Journal of Research of the National Bureau of Standards Vol. 62, No. 3, March 1959 Research Paper 2935

# On a Theorem of M. Riesz [1]

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The structure of the set of skew-symmetric, orthogonal matrices of order 4 with elements in the real field is studied. The first and second regular representations of the quaternions are used to represent these matrices and known properties of the quaternions are exploited to obtain characterizations of the skew-symmetric matrices. In the second part of this paper, the underlying structure of the geometry is changed from the Euclidean to the Lorentzian. It is then shown that there are no skew-symmetric, orthogonal (in the Lorentzian sense) matrices of order 4.

## 1. Introduction

In a series of lectures [6] [3] recently delivered at the University of Maryland (Sept. 1957 to Feb. 1958) dealing with Clifford algebras, M. Riesz proved the following

THEOREM. *If $A$ and $B$ are real skew-symmetric, orthogonal linear transformations on $E_4$ (a real four-dimensional vector space on which there is imposed a Euclidean metric), then $[A, B]=AB-BA$ is a multiple of a real skew-symmetric, orthogonal transformation on $E_4$.*

The main purpose of this paper is to exhibit a proof of this theorem which does not depend on the structure of the Clifford algebra, but only on the properties of the matrices associated with these transformations. Moreover, it will be shown that if the underlying geometry is changed from the Euclidean metric to the Lorentzian metric, then the theorem is vacuously true since no real skew-symmetric, orthogonal (in the Lorentzian sense) matrices of order 4 exist. The main tools used herein are real quaternions and their representations by $4\times4$ matrices. Incidentally, a method is given for constructing all skew-symmetric orthogonal matrices of order 4.

## 2. Regular Representations

Let $\mathfrak{A}$ be an algebra over the (commutative) field $\mathfrak{F}$ and let $e_1, e_2, \ldots, e_n$ be a basis of $\mathfrak{A}$. Multiplication of two elements in $A$ is completely determined by the multiplication of the basal elements. Since $e_1, e_2, \ldots, e_n$ forms a basis for $\mathfrak{A}$,

$$e_ie_j=\sum_{k=1}^{n}c_{ijk}e_k \qquad c_{ijk}\epsilon\mathfrak{F}, \tag{1}$$

and it is well known that associativity in $\mathfrak{A}$ is equivalent to the $n^4$ equations.

$$\sum_{k=1}^{n}c_{ijk}c_{klm}=\sum_{k=1}^{n}c_{ikm}c_{jlk} \qquad (i, j, l, m=1, 2, \ldots, n). \tag{2}$$

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[3] Figures in brackets indicate the literature references at the end of this paper.

If we define the $n\times n$ matrices $R_i=(c_{isr})$ and $S_i=(c_{ris})$ (where $r$ denotes the row index and $s$ denotes the column index), then, by successively setting (i) $m=r$, $l=s$; (ii) $i=r$, $m=s$; and (iii) $j=r$, $m=s$, we have

$$\text{(i)}\ R_iR_j=\sum_{k=1}^{n}c_{ijk}R_k \qquad \text{(ii)}\ S_iS_j=\sum_{k=1}^{n}c_{ijk}S_k \tag{3}$$

and

$$\text{(iii)}\ R_i'S_j=S_jR_i', \tag{4}$$

where $R_i'$ is the transpose of $R_i$.

Every element $a$ of $\mathfrak{A}$ can be written uniquely in the form

$$a=\alpha_1e_1+\alpha_2e_2+\ \ldots\ +\alpha_ne_n \qquad \alpha_i\epsilon\mathfrak{F}. \tag{5}$$

If we define

$$R(a)=\alpha_1R_1+\alpha_2R_2+\ \ldots\ +\alpha_nR_n,\ S(a)=\alpha_1S_1+\alpha_2S_2+\ \ldots\ +\alpha_nS_n \tag{6}$$

then the mappings $a\rightarrow R(a)$ and $a\rightarrow S(a)$ of $\mathfrak{A}$ into the algebras of matrices, $\mathfrak{R}$ and $\mathfrak{S}$, are homomorphisms, called *the first and second regular representations* of $\mathfrak{A}$. If we further assume that $\mathfrak{A}$ has an identity, then these mappings are isomorphisms.

