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Propagation of Very-Low-Frequency Pulses to Great Distances[1]

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A theoretical study is presented for the propagation of electromagnetic pulses at very low frequencies to large distances. The space between the earth and the ionosphere is represented as a wave guide with sharply bounded and concentric spherical boundaries. The concept of phase and group velocity and its application to the present problem is discussed in some detail. The influence of the propagation medium on the shape of the envelope of a quasi-monochromatic pulse is also considered. Using an alternative approach, the response of an impulsive source is also calculated and is shown to be a damped oscillatory function of time with a quasi-half-period varying in a predictable manner with distance of travel in agreement with the observations of Norinder and Hepburn.

1. Introduction

In recent years there has been a renewed interest in the propagation of VLF (very-low-frequency) radio waves. The wave-guide model has been remarkably successful for the prediction of the field strength as a function of distance and frequency. In most investigations, both experimental and theoretical, the source varies essentially in a sinusoidal manner with time.[2] It is of great interest, however, to have a knowledge of the transient characteristics of a pulse and the way it is influenced by the propagation medium. For example, if the received field strength of a lightning stroke is recorded as a function of time, one may ask how this waveform is related to the source current, distance of travel, electrical constants of the ground, and height and electrical properties of the ionosphere. It is the purpose of this paper to attempt an answer to this question on the basis of a theoretical study.

A general analysis is presented in section 4, applying and extending the classical concepts [3,4] such as phase and group velocity for propagation in dispersive media. Such an approach is particularly suitable for a quasi-monochromatic source wherein the spectral components are centered in a narrow band about some central or reference frequency. In section 5, an alternative method is described that is applicable to broad-band sources which contain many spectral components. Extensive numerical results are presented for the parameters characterizing the shape of the radiated pulses. Finally, some reference is made to published experimental data concerning the waveforms of radio atmospherics.

2. Quasi-Monochromatic Pulse

In the present study, the earth is represented by a homogeneous sphere of radius a, and the ionosphere is idealized as a sharply bounded concentric reflecting layer at height h. The source of the field is always assumed to be equivalent to a vertical (radially oriented) electric dipole. At large ranges, the field can then be represented as a sum of waveguide modes.

The vertical electric field of any one mode at a range d can be written in the form (see section 7.1)

$$e(\omega)=A_n e^{-\frac{2\pi d}{h}u_n} e^{-i\frac{2\pi d}{\lambda}s_n} e^{i\omega t}, \tag{1}$$

where A_n is a slowly varying function of frequency ω, u_n is a measure of the attenuation of the

[1] The results in this paper were reported at the commission 4 and 6 sessions of the International Scientific Radio Union General Assembly held in Boulder, Colo., August 22 to September 5, 1957.

[2] See VLF issue of Proc. Inst. Radio Engrs. **45**, June 1957, for papers by Budden, Pierce, Wait, and Watt.

[3] J. A. Stratton, Electromagnetic theory, p. 292 (McGraw-Hill Publishing Co., Inc., New York, N. Y., 1941).

[4] IA. L. Al'pert, V. L. Ginzburg, and E. L. Feinberg, Radiowave propagation, pt. II, p. 364 (Moscow, 1953).

mode per unit distance, and s_n is a dimensionless phase factor. The phase velocity is then given by

$$v_p = c/s_n, \tag{2}$$

where c is the velocity of light. On the other hand, the group velocity of a mode is given by

$$v_g = \frac{c}{\dfrac{d}{d\omega}(\omega s_n)}, \tag{3}$$

which can be written in terms of $H(=h/\lambda)$ as follows:

$$v_g = \frac{c}{s_n + H\dfrac{ds_n}{dH}}. \tag{4}$$

The group velocity, so defined, is a measure of the velocity of the envelope of the quasi-monochromatic pulse or group. For example, if the propagation medium were nondispersive, the signal might be represented by the real part of $E_0(t) = A(t)e^{i\omega_0 t}$, where ω_0 is the carrier frequency, and $A(t)$ is the shape of the envelope as a function of time. In the simple case of a broken sinusoid,

$$A(t) = 1 \quad \text{for} \quad -\frac{T}{2} < t < \frac{T}{2},$$

$$A(t) = 0 \quad \text{for} \quad t > T/2 \quad \text{and} \quad t < -\frac{T}{2}.$$

The group velocity concept is valid only when the frequency spectrum,

$$G(\omega) = \int_{-\infty}^{+\infty} E_0(t)e^{-i\omega t}dt, \tag{5}$$

differs substantially from zero only in a small frequency band near the signal carrier-frequency ω_0. In the present case

$$G(\omega) = 2\frac{\sin(\omega - \omega_0)T/2}{(\omega - \omega_0)}. \tag{6}$$

The spectral width is given by $\Delta\omega \sim 1/(2T)$ and this must be small compared to ω_0. In other words, the period of the envelope must be long compared to the period of the carrier frequency.

