i-a^*_i)\varphi_i(x)=0 \text{ for } x \in [0,1].$$

This possibility is precluded by one of the initial assumptions; so a contradiction has been reached.

The following example shows that the assumption that no linear combination of the $\varphi_i(x)$ be identically zero is necessary for the validity of theorem 1. Let $\varphi_1(x)=1$, $\varphi_2(x)=1$, and let P be the subset of E_2 consisting of $\{(x,0)|x\neq 1\}$ and $(0,1)$. Every continuous function has a unique best approximation, but P is not closed.

3. In order to answer the second question, three lemmas will be established. The first is

LEMMA 1: *If theorem B is valid, then F has property Z.*

PROOF: Assume $F(a,x)-F(a^*,x)$ has n zeros. Let

$$M(x)=\tfrac{1}{2}[F(a,x)+F(a^*,x)];$$

$$d=\tfrac{1}{2}\max |F(a,x)-F(a^*,x)|,$$

and let $0\leq x_1<x_2<, \ldots, <x_{n+1}\leq 1$ be n points where $F(a,x)-F(a^*,x)=0$, along with one point where $F(a,x)-F(a^*,x)\neq 0$, say $F(a^*,x_2)>F(a,x_2)$ for concreteness. Set

$$\delta=\tfrac{1}{4}\min (x_1, x_2-x_1, \ldots, x_{n+1}-x_n, 1-x_{n+1}),$$

with x_1 or $1-x_{n+1}$ omitted if they are zero.

A function $f(x)$ continuous on $[0,1]$ will be defined as follows:

$f(x)=M(x)$ in the intervals $[x_j+\delta, \ x_{j+1}-\delta]$, $j=1,2, \ldots, n$;

$f(x)=M(x)$ in $[0,x_1-\delta]$ if $x_1>0$;

$f(x)=M(x)$ in $[x_{n+1}+\delta, 1]$ if $x_{n+1}<1$;

in the remaining intervals $f(x)$ satisfies

$$f(x_j)=F(a^*,x_j)+(-1)^{j+1}(3d/2)$$

and

$$|F(a,x)-f(x)|<3d/2, \ |F(a^*,x)-f(x)|<3d/2 \text{ for } x\neq x_j.$$

Now $\max |F(a^*,x)-f(x)|$ alternates exactly n times and $\max |F(a,x)-f(x)|$ alternates exactly $n-2$ times. By theorem B, $F(a^*,x)$ is a best approximation to $f(x)$, and since

$$\max |F(a^*,x)-f(x)|=\max |F(a,x)-f(x)|,$$

so is $F(a,x)$. This contradicts the validity of theorem B.

For the purposes of the next lemma an *axial set* will be defined. A set in E_n is said to be axial if every neighborhood of each point a of the set contains a point of the set on each half-axis (positive and negative) of the E_n coordinate system with origin translated to a.

Since the validity of theorem B implies that F has property Z, the parameter space P may be identified with the function values of F at n fixed distinct points in $[0,1]$. Let $0<x_1<x_2<, \ldots, <x_n<1$ be n such points and let P' denote the set identified with P.

LEMMA 2: *If theorem B is valid and P′ is closed, then P′ is an axial set.*

PROOF: Let $\epsilon>0$ and a point $b^*\in P'$ be given. $F(b^*,x)$ shall denote $F(a^*,x)$ where $a^*\in P$ is identified with b^*. Let

$$d=\min [x_1,x_2-x_1, \ldots, x_n-x_{n+1},1-x_n].$$

A sequence of functions $\{f_m(x)|m=2,3, \ldots\}$, each continuous on $[0,1]$, will be defined as follows:

$$f_m(x)=F(b^*,x)+(-1)^j\epsilon/3 \quad \text{for} \quad x\in[0,x_1-d/m],$$
$$x\in[x_j+d/m,x_{j+1}-d/m], \quad j=1,2, \ldots, k-2;$$

$$f_m(x)=F(b^*,x)+(-1)^{k-1}\epsilon/3 \quad \text{for}$$
$$x\in[x_{k-1}+d/m,x_{k+1}-d/m];$$

$$f_m(x)=F(b^*,x)+(-1)^{j+1}\epsilon/3 \text{ for } x\in[x_n+d/m,1],$$
$$x\in[x_j+d/m,x_{j+1}-d/m], \quad j=k+1, \ldots, n;$$

$F(x)$ is defined in the remaining intervals so that $|f_m(x)-F(b^*,x)|<\epsilon/3$ and $f_m(x)$ is continuous in $[0,1]$.