We shall be interested in the particular case where $\mathfrak{A}$ is the algebra $\mathfrak{Q}$ of real quaternions, that is, the algebra over the real field with basis elements $e_0$, $e_1$, $e_2$, $e_3$ satisfying the following conditions:

$$\begin{aligned}&\text{(i)}\quad e_0 \text{ is the identity element of } \mathfrak{Q},\\ &\text{(ii)}\ \ e_1e_2=-e_2e_1=e_3,\ e_2e_3=-e_3e_2=e_1,\ e_3e_1=-e_1e_3=e_2,\\ &\text{(iii)}\ e_1^2=e_2^2=e_3^2=-e_0.\end{aligned} \tag{7}$$

Then $R_0=S_0=I$ (the identity matrix) and

$$R_1=\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}\quad R_2=\begin{bmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{bmatrix}\quad R_3=\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}. \tag{8}$$

$$S_1=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}\quad S_2=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{bmatrix}\quad S_3=\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{bmatrix}. \tag{9}$$

Clearly, $R_1$, $R_2$, $R_3$, $S_1$, $S_2$, $S_3$ are each skew-symmetric and hence $\mathfrak{Q}_0$, the additive group of all linear combinations of the $R$'s and $S$'s with real coefficients, is contained in $\mathfrak{Q}_1$, the additive group of all $4\times4$ real skew-symmetric matrices. However,

$$Q=\begin{bmatrix}0&a&b&c\\-a&0&d&e\\-b&-d&0&f\\-c&-e&-f&0\end{bmatrix}=\tfrac{1}{2}[a(S_1-R_1)+b(S_2-R_2)+c(S_3-R_3)-d(S_3+R_3)\\ +e(S_2+R_2)-f(S_1+R_1)], \tag{10}$$

that is, every skew-symmetric matrix $Q$ can be expressed as a linear combination of $R_1$, $R_2$, $R_3$, $S_1$, $S_2$, $S_3$. Thus, $\mathfrak{Q}_1\subseteq\mathfrak{Q}_0$ and hence $\mathfrak{Q}_1=\mathfrak{Q}_0$. Since $\mathfrak{Q}_1$ has dimension six (when considered as a real vector space), it follows that $R_1$, $R_2$, $R_3$, $S_1$, $S_2$, $S_3$ are linearly independent and form a basis for $Q_1$. Hence, we have shown that *every* $4\times 4$ *skew-symmetric matrix* $Q$ *can be expressed as* $R(a)+S(b)$ *for a unique pair of quaternions,* $a=\alpha_1 e_1+\alpha_2 e_2+\alpha_3 e_3$ and $b=\beta_1 e_1+\beta_2 e_2+\beta_3 e_3$. A quaternion whose $e_0$-coefficient is zero is called a *pure quaternion.*

## 3. Skew-Symmetric, Orthogonal Matrices

We are particularly interested in those matrices $Q$ of $\mathfrak{Q}_1$ which are also orthogonal, and we shall be able to characterize them in terms of the pure quaternions $a$ and $b$.

Since each of the $R$'s and $S$'s is skew-symmetric, (4) may be replaced by

$$R(a)\ S(b)=S(b)\ R(a). \tag{11}$$

We define the *norm,* $N(a)$, of the pure quaternion $a=\alpha_1 e_1+\alpha_2 e_2+\alpha_3 e_3$ to be $\alpha_1^2+\alpha_2^2+\alpha_3^2$. Thus, $N(a)\geq 0$ and $N(a)=0$ if and only if $a=0$. Furthermore, $N(\alpha a)=\alpha^2 N(a)$ for every real number $\alpha$. It follows from (7) that $a^2=-N(a)e_0$ and hence

$$R(a)^2=S(a)^2=-N(a)\ I. \tag{12}$$

Furthermore, if $a\neq 0$, then $c=a/\sqrt{N(a)}$ has norm 1 and is called the unit quaternion in the direction of $a$.

We shall now derive some useful properties of the matrices $R(a)$ and $S(a)$ by means of the mapping $\sigma$ of the real vector $\vec{a}=(\alpha_1,\ \alpha_2,\ \alpha_3)$ into the quaternion $a=\sigma(\vec{a})$. This mapping is clearly an additive isomorphism. Furthermore,

$$\begin{aligned} ab=\sigma(\vec{a})\sigma(\vec{b})=&-(\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3)e_0+(\alpha_1\beta_2-\alpha_2\beta_1)e_3\\ &+(\alpha_3\beta_1-\alpha_1\beta_3)e_2+(\alpha_2\beta_3-\alpha_3\beta_2)e_1=-(\vec{a}\cdot\vec{b})e_0+\sigma(\vec{a}*\vec{b}) \end{aligned} \tag{13}$$

where $\vec{a}\cdot\vec{b}$ is the scalar or dot product of $\vec{a}$ and $\vec{b}$ and $\vec{a}*\vec{b}$ is the vector product of $\vec{a}$ and $\vec{b}$. Since the mappings a $\rightarrow R(a)$ and a $\rightarrow S(a)$ are isomorphisms, we have

$$R(a)R(b)=-(\vec{a}\cdot\vec{b})I+R[\sigma(\vec{a}*\vec{b})], \qquad S(a)S(b)=-(\vec{a}\cdot\vec{b})I+S[\sigma(\vec{a}*\vec{b})]. \tag{14}$$

