

Upper and Lower Bounds for the Center of Flexure

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(February 1, 1960)

There is disagreement in the literature as to how the center of flexure of an isotropic elastic beam should be defined. In either of the two most widely accepted definitions, upper and lower bounds for the coordinates of the center of flexure are obtained.

1. Introduction

We consider a cantilever beam of uniform cross section and choose the z -axis to coincide with the line of cross-section centroids. The plane end $z=0$ is assumed fixed and the other end $z=l$ is loaded by some distribution of forces which is statically equivalent to a single force with components $(W_x, W_y, 0)$ acting at a load point (x_L, y_L, l) . In the treatment of the flexure problem [9]¹ (a similar treatment appears in [3]) we seek a load point (x_f, y_f, l) which has the property that any load $(W_x, W_y, 0)$ applied through this point produces no local twist at the centroid of the section, i.e., the mean value of the local twist over the cross section vanishes (see [9] p. 200). This point (x_f, y_f, l) is referred to as the center of flexure [9]. The determination of (x_f, y_f, l) permits us to break up the general flexure problem into two separate problems—one of pure torsion and one of pure flexure. This point is sometimes referred to as the center of shear (see, for instance, [3]), and it is well known that there is difference of opinion in the literature as to the way in which the point should be defined (see, for instance, [1, 2, 4, 10, 11]). A discussion of these differences has been given in a recent paper by Pearson [7], who demonstrates that in general the two expressions for the center of shear yield results which are very near to one another.

It is not our point here to enter into the discussion of the relative merits of the two definitions. We merely point out that the expression of Trefftz [10] is simpler and independent of Poisson's ratio. The center of flexure of [9], on the other hand, permits an easy decomposition of the general flexure problem.

In order to avoid confusion we shall hereafter refer to the point defined in [9] as the center of flexure and that given by Trefftz [10] as the center of shear.

In this paper then we derive upper and lower bounds for the coordinates of the center of flexure (x_f, y_f, l) and the center of shear (x_s, y_s, l) . In the first case we use the formulation of the flexure problem derived in [9] and thus concern ourselves with the problem of pure flexure.

It is of course well known (see, for instance, [7]) that if we have the solution to the pure torsion problem for the beam in question then we can determine both the center of shear and the center of flexure without knowledge of the solution to the flexure problem. As would be expected it is possible in each case to obtain upper and lower bounds for the coordinates of the point by approximating the solution to the torsion problem alone.

2. Bounds for the Center of Flexure

We denote by \mathcal{D} the cross section of the beam and let C be its boundary. (The region may be simply or multiply connected.) The origin of a rectangular coordinate system is taken at the centroid of \mathcal{D} . For simplicity the x and y axes are chosen to be principal axes of inertia, i.e.,

$$\iint_{\mathcal{D}} xy dA = 0. \quad (2.1)$$

¹ Figures in brackets indicate the literature references at the end of this paper.

According to [9] the pure flexure problem is solved once the solutions to the following boundary value problems have been determined:

$$\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial y^2} = 0, \quad \frac{\partial^2 \varphi_2}{\partial x^2} + \frac{\partial^2 \varphi_2}{\partial y^2} = 0 \quad \text{in } \mathcal{D} \quad (2.2)$$

$$\frac{\partial \varphi_1}{\partial n} = [(1+\sigma)x^2 - \sigma y^2]n_x \quad (2.3)$$

$$\frac{\partial \varphi_2}{\partial n} = [(1+\sigma)y^2 - \sigma x^2]n_y \quad \text{on } C$$

where σ is Poisson's ratio and n_x and n_y are the x and y components of the unit normal on C . (The normal is assumed directed outward from \mathcal{D} .) The coordinates of the center of flexure (x_f, y_f) are defined by:

$$x_f = \frac{1}{2(1+\sigma)I_x} \iint_{\mathcal{D}} [x \frac{\partial \varphi_2}{\partial y} - y \frac{\partial \varphi_2}{\partial x} - (1+\sigma)xy^2 + \sigma x^3] dA, \quad (2.4)$$

$$y_f = \frac{1}{2(1+\sigma)I_y} \iint_{\mathcal{D}} [x \frac{\partial \varphi_1}{\partial y} - y \frac{\partial \varphi_1}{\partial x} + (1+\sigma)x^2y - \sigma y^3] dA \quad (2.5)$$

where

$$I_x = \iint_{\mathcal{D}} y^2 dA \quad (2.6)$$

$$I_y = \iint_{\mathcal{D}} x^2 dA. \quad (2.7)$$

Thus

$$2(1+\sigma)I_x x_f = \iint_{\mathcal{D}} \left[\frac{\partial}{\partial y} (x\varphi_2) - \frac{\partial}{\partial x} (y\varphi_2) - (1+\sigma)xy^2 + \sigma x^3 \right] dA. \quad (2.8)$$