Now when the pulse propagates through the medium by a distance d, the field $E_0(t)$ of a mode becomes transformed to $E(t)$ whence

$$E(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(\omega)e^{i[\omega t - \phi(\omega)]}e^{-\alpha(\omega)}G(\omega)d\omega, \tag{7}$$

where

$$\phi(\omega) = \frac{\omega}{c}s_n d = s_n\frac{2\pi d}{\lambda}$$

and

$$\alpha(\omega) = \frac{2\pi d}{h}u_n.$$

The coefficients $\alpha(\omega)$ and $\phi(\omega)$ are phase and attenuation factors of a mode as a function of frequency for the section of path of length d.

Utilizing well-known properties of Fourier integrals,[1 2] the spectral representations of the signals $E_0(t)$ and $E(t)$ can be written

$$E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(t')\exp\{i(\omega_0 - \omega)t' + \omega t\}d\omega dt', \tag{8}$$

and

$$E(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}A(t')\exp\{i(\omega_0-\omega)t'+\omega t-\phi(\omega)\}G(\omega)e^{-\alpha(\omega)}d\omega dt'. \tag{9}$$

The phase function $\phi(\omega)$ is now written in a Taylor expansion,

$$\phi(\omega)=\phi(\omega_0)+\Omega\phi'(\omega_0)+\frac{\Omega^2}{2}\phi''(\omega_0)+\ldots, \tag{10}$$

where $\Omega=\omega-\omega_0$. Neglecting terms containing Ω^3, etc., it follows that

$$E(t)=\exp[i(\omega_0 t-\phi(\omega_0))]G(\omega_0)e^{-\alpha(\omega_0)}\times$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}A(t')\exp\left\{i\Omega[t-t'-\phi'(\omega_0)]-i\frac{\phi''(\omega_0)}{2}\Omega^2\right\}dt'd\Omega, \tag{11}$$

where the slowly varying factors have been taken outside the integrand. Introducing a new variable z defined by

$$z=\sqrt{\phi''(\omega_0)/\pi}\left[\Omega+\frac{t'-t+\phi'(\omega_0)}{\phi''(\omega_0)}\right], \tag{12}$$

in place of Ω it follows that

$$E(t)=\frac{1-i}{2\pi}\exp\left\{i\left[\omega_0 t-\phi(\omega_0)\right]\right\}G(\omega_0)e^{-\alpha(\omega_0)}\times$$

$$\int_{-\infty}^{+\infty}\left[\frac{\pi}{\phi''(\omega_0)}\right]^{1/2}A(t')\exp\left[i\frac{(t'-t+\phi'(\omega_0))^2}{2\phi''(\omega_0)}\right]dt', \tag{13}$$

since

$$\int_{-\infty}^{+\infty}\exp\left[-i\frac{\pi}{2}z^2\right]dz=1-i. \tag{14}$$

After a further change of variable, via

$$\frac{t'-t+\phi'(\omega_0)}{\phi''(\omega_0)}=\sqrt{\pi}u, \tag{15}$$

we finally arrive at

$$E(t)=\frac{1-i}{2}\exp\{i[\omega_0 t-\phi(\omega_0)]\}G(\omega_0)e^{-\alpha(\omega_0)}\times\int_{-\infty}^{+\infty}A[t-\phi'(\omega_0)-\sqrt{\pi\phi''(\omega_0)}u]\exp\left(i\frac{\pi}{2}u^2\right)du. \tag{16}$$

In the above, it has been assumed that $\phi''(\omega_0)>0$; if $\phi''(\omega_0)<0$, then by changing the sign in the substitution of variables one would have $\sqrt{\pi|\phi''(\omega_0)|}$ in place of $\sqrt{\pi\phi''(\omega_0)}$.

