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Exponential Integral $\int_1^\infty e^{-xt}t^{-n}dt$ for Large Values of n [1]

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An asymptotic expansion is given which is well suited for numerical computation when n is large and x arbitrary positive.

1. Let

$$E_n(x)=\int_1^\infty e^{-xt}\,t^{-n}dt; \qquad x>0; \qquad n=1, 2, 3, \ldots . \tag{1}$$

By means of four integrations by parts, G. Blanch[2] has found the approximation

$$E_n(x)\approx\frac{e^{-x}}{x+n}\left[1+\frac{n}{(x+n)^2}+\frac{n(n-2x)}{(x+n)^4}+\frac{n(6x^2-8nx+n^2)}{(x+n)^6}\right]. \tag{2}$$

She also gives an integral representation for the error. Formula (2) has proved very efficient for computing $E_n(x)$ for large values of n in the whole range $x>0$. In what follows, the complete expansion is given, as well as error estimates.

Denote by $h_k(u)$ the polynomial (of degree k–1 if $k>0$), defined recursively by

$$h_{k+1}(u)=(1-2ku)h_k(u)+u(1+u)h_k'(u) \quad (k=0, 1, 2, \ldots), \; h_0(u)=1. \tag{3}$$

Let

$$H_k(u)=\frac{h_k(u)}{(1+u)^{2k}}, \tag{4}$$

and let α_k, β_k be lower and upper bounds, respectively, for $H_k(u)$ in the interval $u\geq 0$:

$$\alpha_k\leq H_k(u)\leq\beta_k \qquad (u\geq 0). \tag{5}$$

Then it will be proved that

$$E_n(x)=\frac{e^{-x}}{x+n}\left[\sum_{\kappa=0}^{k-1}H_\kappa\left(\frac{x}{n}\right)n^{-\kappa}+R_k(x,n)\right], \tag{6}$$

$$\alpha_k n^{-k}\leq R_k(x,n)\leq\beta_k\left(1+\frac{1}{x+n-1}\right)n^{-k}. \tag{7}$$

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[2] G. Blanch, An asymptotic expansion for $E_n(x)=\int_1^\infty (e^{-xu}/u^n)du$, NBS Applied Math. Series **37**, 61 (1954).

For reference, the first eight polynomials $h_k(u)$ and corresponding values of α_k, β_k are listed:[3]

$$h_0(u)=h_1(u)=1$$

$$h_2(u)=1-2u$$

$$h_3(u)=1-8u+6u^2$$

$$h_4(u)=1-22u+58u^2-24u^3$$

$$h_5(u)=1-52u+328u^2-444u^3+120u^4$$

$$h_6(u)=1-114u+1452u^2-4400u^3+3708u^4-720u^5$$

$$h_7(u)=1-240u+5610u^2-32120u^3+58140u^4-33984u^5+5040u^6$$

$$\alpha_1=0,\quad \alpha_2=-0.07,\quad \alpha_3=-0.18,\quad \alpha_4=-0.36,\quad \alpha_5=-0.60,\quad \alpha_6=-0.94,\quad \alpha_7=-1.4$$

$$\beta_1=\beta_2=\beta_3=\beta_4=\beta_5=\beta_6=1,\quad \beta_7=1.8.$$

2. Consider, more generally, the integral

$$I=\int_a^b e^{f(t)}dt, \tag{8}$$

where $f(t)$ is a real function defined on the finite or infinite interval (a, b). It is assumed that $f(t)$ has derivatives of any order in (a, b) and that $f'(t)\neq 0$. Following van der Corput and Franklin,[4] we define the sequence $g_k(t)$ by

$$g_0(t)=\frac{1}{f'(t)},\qquad g_{k+1}(t)=\frac{g_k'(t)}{f'(t)}\qquad (k=0,1,2,\ldots). \tag{9}$$

Setting

$$I_k=\int_a^b g_k(t)f'(t)e^{f(t)}dt,$$

clearly $I_0=I$, and integration by parts yields

$$I_k=[g_k(t)e^{f(t)}]_a^b-\int_a^b g_k'(t)e^{f(t)}dt=v_k-I_{k+1}, \tag{10}$$

where

$$v_k=g_k(b)e^{f(b)}-g_k(a)e^{f(a)}. \tag{11}$$

Hence,

$$I=v_0-v_1+v_2-v_3+\ldots+(-1)^{k-1}v_{k-1}+(-1)^k I_k, \tag{12}$$

$$I_k=\int_a^b g_{k-1}'(t)e^{f(t)}dt. \tag{13}$$

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[3] The author is indebted to Mrs. L. K. Cherwinski and Mrs. B. H. Walter for the calculation of the α_k and β_k.

[4] J. G. van der Corput and Joel Franklin, Approximation of integrals by integration by parts, Nederl. Akad. Wetensch. Proc. Ser. **[A]** **54** 213–219 (1951).

In case of an infinite interval (a, b) it has to be assumed that the values (11) exist for all k. Equation (10) then shows that the existence of the integral I_k implies the existence of I_{k+1}.

3. The integral in (8) is equal to $E_n(x)$ if

$$f(t) = -(xt + n \ln t), \qquad a = 1, \qquad b = \infty.$$

A short computation shows that with this definition of $f(t)$, the sequence $g_k(t)$ in (9) is equal to

$$g_k(t) = \frac{(-1)^{k+1} t}{xt+n} \frac{h_k(u)}{(1+u)^{2k}} n^{-k} = \frac{(-1)^{k+1} t}{xt+n} H_k(u) n^{-k}, \quad u = \frac{xt}{n},$$

where the $h_k(u)$ are the polynomials defined in (3) and $H_k(u)$ the rational functions (4). For the quantities v_k in (11), one obtains

$$v_k = \frac{(-1)^k e^{-x}}{x+n} H_k\left(\frac{x}{n}\right) n^{-k}.$$

Furthermore,

$$g'_{k-1}(t) = (-1)^k H_k(u) n^{-k}, \qquad u = \frac{xt}{n}.$$

Hence, from (12) and (13),

$$E_n(x) = \frac{e^{-x}}{x+n} \left[\sum_{\kappa=0}^{k-1} H_\kappa\left(\frac{x}{n}\right) n^{-\kappa} \right] + n^{-k} \int_1^\infty H_k\left(\frac{xt}{n}\right) e^{-xt} t^{-n} dt. \tag{14}$$

By (5)

$$\alpha_k E_n(x) \le \int_1^\infty H_k\left(\frac{xt}{n}\right) e^{-xt} t^{-n} dt \le \beta_k E_n(x),$$

and using the well-known inequality,[5]

$$\frac{1}{x+n} \le e^x E_n(x) \le \frac{1}{x+n-1} \qquad (x \ge 0),$$

one gets

$$\alpha_k \frac{e^{-x}}{x+n} \le \int_1^\infty H_k\left(\frac{xt}{n}\right) e^{-xt} t^{-n} dt \le \beta_k \frac{e^{-x}}{x+n-1}.$$

From this and (14) the formulas (6), (7) follow immediately.

It may be observed that the result (6), (7) holds also for nonintegral values of n with $n > 1$.

[5] E. Hopf, Mathematical problems of radiative equilibrium, Cambridge Tracts in Mathematics and Mathematical Physics, No. 31, p. 26 (Cambridge University Press, 1934).

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