Since P' is closed, P is closed and every function has a best approximation by lemma 1 and theorem 1. Let $F(b_m,x)$ be the best approximation to $f_m(x)$. Now it is clear that $\operatorname{sgn}[F(b_m,x_k)-F(b^*,x_k)]=(-1)^{k-1}$ and that $F(b_m,x)-F(b^*,x)$ has a zero in

$$[x_j-d/m, x_j+d/m], j=1,2, \ldots, k-1,k+1, \ldots, n.$$

Since P' is closed (and clearly the b_m are bounded), $\{b_m\}$ contains a convergent subsequence with a limit in P'. Let b_0 be this limit. Now $F(b^*,x_j)-F(b_0,x_j)=0$, $j=1,2, \ldots, k-1,k+1, \ldots, n$. By theorem B, $\max |F(b_mx)-f_m(x)|$ alternates n times; hence

$$\max|F(b_m,x)-F(b^*,x)|>\epsilon/3,$$

since $\max |F(b^*,x)-F_m(x)|$ alternates only $n-1$ times. Therefore,

$$\max |F(b_0,x)-F(b^*,x)|\geq \epsilon/3 \text{ and } b_0\neq b^*.$$

By actual construction it has been shown that there is a $b_0\in P'$ such that b_0 is on the kth coordinate axis (positive or negative according to whether k is odd or even) at a distance less than ϵ for any $\epsilon>0$. Clearly this construction may be modified to obtain a point in P' on the kth axis with opposite sign. Hence P' is axial.

LEMMA 3: *Let P' be a closed axial set; then $P'=E_n$.*

PROOF: The proof is based on the following assertion: Every closed n-sphere whose boundary passes through the origin in E_{n+1} contains a segment of a coordinate axis which is attached to the origin. That this assertion is true is seen as follows. Consider the tangent plane of the sphere at the origin. If the positive kth axis is on one side of the plane, then the negative kth axis is on the other side; i.e., $a_k x_k > 0$ implies $a_k(-x_k) < 0$. Now the tangent plane cannot contain every axis; hence there is an axis intersecting the tangent plane at the origin. Clearly this axis must intersect the sphere also, which establishes the assertion.

If P' is closed, the complement of P' is open. Let c be a point in the complement of P', and let S be the smallest closed sphere with center at c which intersects P', and let b be a point of intersection. Now S contains a segment of some coordinate axis of a coordinate system with origin at b. Since P' is axial, that segment contains a point of P' which contradicts the definition of S and hence the existence of c.

With these three lemmas the second question may be answered by the following theorem.

THEOREM 2: *Theorems A and B are both valid for F if and only if $P=E_n$ and T is a Tchebycheff set.*

PROOF: It is a classical result that theorems A and B are valid if $P=E_n$ and T is a Tchebycheff set.

From lemma 1 it is seen that the validity of theorem B implies that F has property Z. If F has property Z and theorem A is valid, then theorem 1 implies that P is closed. Now there is a one-to-one mapping between P and P', the set defined for lemma 2, which is continuous both ways, since F has property Z. Hence, if P is closed, then so is P'. From lemma 2 it follows that P' is an axial set, and lemma 3 implies that $P'=E_n$. Hence $P=E_n$ and T is a Tchebycheff set.

4. The answer to the third question is given by theorem 3. The proof of this theorem is rather complicated and since it is a special case of a theorem to be published elsewhere, the proof will not be given here. The result is given here for the sake of completeness.

THEOREM 3: *Theorem B is valid for F if and only if (i) F has property Z;*

(ii) given $a^* \in P$, $k<n$, $\{x_j | 0=x_0<x_1<\ldots<x_{k+1}=1\}$ *and* ϵ *with* $0<\epsilon<\frac{1}{2}\min(x_{j+1}-x_j)$, $j=0, 1, \ldots, k$, *then (a) there are* $a_1, a_2 \in P$ *such that, for* $x \in [0,1]$, $F(a^*, x)-\epsilon<F(a_1,x)<F(a^*,x)<F(a_2,x)<F(a^*,x)+\epsilon$, *(b) there are* $a_3, a_4 \in P$ *such that* $|F(a_3,x)-F(a^*,x)|<\epsilon$, $|F(a_4,x)-F(a^*,x)|<\epsilon$, *for* $x \in [0,1]$ *and* $F(a_3,x)-F(a^*,x)$, $F(a_4,x)-F(a^*,x)$ *change sign from* $x_j-\epsilon$ *to* $x_j+\epsilon$, $j=1, 2, \ldots, k$ *and have no zeros outside* $[x_j-\epsilon, x_j+\epsilon]$. *Further* $F(a_3,0)>F(a^*,0)>F(a_4,0)$.

WASHINGTON, D.C. (Paper 64B2–25)