Now if $\phi''(\omega_0)=0$ and noting that

$$\int_{-\infty}^{+\infty}\exp\left(i\frac{\pi}{2}u^2\right)du=1+i, \tag{17}$$

it follows that

$$E(t)=A[t-\phi'(\omega_0)]e^{-\alpha(\omega_0)}G(\omega_0)e^{i[\omega_0 t-\phi(\omega_0)]}. \tag{18}$$

To this approximation, the pulse has not changed shape, although the pulse amplitude has been modified, the phase has changed by an amount $\phi(\omega_0)$, and the pulse as a whole is delayed by the "group time" $\phi'(\omega_0)$. The effect of a nonzero value of $\sqrt{\pi\phi''(\omega_0)}$ is to modify the pulse shape (see footnote 4). To evaluate this phenomenon, the envelope must be specified and the integration indicated in eq (16) must be carried out. For example, in the case of the broken sinusoid extending from $-T/2$ to $T/2$, the integral in eq (16) can readily be transformed to a Fresnel integral,

$$E(t)=\frac{1-i}{2}\exp\{i[\omega_0 t-\phi(\omega_0)]\}\int_{u_1}^{u_2}\exp\left[i\frac{\pi}{2}u^2\right]du, \tag{19}$$

where

$$u_1=\frac{-\theta}{\sqrt{\pi\phi''(\omega_0)}}; \qquad u_2=\frac{T-\theta}{\sqrt{\pi\phi''(\omega_0)}},$$

and $\theta=(T/2)+t-\phi'(\omega_0)$. θ is the time measured from the instant $-T/2+\phi'(\omega_0)$. In the case of negligible dispersion of the pulse shape, $\phi''(\omega_0)$ must be sufficiently small to enable u_1 and u_2 to be replaced by $-\infty$ and $+\infty$, and then

$$\begin{aligned} E(t) &\cong \exp\{i[\omega_0 t-\phi(\omega_0)]\} \quad \text{for} \quad 0<\theta<T \\ &\cong 0 \quad \text{for} \quad \theta<0 \quad \text{and} \quad \theta>T. \end{aligned} \tag{20}$$

To illustrate the influence of dispersion, it will be assumed that the signal duration T is large compared to $\sqrt{\pi\phi''(\omega_0)}$. Then the form of the leading edge of the signal is determined by

$$|E(t)|\cong\frac{1}{\sqrt{2}}\left|\int_{u_1}^{\infty}\exp\left[i\frac{\pi}{2}u^2\right]du\right|. \tag{21}$$

The envelope $|E(t)|$ is plotted versus the time parameter $\theta/\sqrt{\pi\phi''(\omega_0)}$ in figure 1, a. It is now convenient to define a build-up parameter t_b, which is the time measured from $\theta=0$ for the envelope $|E(t)|$ to approach within 5 percent of unity. From inspection of figure 1 it is seen that

$$t_b\cong 4\sqrt{\pi\phi''(\omega_0)}\ \text{sec}.$$

For numerical presentation it is convenient to introduce a dimensionless build-up parameter T_b defined by

$$t_b\cong\frac{4\sqrt{dh/2}}{c}T_b, \tag{22}$$

where

$$T_b=\left|2\frac{\partial s_n}{\partial H}+H\frac{\partial^2 s_n}{\partial H^2}\right|^{1/2}.$$

The build-up parameter T_b and the build-up time t_b are related by eq (22). The chart in figure 1, b, is to facilitate the conversion for a typical height of 70 km, and for various ranges d in kilometers.

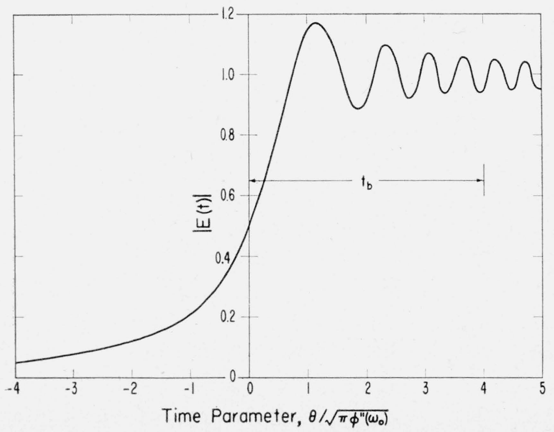

FIGURE 1a. *Leading edge of signal in dispersive media.*

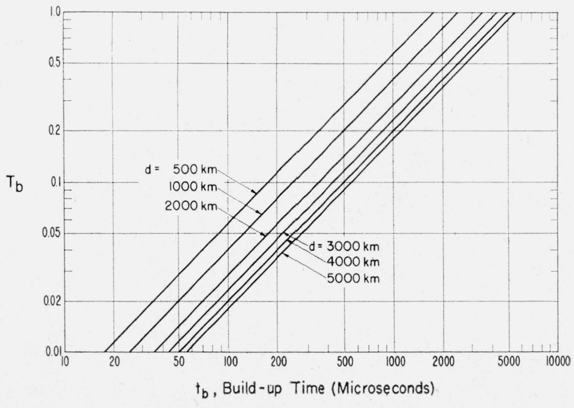

FIGURE 1b. *Build-up parameter.*

h=70 km.

