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Zeros of Certain Polynomials

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Let P be a real parameter. It is proved that all roots of $z^{n+1}-z^n+P=0$ lie in the open unit disk, if and only if $0<P<2\sin\pi/(4n+2)$.

1. *Introduction.* Let n be a fixed non-negative integer. The following problem will be solved:

PROBLEM. *Find the set S_n of all values of the real parameter P for which all roots of*

$$z^{n+1}-z^n+P=0 \tag{1}$$

lie in the open unit disk $|z|<1$.

In view of the substantial literature relating the coefficients of a polynomial to the locations of its zeros,[1] one would expect that some standard algorithm could be applied to so specific a problem. This seems to be only partly true; the appropriate algorithm is that given by the Schur-Cohen Criterion,[2] which yields only an implicit characterization of S_n as the solution-set of a system of $n+1$ polynomial inequalities in P. It appears nontrivial to derive from this an *explicit* characterization of S_n, and so we give instead an elementary self-contained solution. The result is the

THEOREM. $S_n=\{P|\ 0<P<2\sin\pi/(4n+2)\}$.

The problem arose in connection with the generating function [3]

$$F(s)=\frac{p^n s^n(1-ps)}{1-s+(1-p)p^n s^{n+1}}$$

of the recurrence times for runs of n successes in a sequence of Bernoulli trials with "success probability" p. Rigorous justification of the usual probability theory manipulations of the power series for $F(s)$ is easy if $F(s)$ has no singularities for $|s|\leq 1$, and this is in fact true and is equivalent (upon setting $z=1/s$) to the assertion that all roots of $z^{n+1}-z^n+(1-p)p^n=0$ lie in $|z|<1$ for $0<p<1$. The last statement shows that S_n *includes* the interval $\{P|\ 0<P<n^n/(n+1)^{n+1}\}$, and it was natural to inquire whether this expression gave S_n exactly. It follows from our theorem that S_n is larger than this interval for all $n>0$, and is approximately $\tfrac{1}{2}\pi e\approx 4.27$ times as long for large n.

[1] Marden, The geometry of the zeros of a polynomial in a complex variable, Am. Math. Soc. Math. Survey No. 3 (New York, N.Y., 1949).
[2] Ibid, p. 152.
[3] Feller, Probability theory and its applications, p. 265 (John Wiley & Sons, Inc. ,New York, N.Y., 1950

2. *Solution.* In this section x, r, and θ denote real variables obeying $r>0$, $0\leq\theta<2\pi$. Since the theorem stated above is obviously true for $n=0$, it is assumed that $n>0$ in what follows. It is convenient to define

$$\begin{aligned}
f(\theta)&=\sin^n n\theta\,\sin\theta/\sin^{n+1}(n+1)\theta,\\
A&=\{x^n-x^{n+1}|\ |x|\geq 1\},\\
B&=\{\theta|\ 0<\theta<\pi,\quad \sin n\theta\geq\sin(n+1)\theta>0\},\\
C&=\{f(\theta)|\ \theta\text{ in }B\}.
\end{aligned}$$

We can omit the easy proof of

LEMMA 1. $A=\{P|\ P\leq 0\ \textit{or}\ P\geq 2\}$ *if n is even,*
$A=\{P|\ P\leq 0\}$ *if n is odd.*

LEMMA 2. $C=\{P|\ P\geq 2\sin\pi/(4n+2)\}$.

PROOF. (a) From the formula

$$f'(\theta)=\frac{\sin^{n-1}n\theta[(n\sin\theta-\sin n\theta)^2+2n\sin\theta\sin n\theta(1-\cos(n+1)\theta)]}{\sin^{n+2}(n+1)\theta}$$

we conclude that $f(\theta)$ is increasing on each subinterval of B.

(b) Suppose $\sin(n+1)\theta^*=0$ at a left endpoint θ^* of some maximal subinterval of B. If $\sin n\theta^*\neq 0$, then $\sin\theta^*\neq 0$ and so $f(\theta^*+)=(+\infty)$, contradicting (a). If $\sin n\theta^*=0$, then $\theta^*=0$, contradicting the requirement that

$$\sin n(\theta^*+\delta)\geq\sin(n+1)(\theta^*+\delta)$$

for all sufficiently small $\delta>0$. Thus the supposition is untenable.