It is well known that $\vec{a}*\vec{b}=-\vec{b}*\vec{a}$ and since $\sigma(-a)=-\sigma(a)$, it follows immediately from (14) that

LEMMA 1. *If* a *and* b *are orthogonal pure quaternions (or equivalently, if* $\vec{a}\cdot\vec{b}=0$), *then* $R(a)R(b)=-R(b)R(a)$. *Similarly, if a and b are orthogonal pure quaternions, then*

$$S(a)S(b)=-S(b)S(a).$$

We define the *conjugate,* $\bar{Q}$, of the skew-symmetric matrix $Q=R(a)+S(b)$ to be $R(a)-S(b)$, and the *norm,* $N(Q)$, of $Q$ to be $N(b)-N(a)$. It follows from (7) and (11) that

$$\begin{aligned} Q\bar{Q}&=[R(a)+S(b)]\ [R(a)-S(b)]=[R(a)]^2-[S(b)]^2=\bar{Q}Q\\ &=[-N(a)+N(b)]\ I=N(Q)\ I=N(\bar{Q})\ I. \end{aligned} \tag{15}$$

Now, let $\vec{c}$ be any vector which is orthogonal to $\vec{b}$. Then $c$ is orthogonal to $b$ and it follows from (11) and Lemma 1 that

$$S(c)QS(c)^{-1}=S(c)[R(a)+S(b)]S(c)^{-1}=R(a)-S(b)=\bar{Q}. \tag{16}$$

Thus, $Q$ and $\bar{Q}$ are similar matrices and consequently have the same rank. Furthermore,

$\det(Q)=\det(\bar{Q})$ and by taking the determinants of the terms in (15) we have $\det(Q)\det(\bar{Q})=\det[N(Q)\,I]=N(Q)^4$. Since the determinant of every real skew-symmetric matrix is non-negative, we have proven

Lemma 2. Det $(Q)=$ Det $(\bar{Q})=N(Q)^2$.

We are now in a position to prove the main theorem of this paper. Riesz' Theorem, stated at the beginning of this paper, follows immediately as a corollary.

Theorem 1. *The skew-symmetric matrix $Q$ is orthogonal if and only if one of the following conditions is satisfied:*

(i) $N(Q)=-1$ *and* $b=0$, (*and thus* $Q=R(a)$).
(ii) $N(Q)=+1$ *and* $a=0$, (*and thus* ($Q=S(b)$)).

Proof. Let $Q$ be orthogonal. Then $Q^{-1}=-Q$ and since, by (15), $\bar{Q}=N(Q)\,Q^{-1}$, it follows that

$$\bar{Q}=-N(Q)\,Q. \tag{17}$$

Taking the norm of each side of this equation, we have

$$N(Q)=N(\bar{Q})=N[-N(Q)]=[-N(Q)]^2\,N(Q)=[N(Q)]^3. \tag{18}$$

Since $Q$ is nonsingular, $N(Q)\neq 0$ and hence $N(Q)=\pm 1$.

*Case* 1. $N(Q)=+1$. Substituting this value in (17), we have

$$0=Q+\bar{Q}=[R(a)+S(b)]+[R(a)-S(b)]=2\,R(a). \tag{19}$$

Thus, $R(a)=0$ and hence $a=0$.

*Case* 2. $N(Q)=-1$. Again, by (17), we have

$$0=Q-\bar{Q}=[R(a)+S(b)]-[R(a)-S(b)]=2\,S(b). \tag{20}$$

As before, $b=0$.

Conversely, if $b=0$ then $Q=\bar{Q}$ and it follows from (15) that if $N(Q)=-1$ then $Q\bar{Q}=Q^2=-I$. Thus $Q$ is orthogonal. Similarly, if $a=0$ then $Q=-\bar{Q}$ and it follows from $N(Q)=+1$ that $Q\bar{Q}=Q^2=-I$. Again, $Q$ is orthogonal.

To prove Riesz' Theorem, let $P$ and $Q$ be two skew-symmetric, orthogonal matrices. Clearly, we can assume that $P=R(a)$ for some pure unit quaternion $a$. If $Q=S(b)$ for some $b$, then, by (11), $[P,Q]=PQ-QP=0$. On the other hand, if $Q=R(b)$ for some $b$, then, using (14), we have

$$[P,\,Q]=R(a)R(b)-R(b)R(a)=2\,R[\sigma(\vec{a}*\vec{b})]=R[\sigma(2(a*b))]. \tag{21}$$

If $\vec{a}=\pm\vec{b}$, then $\vec{a}*\vec{b}=0$ and $[P,\,Q]=0$. If $\vec{a}\neq\pm\vec{b}$, then $\vec{a}*\vec{b}\neq 0$. Since $a$ and $b$ each have norm 1, $N[\sigma(2(\vec{a}*\vec{b}))]=4N[\sigma(\vec{a}*\vec{b})]=4\delta$ where $1\geq\delta>0$. If we now set $\vec{c}=2(\vec{a}*\vec{b})/\sqrt{4\delta}=(\vec{a}*\vec{b})/\sqrt{\delta}$, then $R[\sigma(2(\vec{a}*\vec{b}))]=4\delta R(c)$. Since $c$ is a unit quaternion, $R(c)$ is a skew-symmetric, orthogonal matrix and the theorem is proved.