3. Numerical Presentation of Waveguide Parameters

The important parameters in specifying the characteristics of pulse propagation in the earth-ionosphere waveguide are attenuation rate, phase velocity, group velocity, and the build-up parameter. In general, these are functions of the dimensions of the guide and the electrical properties of the bounding walls. Those four quantities (u_n, v_p/c, v_g/c, and T_b) are plotted in figures 2 and 3 for mode number 1 (i. e., the dominant mode). The abscissa is

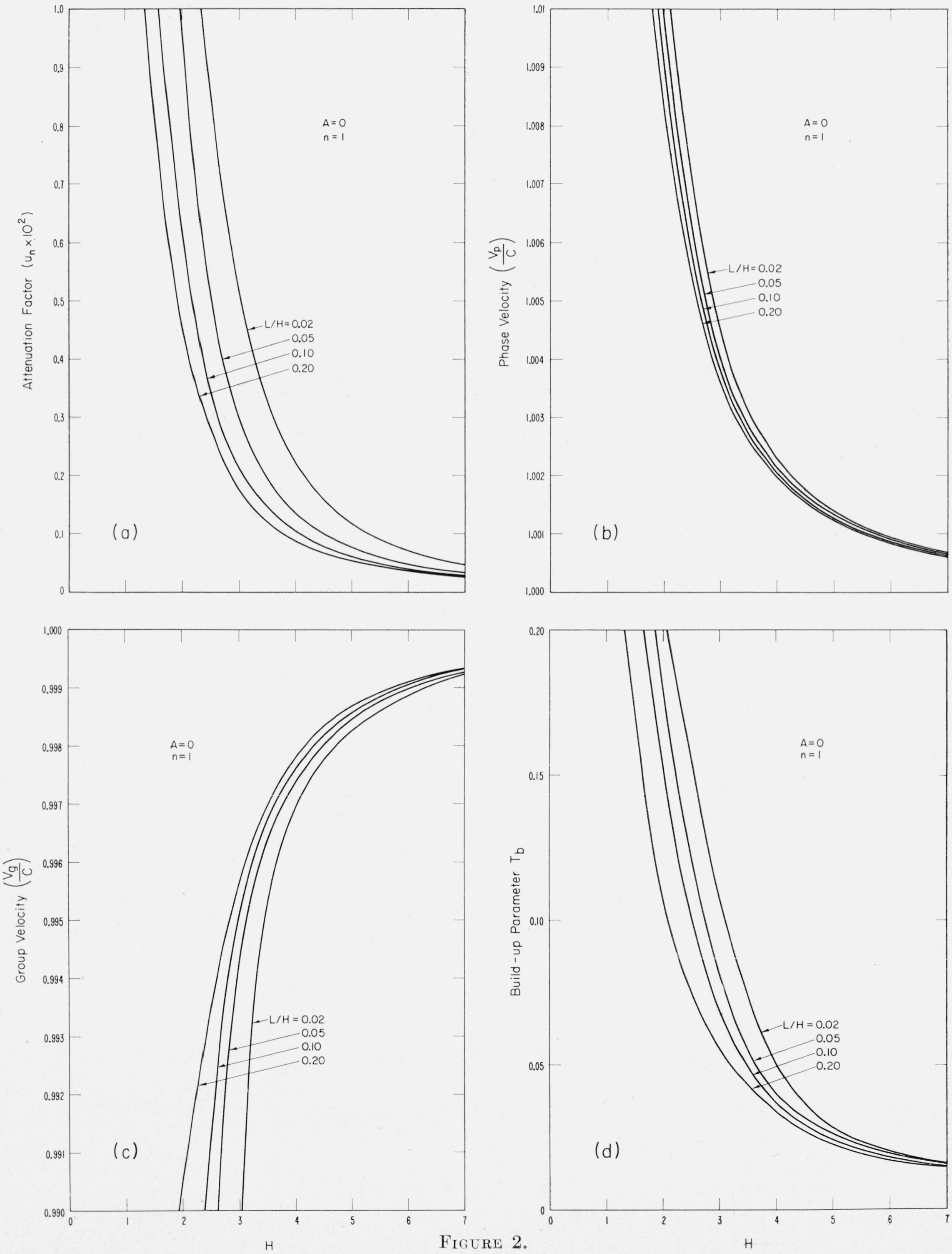

FIGURE 2.

$H(=h/\lambda)$, which is the height of the ionosphere in wavelengths and may be regarded as a frequency parameter.

