Journal of Research of the National Bureau of Standards Vol. 62, No. 3, March 1959 Research Paper 2937

Heat Transfer in Laminar Flow Through a Tube[1]

Milton Abramowitz,[2] William F. Cahill,[3] and Clarence Wade, Jr.[3]

The problem of heat transfer due to laminar flow of a viscous fluid in a channel is studied under the assumption that there is a parabolic distribution of velocity. The effect of axial temperature changes are considered and the solution is based on the simpler situation where axial effects are discussed. The solution, obtained by the method of least squares, is represented in terms of a set of nonorthogonal characteristic functions. These functions and the corresponding characteristic values are determined by numerical integration employing the Runge-Kutta procedure. Finally, asymptotic developments are obtained which are useful in the limiting cases.

1. Statement of Problem

The present paper is concerned with the steady state problem of heat transfer in a tube. The fluid moves with a prescribed velocity profile, which is parabolic in the present case. Thus the end effect is not considered in the velocity profile. The heat transfer is small and is supposed not to influence the fluid motion. The wall of the tube is assumed to be kept at a fixed temperature, θ_1, while the temperature of the fluid entering the tube is fixed to be θ_0. The velocity distribution in the tube whose radius is unity is defined by

$$v_x = 2v_m(1\text{-}r^2),\ 0 \leq r \leq 1, \tag{1.1}$$

where r is the radial distance from the center and v_x is taken along the x-axis which is the axis of symmetry of the tube and v_m is the average value of v_x. The equation of heat transfer under these conditions is

$$v_x \frac{\partial \theta}{\partial x} = K \left\{ \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial x^2} \right\} \tag{1.2}$$

with the boundary conditions

$$\begin{aligned} \theta &= \theta_0 \text{ for } x=0 \\ \theta &\text{ finite for } x=\infty \\ \theta &= \theta_1 \text{ for } r=1. \end{aligned} \tag{1.3}$$

This problem has been considered [1][4] under the assumption that temperature changes in the x-direction will be negligible so that the term $\partial^2\theta/\partial x^2$ may be omitted. However, recent studies [2,3], which have included the effects of this term have exhibited solutions based on approximating the temperature distribution by a series of Bessel functions. In the present paper we propose to obtain a solution by the method of least squares. The boundary value problem will be solved by numerical integration of the differential equation employing the Runge-Kutta method. This procedure avoids the use of special functions but employs the basic concepts necessary to solve the boundary value problem. The results obtained indicate that the present technique may be employed in other problems of a similar nature.

Let us first put $\lambda = 2v_m/K$ and $\lambda\xi = x$ so that (1.2) becomes

$$(1-r^2) \frac{\partial \theta}{\partial \xi} = \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{1}{\lambda^2} \frac{\partial^2 \theta}{\partial \xi^2} \tag{1.4}$$

while the boundary conditions (1.3) remain unchanged except that x is replaced by ξ. We see from (1.4) that when λ increases the contribution from the term $\partial^2\theta/\partial\xi^2$ will decrease. By analogy with the solution for $\lambda = \infty$ it is assumed that

$$\frac{\theta - \theta_1}{\theta_0 - \theta_1} = \sum_{n=1}^{\infty} A_n \exp(z_n \xi) y(r, z_n), \tag{1.5}$$

where the constants z_n are the eigenvalues and the functions $y(r,z_n)$ are the eigenfunctions of the boundary value problem

$$y'' + \frac{1}{r} y' + \left\{ \frac{z_n^2}{\lambda^2} - z_n(1-r^2) \right\} y = 0, \tag{1.6}$$

$$y(r,\lambda,z) = 0 \text{ for } r=1. \tag{1.7}$$

[1] The preparation of this paper was sponsored in part by the Office of Naval Research, United States Navy.
[2] Deceased.
[3] Present address, Theoretical Division, National Aeronautics and Space Administration.
[4] Figures in brackets indicate the literature references at the end of this paper.