Using the divergence theorem we have

$$2(1+\sigma)I_x x_f = \oint_C \varphi_2 (xn_y - yn_x) ds - J \quad (2.9)$$

where

$$J = \iint_{\mathcal{D}} [(1+\sigma)xy^2 - \sigma x^3] dA. \quad (2.10)$$

Since J is a geometric quantity it can be computed explicitly.

We note now that the coefficient of φ_2 in the boundary integral in (2.9) is proportional to the boundary value of the normal derivative of the warping function φ in the pure torsion problem. In fact

$$\frac{\partial \varphi}{\partial n} = yn_x - xn_y \quad \text{on } C. \quad (2.11)$$

Thus if we let

$$\chi = 2(1+\sigma)I_x x_f + J \quad (2.12)$$

we find

$$\chi = - \oint_C \varphi_2 \frac{\partial \varphi}{\partial n} ds = -D(\varphi, \varphi_2) \quad (2.13)$$

where

$$D(\varphi, \varphi_2) = \iint_{\mathcal{D}} \left[\frac{\partial \varphi}{\partial x} \frac{\partial \varphi_2}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial \varphi_2}{\partial y} \right] dA. \quad (2.14)$$

The determination of bounds for x_f is thus equivalent to the determination of bounds for χ .

Note that if the warping function φ is known the value of x_f can be determined explicitly since

$$\chi = -\oint_C \varphi [(1+\sigma)y^2 - \sigma x^2] n_y ds. \quad (2.15)$$

If φ is not known we introduce two arbitrary harmonic functions u and u_2 and define

$$a_1 = -D(\varphi, u_2) = -\oint_C u_2 (y n_x - x n_y) ds, \quad (2.16)$$

$$a_2 = -D(u, \varphi_2) = -\oint_C u [(1+\sigma)y^2 - \sigma x^2] n_y ds, \quad (2.17)$$

$$a_3 = -D(u, u_2). \quad (2.18)$$

With u and u_2 prescribed we can determine a_1 , a_2 , and a_3 explicitly. We now form the expression

$$\begin{aligned} \chi - a_1 - a_2 + a_3 &= -D(\varphi - u, \varphi_2 - u_2) \\ &= -\oint_C (\varphi - u) \left\{ [(1+\sigma)y^2 - \sigma x^2] n_y - \frac{\partial u_2}{\partial n} \right\} ds. \end{aligned} \quad (2.19)$$

An application of Schwarz's inequality yields

$$(\chi - a_1 - a_2 + a_3)^2 \leq \oint_C (\varphi - u)^2 ds \oint_C \left\{ [(1+\sigma)y^2 - \sigma x^2] n_y - \frac{\partial u_2}{\partial n} \right\}^2 ds. \quad (2.20)$$

It remains now to approximate the first boundary integral on the right of (2.20) in terms of known quantities.

By adding an appropriate constant (if necessary) it is possible to choose $\varphi - u$ in such a way that

$$\oint_C (\varphi - u) ds = 0. \quad (2.21)$$

With $\varphi - u$ so chosen we shall estimate the first integral on the right of (2.20) in terms of the boundary integral of the square of its normal derivative. To this end we note that

$$\frac{\oint_C \left(\frac{\partial}{\partial s} (\varphi - u) \right)^2 ds}{\oint_C (\varphi - u)^2 ds} \geq \min_{\oint_C v ds = 0} \frac{\oint_C \left(\frac{\partial v}{\partial s} \right)^2 ds}{\oint_C v^2 ds} = \frac{4\pi^2}{L^2} \quad (2.22)$$

where L denotes the length of the perimeter of C , $\partial/\partial s$ denotes the tangential derivative on C , and v is any function continuous on C and satisfying the condition $\oint_C v ds = 0$. The quantity on the right of (2.22) is just the eigenvalue of the vibrating string equation, the differential equation which arises as the Euler equation for the minimum of the quotient $\left(\oint_C (\partial v / \partial s)^2 ds \right) / \oint_C v^2 ds$. It follows then that

$$\oint_C (\varphi - u)^2 ds \leq \frac{L^2}{4\pi^2} \oint_C \left[\frac{\partial}{\partial s} (\varphi - u) \right]^2 ds. \quad (2.23)$$