Corollary 1. *Let $P$, $Q$ be skew-symmetric, orthogonal matrices such that $P\neq\pm Q$. Then $P$ and $Q$ commute if and only if $N(P)N(Q)=-1$.*

Corollary 2. *Every skew-symmetric matrix $Q$ can be written as $\alpha R+\beta S$ for unique $\alpha$, $\beta$, $R$, $S$, where $\alpha$, $\beta$ are real numbers and $R$, $S$ are skew-symmetric orthogonal matrices.*

## 4. The Lorentzian Metric

The Lorentzian metric on a real four-dimensional vector space $V$ is defined by the matrix

$$J=\begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{22}$$

We shall denote this space by $\mathfrak{L}_4$.

The norm, $N(x)$, of a vector $x=(x_1, x_2, x_3, x_4)$ is given by $x'Jx=x_1^2+x_2^2+x_3^2-x_4^2$. Clearly, $N(x)$ may now take on all real values. If $N(x)=0$, it does not follow that $x=0$. A vector satisfying this equation is said to be isotropic or lightlike. We define a matrix $A$ to be orthogonal with respect to $J$ (or Lorentzian-orthogonal) if $A'JA=J$. An alternate definition for the Lorentzian-orthogonality of a matrix is that it be the matrix of a norm-preserving linear transformation on $\mathfrak{L}_4$.

By analogy with the Euclidean case, we define a matrix $Q$ to be skew-symmetric if it satisfies the equation $x'QJx=0$ for every vector $x$ in $\mathfrak{L}_4$. Then, $Q$ is skew-symmetric in the Lorentzian sense if and only if $QJ$ is skew-symmetric in the Euclidean sense. Furthermore,

Lemma 3. *$A$ is both orthogonal and skew-symmetric (in the Lorentzian sense) if and only if $A=QJ$ for some skew-symmetric (in the Euclidean sense) matrix $Q$ and*

$$(JQ)^2=(QJ)^2=-I. \tag{23}$$

Thus, we have reduced the problem of Lorentzian skew-symmetric, orthogonal matrices to a corresponding problem involving Euclidean skew-symmetric matrices satisfying (23).

Theorem 2. *There does not exist a matrix $Q$ satisfying* (23).

Proof. A simple calculation shows that

$$JR_iJ=\pm S_i, \qquad JS_iJ=\pm R_i, \qquad (+\text{ when } i=3,\ -\text{ when } i=1,2). \tag{24}$$

Suppose there exists a skew-symmetric matrix $Q$ satisfying (23). Then we can set $Q=R(a)+S(b)$ for some $a$ and $b$ and $JQJ=R(c)+S(d)$. By (24), $N(c)=N(b)$ and $N(d)=N(a)$. Hence,

$$N(Q)=N(\overline{Q})=-N(JQJ). \tag{25}$$

Since $(QJ)^2=-I$, we have $(\det Q)^2\ (\det J)^2=(\det Q)^2=(\det(-I))=+1$. Combining this with Lemma 2, we have

$$[N(Q)]^2=\det Q=\det \overline{Q}=+1. \tag{26}$$

Also, it follows from (15) that $Q^{-1}=(1/N(Q))\overline{Q}$ and from (23) that $Q^{-1}=-JQJ$. Equating these two expressions and taking the norm of each term yields

$$N(Q)=-N(JQJ)=-N(Q^{-1})=-N[(1/N(Q))\overline{Q}]=-[1/N(Q)]^2\,N(\overline{Q})=-1/N(Q). \tag{27}$$

Thus, $N(Q)^2=-1$, which contradicts (26).

## 5. Conclusion

The corresponding problem in the two-dimensional case is trivial since there only exist two skew-symmetric orthogonal (Euclidean) matrices, namely, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. These clearly commute since one is the negative of the other. On the other hand, if the quadratic form of the metric is given by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ then it can be shown that no skew-symmetric Lorentzian, orthogonal matrices exist.

Little is known about the structure of higher dimensional skew-symmetric orthogonal matrices. The methods of this paper are isolated and it is not expected that they can be used in a discussion of higher dimensions.

## 6. References


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WASHINGTON, August 21, 1958.