TABLE 1. *The eigenvalues—z_n*

n	λ[a]=0	λ=1	λ=10	λ=100	λ=1000	$\lambda=\infty$
1	2.405	2.644369	6.744051	7.306909	7.313523	7.313587
2	5.520	5.187345	30.76791	44.32333	44.60653	44.60947
3	8.654	8.323452	59.50344	111.9926	113.9009	113.9210
4	11.792	11.46137	89.47665	208.3608	215.1669	215.2447
5	14.931	14.60050	119.9727	330.8685	348.3678	348.564
6	18.071	17.74037	150.7501	476.6471	513.4475	513.890
7	21.212	20.88073	181.6979	642.7513	710.3849	711.217
8	24.352	24.02119	212.7574	826.6381	939.7287	940.551
9	27.49	27.16301	243.9199	1024.875	1196.191	1201.87
10	30.63	30.30613	275.1170	1224.774	1464.309	1495.20
11	33.78	33.44743				
12	36.92	36.50123				
13	40.06	40.06675				

[a] For $\lambda=0$, z_n are the zeros of the Bessel function $J_0(z)$; the values for $\lambda=\infty$ were taken from reference [4].

In order to have axial symmetry it is required that $y(r,\lambda,z)$ be an even function of r. Further, to determine the temperature distribution θ in (1.5), we determine solutions of (1.6) which constitute a nonorthogonal set of characteristic functions $y(r,\lambda,z_n)$. It is noted that since z occurs quadratically in (1.6) there are two sets of characteristic values. It turns out that one set is all positive while the other is all negative. However, in evaluating the temperature distribution, we wish to have a solution which is finite for ξ infinite so we discard the positive set of characteristic values. In order to satisfy the condition $\theta=\theta_0$ at $\xi=0$ we formally assume the solution in the form

$$f(r)\sim\sum_{n=1}^{\infty}A_n y(r,z_n), \tag{1.8}$$

where $f(r)=1$ in the present problem. When $\lambda=\infty$, (1.6) reduces to the Graetz-Nusselt equation and the functions $\sqrt{r(1-r^2)},\ y(r,z_n)$ constitute an orthogonal set and the coefficients are determined accordingly. In the present case where λ is finite the orthogonality property no longer holds. However, we may still determine the coefficients in the sense of least squares. Consequently, we require

$$\int_0^1 r(1-r^2)\left\{f(r)-\sum_{n=1}^{\infty}A_n y(r,z_n)\right\}^2 dr=\text{minimum}. \tag{1.9}$$

The weight function $r(1-r^2)$ has been introduced so that the results shall be consistent with the Graetz-Nusselt ($\lambda=\infty$) case. It should be noted that the solution of (1.6) may be expressed in terms of the confluent hypergeometric function. However, to obtain the required results it would still be necessary to consider the problem of evaluating these functions. Instead, we propose to solve the differential eq (1.6) directly using numerical integration. Differentiating (1.9) with respect to A_n we get the infinite system of equations

$$\alpha A=\beta, \tag{1.10}$$

where

$$\alpha=[\alpha_{mn}]=\left[\int_0^1 r(1-r^2)\ y(r,z_m)\ y(r,z_n)dr\right],$$

$$\beta=[\beta_m]=\left[\int_0^1 r(1-r^2)\ y(r,z_m)\ f(r)dr\right],$$

$$A=[A_n],m,n=1,2,\ .\ .\ .\ . \tag{1.11}$$

2. Determination of z_n and $y(r,z_n)$

The solution of the infinite system is dependent on the determination of the characteristic functions $y(r,z_n)$ and the characteristic values z_n. The power series solution of (1.6) is

$$y(r,z_n)=\sum_{k=0}^{\infty}b_{2k}r^{2k}, \tag{2.1}$$

where

$$b_0=1,b_2=-\frac{1}{4}\left(\frac{z^2}{\lambda^2}-z\right),4k^2b_{2k}+(z^2/\lambda^2-z)b_{2k-2}+zb_{2k-4}=0. \tag{2.2}$$

This representation was used to expedite the numerical integration of (1.6). The Runge-Kutta method [5] was employed starting at $r=0.5$ with the value of $y(0.5,z_n)$ obtained from (2.1) with trial values of z_n. The results are given in table 1 (see figure 1 for the characteristic functions corresponding to $\lambda=10$).