The dimensionless quantity A is a ground conductivity parameter and is defined by $A=G/H$, where $G=\epsilon_0\omega/\sigma_g$ in terms of the conductivity σ_g of the ground. Alternately, one may write

$$A=\frac{1}{60\sigma_g h}.$$

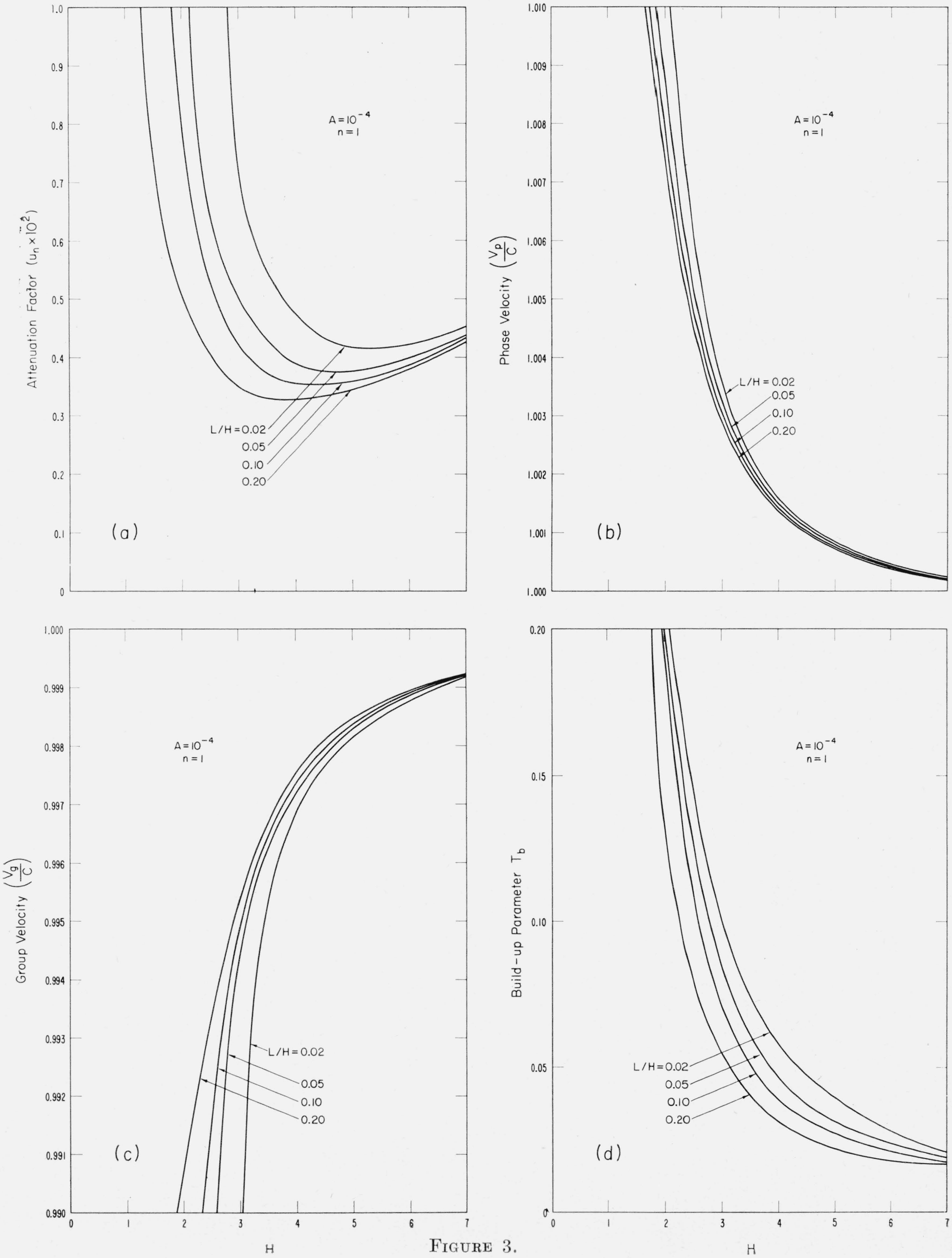

FIGURE 3.

The dimensional quantity B is the corresponding parameter for the (assumed) sharply bounded ionosphere. It is defined by $B=L/H$, where $L=\omega/\omega_r$,

and
$$\omega_r=\omega_0^2/\nu=\frac{(\text{plasma frequency})^2}{\text{collisional frequency}}.$$

In terms of the effective ionospheric conductivity, σ_i,[5] B is given by

$$B=\frac{1}{60\sigma_i h}.$$

The curves in figure 2, a to d, inclusive, are for $A=0$ or $\sigma_g\cong\infty$. This assumption is valid for propagation over sea water for all values of H shown. The curves in figure 3, a to d, inclusive, are for $A=10^{-4}$, which corresponds to moderately conducting ground [2.2 millimhos/m for $h=70$ km].