(c) We next apply the identity

$$\sin n\theta-\sin(n+1)\theta=-2\sin\frac{1}{2}\theta\cos\frac{1}{2}(2n+1)\theta$$

to obtain

$$B=\{\theta|\ 0<\theta<\pi,\quad \sin(n+1)\theta>0,\quad \cos\frac{1}{2}(2n+1)\theta\leq 0\}.$$

Consider now any left endpoint θ^* of a maximal subinterval of B. According to (b), we must have

$$\sin(n\theta^*)=\sin(n+1)\theta^*, \tag{2}$$

so that

$$f(\theta^*)=\sin\theta^*/\sin(n+1)\theta^*. \tag{3}$$

In fact, the conditions determining such an endpoint are, in addition to (2) and $0\leq\theta^*<\pi$, that

$$\sin(n+1)\theta^*>0 \text{ and } \sin n(\theta^*+\delta)>\sin(n+1)(\theta^*+\delta),$$

for all sufficiently small $\delta>0$. Equivalently, θ^* is such an endpoint if and only if

$$\cos\frac{1}{2}(2n+1)\theta^*=0, \tag{4}$$

$$\cos\frac{1}{2}(2n+1)(\theta^*+\delta)<0, \qquad (\delta \text{ as above}) \tag{5}$$

$$\sin(n+1)\theta^*>0, \tag{6}$$

$$0\leq\theta^*<\pi. \tag{7}$$

The points obeying (4), (5), and (7) are precisely the points

$$\theta_j=(4j+1)\pi/(2n+1) \qquad (0\leq 2j<2n);$$

these points also satisfy (6), since

$$\sin(n+1)\theta_j=\cos(4j+1)\pi/(4n+2) \qquad (4j+1<4n+2), \tag{8}$$

and so the θ_j's are precisely the left endpoints of the maximal subintervals of B. From (3) and (8) we have

$$f(\theta_j)=2\sin(4j+1)\pi/(4n+2);$$

together with (a), this shows that $f(\theta)$ reaches its minimum on B at $\theta=\theta_0=\pi/(2n+1)$, this minimum being

$$\min C=2\sin\pi/(4n+2).$$

(d) Finally, we seek the right endpoint $\bar{\theta}$ of that maximal subinterval of B of which θ_0 is the left endpoint. After θ_0, $\cos\frac{1}{2}(2n+1)\theta$ first changes sign at $3\pi/(2n+1)$, but $\sin(n+1)\theta$ changes sign earlier, at $\pi/(n+1)$. Thus $\bar{\theta}=\pi/(n+1)$, and so $f(\bar{\theta}-)=(+\infty)$. This fact, together with (a) and the results of (c), completes the proof.

Our final lemma gives the motivation for lemmas 1 and 2; the three lemmas together immediately imply the theorem stated in the introduction.

LEMMA 3. *S_n is the complement of $A\cup C$.*

PROOF. (a) Clearly eq (1) has a *real* root outside the disk $|z|<1$ if and only if P is in A.

(b) Next we observe that $z=r\exp(i\theta)$ is a nonreal root of eq (1) if and only if

$$r^n\sin n\theta-r^{n+1}\sin(n+1)\theta=0, \tag{9}$$

$$\sin\theta\neq 0, \tag{10}$$

$$r^n\cos n\theta-r^{n+1}\cos(n+1)\theta=P. \tag{11}$$

Now (9) and (10) are equivalent to

$$\sin n\theta/\sin(n+1)\theta=r, \tag{12}$$

and (11) and (12) are equivalent to

$$f(\theta)=P. \tag{13}$$

Thus eq (1) has a nonreal root outside the disk $|z|<1$ if and only if P lies in the set

$$\{f(\theta)|\ \sin n\theta/\sin(n+1)\theta\geq 1\},$$

which (by considering the change $\theta\rightarrow 2\pi-\theta$) is readily seen to be identical with the set

$$\{f(\theta)|\ \sin n\theta\geq\sin(n+1)\theta>0\}.$$

(c) By (a) and (b), S_n is the complement of

$$A\cup\{f(\theta)|\ \sin n\theta\geq\sin(n+1)\theta>0\}.$$

By lemma 1, this union is identical with

$$A\cup\{f(\theta)|\ \sin n\theta\geq\sin(n+1)\theta>0,\quad f(\theta)>0\},$$

which (by the form of $f(\theta)$) is just $A\cup C$.

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