We seek now an inequality which relates the integral on the right-hand side of (2.23) to the boundary integral of the square of the normal derivative of $\varphi - u$. We introduce therefore,

an arbitrary continuous vector function with components f_k and consider the generalized Rellich identity (see [6, 8]). Letting $w = \varphi - u$, we have

$$0 = \iint_{\mathcal{D}} f_k \frac{\partial w}{\partial x_k} \Delta w dA = \oint_C f_k \frac{\partial w}{\partial x_k} \frac{\partial w}{\partial n} ds - \frac{1}{2} \oint_C f_k n_k |\text{grad } w|^2 ds - \iint_{\mathcal{D}} \frac{\partial f_k}{\partial x_l} \frac{\partial w}{\partial x_k} \frac{\partial w}{\partial x_l} dA + \frac{1}{2} \iint_{\mathcal{D}} \frac{\partial f_k}{\partial x_k} |\text{grad } w|^2 dA \quad (2.24)$$

where Δ denotes the Laplace operator ($\Delta = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$) and summation is to be carried out over the repeated indices $k, l = 1, 2$. The f_k are now assumed to be so chosen that

$$p = f_k n_k > 0 \text{ on } C. \quad (2.25)$$

A decomposition of the derivatives of u in the boundary integrals into normal and tangential components yields (see [8])

$$\oint_C p \left(\frac{\partial w}{\partial s} \right)^2 ds = \oint_C p \left(\frac{\partial w}{\partial n} \right)^2 ds + 2 \oint_C f_k \frac{\partial x_k}{\partial s} \frac{\partial w}{\partial s} \frac{\partial w}{\partial n} ds + \iint_{\mathcal{D}} \frac{\partial f_k}{\partial x_k} |\text{grad } w|^2 dA - 2 \iint_{\mathcal{D}} \frac{\partial f_k}{\partial x_l} \frac{\partial w}{\partial x_k} \frac{\partial w}{\partial x_l} dA. \quad (2.26)$$

Since the functions f_k are prescribed, we can find a constant γ such that (see [6, p. 555])

$$\iint_{\mathcal{D}} \frac{\partial f_k}{\partial x_k} |\text{grad } w|^2 dA - 2 \iint_{\mathcal{D}} \frac{\partial f_k}{\partial x_l} \frac{\partial w}{\partial x_k} \frac{\partial w}{\partial x_l} dA \leq \gamma D(w, w). \quad (2.27)$$

We make use also of the ordinary Green's identity

$$D(w, w) = \oint_C w \frac{\partial w}{\partial n} ds, \quad (2.28)$$

which yields by Schwarz's inequality

$$D(w, w) \leq \left(\oint_C w^2 ds \right)^{\frac{1}{2}} \left(\oint_C \left(\frac{\partial w}{\partial n} \right)^2 ds \right)^{\frac{1}{2}}. \quad (2.29)$$

In view of (2.23), eq (2.29) yields

$$D(w, w) \leq \frac{L}{2\pi} \left[\oint_C \left(\frac{\partial w}{\partial s} \right)^2 ds \right]^{\frac{1}{2}} \left[\oint_C \left(\frac{\partial w}{\partial n} \right)^2 ds \right]^{\frac{1}{2}}. \quad (2.30)$$

We now insert (2.27) and (2.30) back into (2.26), make use of the Schwarz inequality and obtain

$$\begin{aligned} \oint_C p \left(\frac{\partial w}{\partial s} \right)^2 ds \leq \oint_C p \left(\frac{\partial w}{\partial n} \right)^2 ds + 2 \left[\oint_C \frac{\left(f^k \frac{\partial x_k}{\partial s} \right)^2}{p} \left(\frac{\partial w}{\partial n} \right)^2 ds \right]^{\frac{1}{2}} \left[\oint_C p \left(\frac{\partial w}{\partial s} \right)^2 ds \right]^{\frac{1}{2}} \\ + \frac{\gamma L}{2\pi p_{\min}^{\frac{1}{2}}} \left[\oint_C p \left(\frac{\partial w}{\partial s} \right)^2 ds \right]^{\frac{1}{2}} \left[\oint_C \left(\frac{\partial w}{\partial n} \right)^2 ds \right]^{\frac{1}{2}}, \end{aligned} \quad (2.31)$$