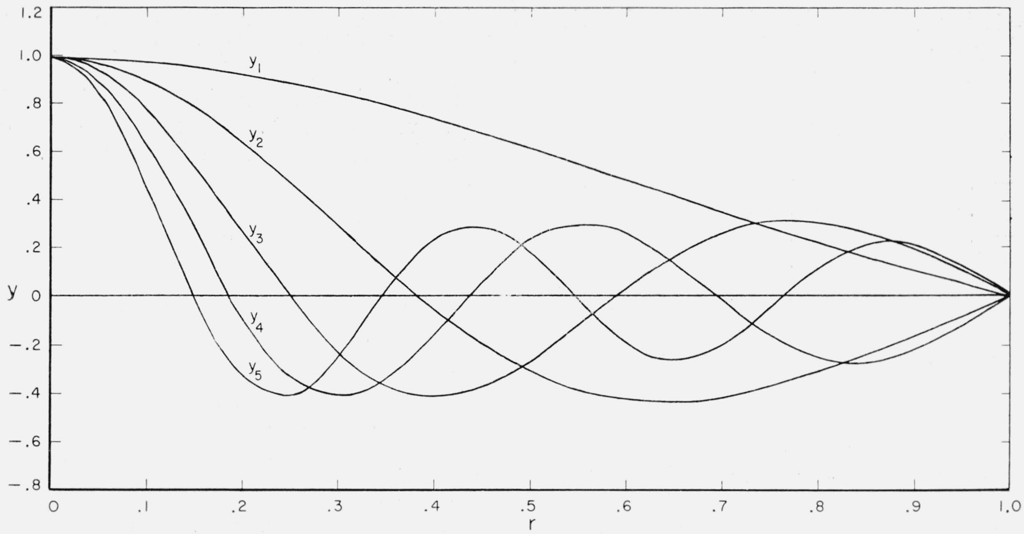

FIGURE 1.

3. Calculation of the Coefficients A_j

The integrals in (1.11) necessary for the determination of the solution A of (1.10) were evaluated with Simpson's rule using the characteristic functions $y(r, z_n)$ calculated previously. We give the results obtained for 3 cases. Since the matrices are symmetric we omit the elements above the main diagonal.

$$\lambda=1, \quad m=n=5$$

$$10^8\alpha=(10^8\alpha_{mn})=\begin{bmatrix} 1018886 & \text{----} & \text{----} & \text{----} & \text{----} \\ 1338862 & 3951938 & \text{----} & \text{----} & \text{----} \\ -242794 & 849201 & 2472921 & \text{----} & \text{----} \\ 77045 & -176352 & 606464 & 1804961 & \text{----} \\ -32588 & 63431 & -133686 & 469689 & 1422473 \end{bmatrix}$$

$$\lambda=10, \quad m=n=5$$

$$10^7\alpha=(10^7\alpha_{mn})=\begin{bmatrix} 947286 & \text{----} & \text{----} & \text{----} & \text{----} \\ 42284 & 382270 & \text{----} & \text{----} & \text{----} \\ -18966 & 48645 & 239335 & \text{----} & \text{----} \\ 8546 & -16907 & 43600 & 175401 & \text{----} \\ -4079 & 7130 & -13791 & 37374 & 138567 \end{bmatrix}$$

$$\lambda=1000, \quad m=n=8$$

$$10^7\alpha=(10^7\alpha_{mn})=\begin{bmatrix} 939335 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{----} \\ 5 & 375196 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{----} \\ -12 & 15 & 234420 & \text{---} & \text{---} & \text{---} & \text{---} & \text{----} \\ 6 & -19 & 26 & 170471 & \text{---} & \text{---} & \text{---} & \text{----} \\ -11 & 11 & -28 & 39 & 133932 & \text{---} & \text{---} & \text{----} \\ 6 & -17 & 19 & -43 & 66 & 110280 & \text{---} & \text{----} \\ 8 & 3 & -32 & 33 & -35 & 38 & 93796 & \text{----} \\ 52 & -14 & -69 & 120 & 157 & 158 & -186 & 81323 \end{bmatrix}$$