4. Analysis for Impulse Response

The foregoing analysis is applicable to the description of wave packets wherein the spectral components are essentially contained in a relatively narrow band about the carrier frequency. In the case of a lightning discharge, the pulse contains many spectral components, and as a matter of illustration, if the current was in the form of a Dirac delta function, all frequencies would be contained in a uniform distribution. In this case and for other similar instances, it is desirable to use an alternative approach. In the following, it is assumed that the source is initiated at $t=0$ and $d=0$.

The most convenient description of the source is the (transient) waveform of the radiation component of the vertical field on a perfectly conducting flat ground plane at distance d. This field is denoted $\overline{e}_0(t)$ as a function of t and its Fourier transform is

$$\overline{E}_0(\omega)=\int_0^\infty \overline{e}_0(t)e^{-i\omega t}dt. \tag{23}$$

If the source is a dipole carrying a current $j(t)$ with a height $h(t)$, both functions of t, then the radiation field on the (hypothetical) flat ground plane is

$$\overline{e}_0(t)=\frac{\mu}{2\pi d}\left[\frac{d}{dt}[j(t)h(t)]\right]_{t=t'}\cdot u(t'), \tag{24}$$

where

$$\left.\begin{aligned} u(t')&=1 \quad \text{for} \quad t'>0 \\ &=0 \quad \text{for} \quad t'<0 \end{aligned}\right\} t'=t-t_0,\ t_0=d/c.$$

The field at range d of the same dipole source when located on a homogeneous spherical earth of radius a with concentric ionospheric reflecting layer at height h is denoted $e(t)$, and its transform is

$$E(\omega)=\int_0^\infty e(t)e^{-i\omega t}dt. \tag{25}$$

The relation between $E(\omega)$ and $\overline{E}_0(\omega)$ has been previously derived (see footnote 2), and the final result is simply quoted here.

$$E(\omega)\cong\left[\frac{d/a}{\sin d/a}\right]^{1/2}\frac{(2\pi cd)^{1/2}}{h}\frac{\overline{E}_0(\omega)}{(i\omega)^{1/2}}\sum_{h=0}^{\infty}s_n^{3/2}\,\delta_n e^{\frac{-2\pi d}{h}u_n}e^{-iF(\omega)}, \tag{26}$$

where u_n and s_n are the attenuation and phase constants of the waveguide mode of order n, as defined in section 7.1, $\delta_0\cong\frac{1}{2}$, and $\delta_n\cong 1$ for $n\neq 0$, and $F(\omega)=(\omega/c)s_n d-\omega t$. The waveform of the electric field $e(t)$ is now obtained from the inverse Fourier transform

$$e(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}E(\omega)e^{i\omega t}d\omega. \tag{27}$$

[5] The concept of an effective isotropic ionospheric conductivity is valid for $\omega<<\nu\cong 10^7$ and highly oblique (low-order) modes.

Unfortunately, the integration, except in certain limiting cases, cannot be carried out in closed form because u_n and s_n are not simple analytical functions of ω. In view of the fact, however, that extensive numerical data is already available for u_n, s_n, and related functions, it seems desirable to evaluate the integral by an approximate saddle-point procedure. The integrals to cope with are of the form

$$J=\int_{-\infty}^{+\infty} G(\omega)e^{-iF(\omega)}d\omega. \tag{28}$$

$G(\omega)$ can be assumed to be slowly varying compared to $F(\omega)$. The saddle point of $F(\omega)$ is at $\omega=\omega_s$, which is a solution of

$$\frac{\partial F(\omega)}{\partial\omega}=0.$$

Expanding $F(\omega)$ in a Taylor series about ω_s leads to $F(\omega)=F(\omega_s)+[(\omega-\omega_s)^2/2]F''(\omega_s)+$terms containing $(\omega-\omega_s)^3$, etc., where

$$F''(\omega_s)=\left[\frac{\partial^2F(\omega)}{\partial\omega^2}\right]_{\omega=\omega_s}.$$