where p_{\min} is the minimum value of p on C . This inequality yields

$$\begin{aligned} \left[\oint_C p \left(\frac{\partial w}{\partial s} \right)^2 ds \right]^{\frac{1}{2}} \leq \left[\oint_C \frac{\left(f^k \frac{\partial x_k}{\partial s} \right)^2}{p} \left(\frac{\partial w}{\partial n} \right)^2 ds \right]^{\frac{1}{2}} + \left[\frac{\gamma^2 L^2}{4\pi^2 p_{\min, C}} \oint_C \left(\frac{\partial w}{\partial n} \right)^2 ds \right]^{\frac{1}{2}} + \left\{ \left(\left[\oint_C \frac{\left(f^k \frac{\partial x_k}{\partial s} \right)^2}{p} \left(\frac{\partial w}{\partial n} \right)^2 ds \right)^{\frac{1}{2}} \right. \right. \\ \left. \left. + \left[\frac{\gamma^2 L^2}{4\pi^2 p_{\min, C}} \oint_C \left(\frac{\partial w}{\partial n} \right)^2 ds \right]^{\frac{1}{2}} \right)^2 + \oint_C p \left(\frac{\partial w}{\partial n} \right)^2 ds \right\}^{\frac{1}{2}}. \end{aligned} \quad (2.32)$$

From (2.32) it is thus always possible to obtain the inequality

$$\oint_C \left(\frac{\partial w}{\partial s} \right)^2 ds \leq \alpha \oint_C \left(\frac{\partial w}{\partial n} \right)^2 ds \quad (2.33)$$

where

$$\alpha p_{\min} = (\beta + [\beta^2 + p_{\max}]^{\frac{1}{2}})^2 \quad (2.34)$$

and

$$\beta = \left\{ \frac{\left| f^k \frac{\partial x_k}{\partial s} \right|}{p^{\frac{1}{2}}} \right\}_{\max} + \frac{\gamma L}{2\pi p_{\min}^{\frac{1}{2}}}. \quad (2.35)$$

If the boundary C is star-shaped with respect to some point (x_0, y_0) in \mathcal{D} , the problem is simplified considerably. The fact that the boundary is star-shaped with respect to (x_0, y_0) is equivalent to the condition that

$$q = (x - x_0)n_x + (y - y_0)n_y > 0 \quad (2.36)$$

at every point on C . Thus, in this case, (2.25) is satisfied if we let

$$f_1 = x - x_0, \quad f_2 = y - y_0. \quad (2.37)$$

The left-hand side of (2.27) then vanishes identically, and hence we take $\gamma \equiv 0$ in (2.31), (2.32), and (2.35). We find in this case instead of (2.33) (see [5]) that

$$\oint_C \left(\frac{\partial w}{\partial s} \right)^2 ds \leq q_{\min}^{-1} \left[1 + \left| \frac{\partial r}{\partial s} \right|_{\max} \right]^2 \oint_C \frac{r^2}{q} \left(\frac{\partial w}{\partial n} \right)^2 ds. \quad (2.38)$$

In the general case then, an inequality of the form

$$\oint_C \left(\frac{\partial(\varphi - u)}{\partial s} \right)^2 ds \leq \oint_C A \left[\frac{\partial}{\partial n} (\varphi - u) \right]^2 ds \quad (2.39)$$

is obtained, where A is a completely determined positive function. Inserting (2.39) and (2.23) into (2.20), we find

$$(x - a_1 - a_2 + a_3)^2 \leq \frac{L^2}{4\pi^2} \oint_C A \left[\frac{\partial u}{\partial n} - yn_x + xn_y \right]^2 ds \oint_C \left\{ \frac{\partial u_2}{\partial n} - [(1 + \sigma)y^2 - \sigma x^2]n_y \right\}^2 ds. \quad (2.40)$$

(It is apparent that the quantity A may be placed in the second integral rather than in the first if it is desirable.) We now use the Rayleigh-Ritz technique to make the two integrals on the right of (2.40) small.

Note that it is possible to derive close upper and lower bounds for x_f without approximating the flexure function at all. In fact, if we use instead of u a conjugate function v defined by²

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad (2.41)$$

we obtain (setting $u_2 \equiv 0$)

$$|x - a_2|^2 \leq \frac{L^2}{4\pi^2} \oint_C \left[\frac{\partial}{\partial s} (v - \frac{1}{2}r^2) \right]^2 ds \oint_C A \{ [(1 - \sigma)y^2 - \sigma x^2]n_y \}^2 ds. \quad (2.42)$$

In this case the A term has been put in the second integral since this expression is a purely geometrical quantity (independent of v) and can be computed explicitly.