Thus for $\lambda=1$, if we start with the approximations $z_2^{(1)}=-5$ and $z_2^{(2)}=-6$ to z_2 the following successive approximations are obtained:

j	$z_2^{(j)}$	$y(1,z_2^{(j)})$
1	-6.0	0.2362897
2	-5.0	$-.6633393\times10^{-1}$
3	-5.219196	$.1110549\times10^{-1}$
4	-5.187761	$.1455328\times10^{-3}$
5	-5.187344	$-.4580215\times10^{-6}$
6	-5.187345	$-.2379238\times10^{-7}$

The improved value of z_n was obtained by linear interpolation as follows:

$$\frac{z_2^{(j+1)}-z_2^{(j)}}{z_2^{(j-1)}-z_2^{(j)}}=\frac{y(1,z_2)-y(1,z_2^{(j)})}{y(1,z_2^{(j-1)})-y(1,z_2^{(j)})};y(1,z_2)=0.$$

The efficiency of the method was dependent on the choice of the initial values of z. Once some of the eigenvalues were known, relatively good starting values were easily found by extrapolation. In performing the integration (1.6) was written in the form

$$y'=\frac{1}{r}u$$

$$u'=-r\{(z^2/\lambda^2-z)+zr^2\}y$$

and the interval $\Delta r=0.002$ was ultimately employed.

It is noted that as λ increases, the matrix α tends to a diagonal matrix, i.e., the set of characteristic functions $y(r, z_n)$ tends to an orthogonal set with respect to the weight factor $r(1-r^2)$. Finally, examination of the integrals which are the elements of α and β indicates that since $f(r)=1$, elements of β follow as a simple byproduct of the computation of the elements of α.

Solving the system $\alpha A=\beta$ we get the values of A_n in table 2.

With the known values of A_n we are able to test the efficiency of the least square approximation. The results are summarized in table 3.

We conclude from the above that the method described yields satisfactory results for practical purposes. In conclusion we shall give asymptotic representations useful in the regions of small and large values of λ. These serve to confirm the numerical results obtained previously.

TABLE 3. $\sum_{n=1}^{\infty} A_n y(r,z_n)\sim f(r)=1$

r	$\lambda=1$		$\lambda=10$		$\lambda=100$		$\lambda=1{,}000$	
	$n=5$	$n=8$	$n=5$	$n=8$	$n=5$	$n=8$	$n=5$	$n=8$
0	0.981	0.918	1.151	0.905	1.188	0.845	1.188	0.864
0.1	.937	1.011	1.047	1.009	1.037	1.042	1.033	1.100
.2	.909	1.004	0.935	0.997	0.921	0.972	0.922	0.961
.3	1.004	0.985	1.001	.992	1.042	1.033	1.186	1.178
.4	1.100	1.023	1.057	1.010	1.040	0.965	1.036	0.963
.5	1.053	0.971	0.969	0.972	0.920	1.023	0.917	1.036
.6	0.965	1.032	.942	1.039	.997	0.997	1.004	0.974
.7	.990	0.969	1.076	0.952	1.139	.964	1.142	.987
.8	.984	1.024	1.081	1.052	0.992	1.121	0.983	1.126
.9	.651	1.004	0.682	0.964	.539	0.758	.528	0.718
1.0	0	0	0	0	0	0	0	0

4. Solution of (1.6) for Small Values of λ

To obtain a solution of (1.6) in terms of Bessel functions valid for small values of λ we put $\alpha\lambda=z$ and obtain

$$y''+r^{-1}y'+[\alpha^2-\alpha\lambda(1-r^2)]y=0. \tag{4.1}$$

Now if $\lambda\rightarrow0$, (4.1) becomes $y''+r^{-1}y'+\alpha^2y=0$. The solution of the limiting problem is $J_0(\alpha_s r)$ where J_0 is the Bessel function of the first kind of order zero and $\alpha_s=j_s$ are the zeros of J_0.