Retaining only the first two terms in the expansion for $F(\omega)$, the integration can be carried out to yield

$$J\cong\left[\frac{\pi}{2iF''(\omega_s)}\right]^{1/2}G(\omega_s)e^{-iF(\omega_s)}+\left[\frac{\pi}{-2iF''(\omega_s)}\right]^{1/2}G^*(\omega_s)e^{iF(\omega_s)}, \tag{29a}$$

where the asterisk denotes a complex conjugate. When the current is in the form of a unit impulse or Dirac delta function, the radiated field $\bar{e}_0(t)$/is a doublet impulse function. That is,

$$\bar{e}_0(t)=E_1\delta'(t) \tag{29b}$$

where

$$\delta'(t)=\lim_{\Delta\to0}\frac{u(t-\Delta)-2u(t)+u(t+\Delta)}{\Delta^2}.$$

Therefore, the Fourier transform is

$$\overline{E_0}(\omega)=E_1i\omega. \tag{29c}$$

In this important case

$$G(\omega)=E_1\left[\frac{d/a}{\sin d/a}\right]^{1/2}\frac{(2\pi cd)^{1/2}}{h}\cdot\frac{1}{2\pi}(i\omega)^{1/2}s_n^{3/2}e^{-\frac{2\pi d}{h}u_n}. \tag{30}$$

Therefore, the transient response for the impulse current source is given by

$$e(t)\cong E_1\left[\frac{d/a}{\sin d/a}\right]^{1/2}(2\pi)^{1/2}\left(\frac{c}{h}\right)^2\sum_{n=0}^{\infty}\delta_ne_n(t), \tag{31}$$

where

$$e_n(t)=(H_s)^{1/2}[s_n(\omega_s)]^{3/2}\frac{\cos F(\omega_s)}{T_b(\omega_s)}\exp\left[\frac{-2\pi d}{h}u_n(\omega_s)\right], \tag{32}$$

because

$$F''(\omega_s)=2\frac{d}{c}\frac{ds_n}{d\omega}+\frac{d}{c}\omega\frac{d^2s_n}{d\omega^2}=\frac{hd}{c^2}[T_b(H_s)]^2. \tag{33}$$

The quantities H_s, $s_n(\omega_s)$, $u_n(\omega_s)$, and $T_b(H_s)$ are the values of H, s_n, u_n, and T_b evaluated at the saddle point $\omega=\omega_s$.

It can be readily seen that the equation for determining the saddle point is equivalent to finding the value of H (or ω) that satisfies

$$\frac{1}{s_n}=\frac{v_g(H)}{c}=\frac{d}{ct}. \tag{34}$$

Because $v_g(H)/c$ is a smooth monotonic function of H, the saddle-point value H_s (or ω_s) is readily determined by using a simple graphical procedure for any specified value of t.

5. Application of the Impulse Responses

The response of the waveguide to an impulsive current source is shown in figure 4 for various ranges. The quantity plotted is $e_1(t)$ given by eq (32) with $n=1$. This is the waveform of the dominant mode for the typical daytime conditions ($h=70$ km and $L/H=0.1$) and propagation over sea water ($A\simeq 0$). It is interesting to note that the pulses have the appearances of damped sinusoids. The oscillating nature of the curves is not due to the source which is impulsive, but rather is a result of the modal characteristics of propagation medium. In general, the initial quasi-period (temporal length of the first half-cycle) is becoming progressively shortened with increasing range, whereas the oscillatory nature of the pulse is becoming enhanced. In each case the frequency of the latter part of the waveform seems to be approaching about 9 kc (i. e., a half-period of about 56 μsec).

It is desirable to repeat this calculation for a source current of finite duration. For example, if the current dipole moment is proportional to $\alpha e^{-\alpha t}$ then the Fourier transform of the primary field $\bar{e}_0(t)$ is

$$\overline{E}_0(\omega)=E_1\cdot\frac{\alpha i\omega}{\alpha+i\omega}, \tag{35}$$

which replaces eq (29c). The transient response, denoted by $e_n(\alpha,t)$, is thus given by

$$e_n(\alpha,t)=(H_s)^{1/2}[s_n(\omega_s)]^{3/2}\frac{\alpha}{(\alpha^2+\omega^s)^{\frac{1}{2}}}\frac{\cos\,[F(\omega_s)+\psi(\omega_s)]}{T_b(\omega_s)}\exp\left[\frac{-2\pi d}{h}u_n(\omega_s)\right], \tag{36}$$

where $\psi(\omega_s)=\arctan\,(\omega_s/\alpha)$. As α tends to ∞, $e_n(\alpha,t)$, of course, approaches $e_n(t)$, which is the impulse response.[6]

The response $e_1(\alpha,t)$ for the dominant mode at $d=3{,}000$ km is indicated in figure 5 for the same daytime conditions as the curves in figure 4. Various values of the time constant $1/\alpha$

[6] Note that $\delta(t)=\lim\limits_{\alpha\to\infty}\alpha e^{-\alpha t}$ for $t>0$.