² The function v is not to be confused with that used in (2.22).

In an entirely analogous way we find that

$$-2(1+\sigma)I_y y_f + \iint_{\mathcal{D}} [(1+\sigma)x^2 y - \sigma y^3] dA = D(\varphi, \varphi_1). \quad (2.43)$$

By introducing the two arbitrary harmonic functions u and u_1 and proceeding as before, we obtain upper and lower bounds for y_f . Note that if we choose $u_1, u_2 \equiv 0$ and merely approximate the warping function (or the conjugate function) on the right, then the same minimizing function u (or v) minimizes the error term in each case.

3. Bounds for the Center of Shear

We look now at the expression for the center of shear (x_s, y_s, l) as defined by Trefftz [10]. The coordinates x_s and y_s are given by

$$x_s = -\frac{1}{I_x} \iint_{\mathcal{D}} \varphi y dA \quad (3.1)$$

$$y_s = \frac{1}{I_y} \iint_{\mathcal{D}} \varphi x dA. \quad (3.2)$$

Equation (3.2) may be rewritten as

$$y_s = \frac{1}{6I_y} \iint_{\mathcal{D}} (\Delta x^3) \varphi dA. \quad (3.3)$$

By Green's formula we have

$$\begin{aligned} y_s &= \frac{1}{6I_y} \oint_C \left[3x^2 n_x \varphi - x^3 \frac{\partial \varphi}{\partial n} \right] ds \\ &= \frac{1}{6I_y} \oint_C [3x^2 n_x \varphi - x^3 (y n_x - x n_y)] ds. \end{aligned} \quad (3.4)$$

The divergence theorem then yields

$$y_s + \frac{1}{2I_y} \iint_{\mathcal{D}} x^2 y dA = \frac{1}{2I_y} \oint_C x^2 n_x \varphi ds. \quad (3.5)$$

We introduce the notation

$$\Phi = y_s + \frac{1}{2I_y} \iint_{\mathcal{D}} x^2 y dA, \quad (3.6)$$

and two harmonic functions h_1 and h_2 . Let

$$b_1 = \oint_C h_1 x^2 n_x ds, \quad (3.7)$$

$$b_2 = \oint_C \varphi \frac{\partial h_2}{\partial n} ds = \oint_C h_2 [y n_x - x n_y] ds, \quad (3.8)$$

$$b_3 = \oint_C h_1 \frac{\partial h_2}{\partial n} ds. \quad (3.9)$$

Combining (3.5) to (3.9) we obtain

$$\Phi - b_1 - b_2 + b_3 = \oint_C (\varphi - h_1)(x^2 n_x - h_2) ds. \quad (3.10)$$

Then by Schwarz's inequality

$$[\Phi - b_1 - b_2 + b_3]^2 \leq \oint_C (\varphi - h_1)^2 ds \oint_C \left(x^2 n_x - \frac{\partial h_2}{\partial n}\right)^2 ds. \quad (3.11)$$

Again it is desirable to choose $(\varphi - h_1)$ in such a way that

$$\oint_C (\varphi - h_1) ds = 0. \quad (3.12)$$

Using (2.23) and (2.39), we obtain finally

$$[\Phi - b_1 - b_2 + b_3]^2 \leq \frac{L^2}{4\pi^2} \oint_C A \left(\frac{\partial \varphi}{\partial n} - \frac{\partial h_1}{\partial n}\right)^2 ds \oint_C \left(x^2 n_x - \frac{\partial h_2}{\partial n}\right)^2 ds. \quad (3.13)$$

We insert the known boundary value for φ on the right-hand side and use the Rayleigh-Ritz technique to obtain close upper and lower bounds for y_s . Note that if the boundary value problem

$$\begin{aligned} \Delta h_2 &= 0 & \text{in } \mathcal{D} \\ \frac{\partial h_2}{\partial n} &= x^2 n_x & \text{on } C \end{aligned} \quad (3.14)$$

can be solved, it is then possible to determine y_s explicitly without solving either the torsion or the flexure problem.

In a similar way it is possible to obtain upper and lower bounds for the coordinate x_s .

4. References

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(Paper 64B2-28)