TABLE 2. *Values of A_n*

n	$\lambda=1$		$\lambda=10$		$\lambda=100$		$\lambda=1{,}000$	
	$n=5$	$n=8$	$n=5$	$n=8$	$n=5$	$n=8$	$n=5$	$n=8$
1	1.5779	1.6003	1.5370	1.5423	1.478413	1.479273	1.476457	1.476472
2	−0.9629	1.0527	−0.9563	−0.9778	−0.8126151	−0.813778	−0.806192	−0.806222
3	.5217	0.8172	.7559	.8062	.599728	.606701	.588885	.588874
4	−.4547	−.6658	−.5894	−.6865	−.488371	−.501968	−.475988	−.475937
5	.2995	.5489	.4038	.5805	.411076	.419506	.405103	.405024
6		−.4462		−.4797		−.390254		−.355635
7		.3450		.3765		.348450		.318681
8		−.2289		−.2565		−.303294		−.287201

Thus, if we assume

$$y(r)=y_0+\lambda y_1+\lambda^2 y_2+\ .\ .\ .$$

$$\alpha_s(\lambda)=\alpha_{0s}+\lambda\alpha_{1s}+\lambda^2\alpha_{2s}+\ .\ .\ . \tag{4.2}$$

and substitute in (4.1) we find

$$y_1(r)=\left(\frac{1}{6}-\frac{1}{3j_s^2}\right)r\ J_1(j_s r)-\frac{1}{2}\frac{r^2}{j_s^2}\ J_2(j_s r)+\frac{r^3}{6}\ J_3(j_s r)$$

$$\alpha_{1s}=\frac{1}{3}+\frac{1}{3j_s^2},\ \alpha_{2s}=\frac{1}{15j_s}-\frac{7}{90j_s^3}+\frac{59}{90j_s^5}. \tag{4.3}$$

Omitting further approximations we show the comparison considering the quantities α_{1s} and α_{2s}.

λ	$-z_1(\lambda)$	*Approx.*	$-z_2(\lambda)$	*Approx.*
1	2. 04437	2. 04417	5. 187345	5. 18755
2	3. 48773	3. 44802	9. 75388	9. 7570
5	5. 68968	6. 0350	20. 4943	20. 462
10	6. 744051	15. 2	30. 7679	32. 52

5. Solution of (1.6) for Large Values of λ

Let us assume that

$$y(r)=y_0+\lambda^{-2}y_1+\lambda^{-4}y_2+\ .\ .\ .$$

$$z_s(\lambda)=-z_{0s}+z_{1s}\lambda^{-2}+z_{2s}\lambda^{-4}+\ .\ .\ ..$$

We then find that $y_0(r)$ is the solution of the Graetz-Nusselt equation and z_{0s} are the corresponding eigenvalues which have been listed in table 1. Substituting in (1.6) we obtain a series of characteristic values from which we find

s	z_{1s}	z_{2s}
1	-66.9260	1218. 6
2	-2897.8	374×10^3

A comparison of the results taking into account $z_1{}^s$ and z_{2s} follows:

λ	$-z_1(\lambda)$		$-z_2(\lambda)$	
	True value	*Approx.*	*True value*	*Approx.*
10	6. 744	6. 766	30. 8	53. 0
100	7. 307	7. 307	44. 3	44. 323
1000	7. 3135	7. 313	44. 6065	44. 606
∞	7. 313587		44. 60947	

The authors acknowledge suggestions and assistance of Miss Irene A. Stegun in the asymptotic developments obtained.

6. References

[1] M. Jakob, Heat transfer, vol. 1 (John Wiley & Sons, Inc., New York, N.Y., 1949), p. 451.

[2] K. Millsaps and K. Pohlhausen, Heat transfer to Hagen-Poiseuille flows, Proceedings of the Conference on Differential Equations (University of Maryland, College Park, Md., 1955).

[3] S. N. Singh, Heat transfer by laminar flow in cylindrical tube (private communication).

[4] M. Abramowitz, On the solution of the differential equation occurring in the problem of heat convection in laminar flow through a tube, J. Math. Phys. **32**, 184 (1953).

[5] William E. Milne, Numerical solution of differential equations, Ch. 5 (John Wiley & Sons, Inc., New York, N.Y., 1953).

[6] H. A. Lauwerier, The use of confluent hypergeometric functions in mathematical physics and the solution of the eigenvalue problem, Appl. Sci. Research, [A] **2**, 184 (1951).

Washington, June 2, 1958.