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# Kantorovich's Inequality [1]

Morris Newman

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An elementary proof with a generalization of an inequality of Kantorovich is given.

Let $A$ be a hermitian positive definite matrix with smallest eigenvalue $\alpha$ and largest eigenvalue $\beta$. Then Kantorovich's inequality [2] states that for all vectors $x$ of unit norm,

$$(Ax, x)(A^{-1}x, x) \leq \frac{1}{4}\left\{\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}+\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}\right\}^2. \qquad (1)$$

In this note we give an elementary proof of (1) which allows an easy generalization. The results obtained do not depend on the order of $A$ and this is accordingly left unspecified.

We assume only that $A$ is hermitian, and that the eigenvalues of $A$ are contained in the closed interval $m \leq t \leq M$.

Let $f(t)$, $g(t)$ be real functions such that

$$0 < f(t),\ g(t) < \infty \qquad \text{for} \quad m \leq t \leq M, \qquad (2)$$

$$f(t),\ g(t) \quad \text{are convex} \qquad \text{for} \quad m \leq t \leq M. \qquad (3)$$

Then (2) implies that the matrices $F=f(A)$, $G=g(A)$ are well-defined and are hermitian positive definite. We shall prove:

THEOREM. *Let $x$ be any vector of unit norm. Put*

$$K=(Fx,x)(Gx,x).$$

*Then for every positive $c$,*

$$2K^{\frac{1}{2}} \leq max\left(cf(m)+\frac{1}{c}g(m), cf(M)+\frac{1}{c}g(M)\right). \qquad (4)$$

*If in addition $f(M)-f(m)$ and $g(M)-g(m)$ are of opposite sign, then*

$$2K^{\frac{1}{2}} \leq rf(m)+\frac{1}{r}\,g(m), \qquad (5)$$

*where*

$$r=\left(-\frac{g(M)-g(m)}{f(M)-f(m)}\right)^{\frac{1}{2}}.$$

PROOF. Since $A$ is hermitian, $A$ is unitarily equivalent to a diagonal matrix. We may therefore assume without loss of generality that $A=(a_i\delta_{ij})$, where $m \leq a_i \leq M$. Then also $F=(f(a_i)\delta_{ij})$ and $G=(g(a_i)\delta_{ij})$. Suppose that the $i^{th}$ coordinate of $x$ is $x_i$ and set $t_i=|x_i|^2$, so that

$$t_i \geq 0, \qquad \sum t_i=1, \qquad (6)$$

$x$ being of unit norm. Then $K$ becomes

$$K=\sum f(a_i)t_i \sum g(a_i)t_i.$$

Let $c>0$ be arbitrary and rewrite $K$ as

$$K=\sum cf(a_i)t_i \sum \frac{1}{c}\,g(a_i)t_i.$$

Then

$$2K^{\frac{1}{2}}=2\left(\sum cf(a_i)t_i\right)^{\frac{1}{2}}\left(\sum \frac{1}{c}\,g(a_i)t_i\right)^{\frac{1}{2}}$$

$$\leq \sum cf(a_i)t_i+\sum \frac{1}{c}\,g(a_i)t_i$$

$$\leq \max\left(cf(a_i)+\frac{1}{c}\,g(a_i)\right),$$

by (6).

We now consider the function

$$u=cf(t)+\frac{1}{c}g(t), \qquad m \leq t \leq M.$$

Because of assumptions (2) and (3) and the fact that $c$ is positive, we have that $u$ is convex and positive for $m \leq t \leq M$. This implies that

$$\max_{m \leq t \leq M} u=\max\ (u(m), u(M))$$

which proves (4).

If in addition $f(M)-f(m)$ and $g(M)-g(m)$ are of opposite sign, the choice $c=r$ is permissible and makes

$$rf(m)+\frac{1}{r}g(m)=rf(M)\frac{1}{r}g(M),$$

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[2] L. V. Kantorovich, Functional analysis and applied mathematics, Uspekhi Math. Nauk **3**, 89 (1948).

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which gives (5). This completes the proof of the theorem.

Kantorovich's inequality (1) is the case $f(t)=t$, $g(t)=\frac{1}{t}$ for which $r=(mM)^{-1/2}$ (which is permissible when $A$ is positive definite).

Ky Fan in a written communication to the author points out that the theorem can be generalized as follows:

*Let $A$ be a hermitian matrix of order $n$ with all its eigenvalues contained in the closed interval $\alpha\leq t\leq\beta$. Let $x_1, x_2, \ldots, x_k$ be vectors in unitary $n$-space such that*

$$\sum_{i=1}^{k}||x_i||^2=1.$$

*Let $f_1, f_2, \ldots, f_m$ be positive convex functions for $\alpha\leq t\leq\beta$.*

*If $c_1, c_2, \ldots, c_m$ are positive numbers satisfying $c_1c_2 \ldots c_m=1$, then*

$$\prod_{i=1}^{m}\left\{\sum_{j=1}^{k}(f_i(A)x_j, x_j)\right\}^{1/m} \leq\frac{1}{m}\max\left\{\sum_{i=1}^{m}c_if_i(\alpha), \sum_{i=1}^{m}c_if_i(\beta)\right\}.$$

Furthermore if $A$ is positive definite and $\alpha>0$, Kantorovich's inequality can be replaced by

$$\left\{\sum_{i=1}^{k}(Ax_i, x_i)\right\}\left\{\sum_{i=1}^{k}(A^{-1}x_i, x_i)\right\}\leq\frac{1}{4}\left\{\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}+\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}\right\}^2.$$

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