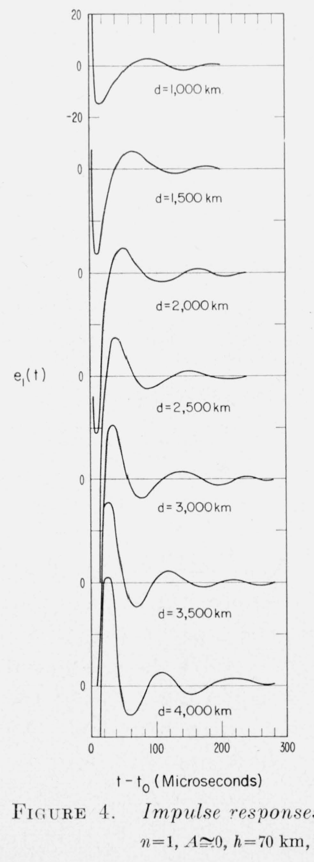

FIGURE 4. *Impulse responses for dominant mode.*
$n=1$, $A\simeq 0$, $h=70$ km, $L/H=0.1$.

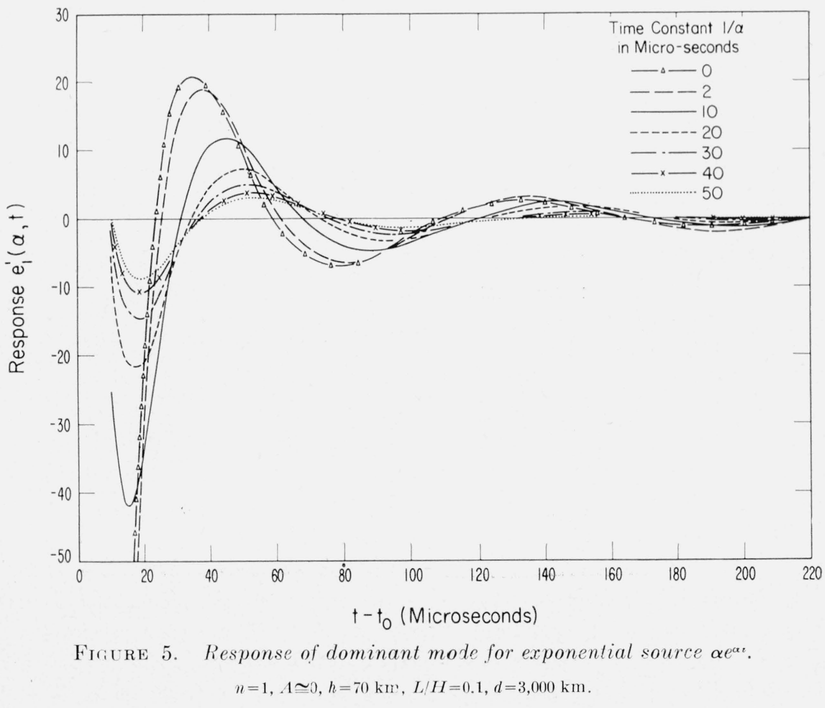

FIGURE 5. *Response of dominant mode for exponential source* $\alpha e^{\alpha t}$.
$n=1$, $A\simeq 0$, $h=70$ km, $L/H=0.1$, $d=3{,}000$ km.

are indicated on the curves. The impulse response corresponds to the case $1/\alpha=0$. The general nature of the curves is very similar. There appears to be a shift in the phase of the cycles but the quasi-half-periods are essentially unchanged.

A more realistic waveform for the current dipole moment $S(t)$ of a lightning stroke is a pulse which rises up smoothly from zero to a peak value in about 10 or 20 μsec and then decays to zero somewhat more slowly. Examples of such pulses are shown in figure 6, where $S(t)$ is of the form

$$S(t)=\alpha[e^{-\alpha t}-e^{-\alpha_i t}], \tag{37}$$

with $\alpha_i>\alpha$. The response $\overline{e}_1(t)$ to this composite exponential source is then simply obtained by superposition of the exponential responses. For example,

$$\overline{e}_1(t)=e_1(\alpha,t)-\frac{\alpha}{\alpha_i}e_1(\alpha_i,t). \tag{38}$$

Employing the three particular forms of source current $S(t)$, the response $\overline{e}_1(t)$ for the dominant mode is shown plotted in figure 7 for typical daytime conditions over sea water. The curves

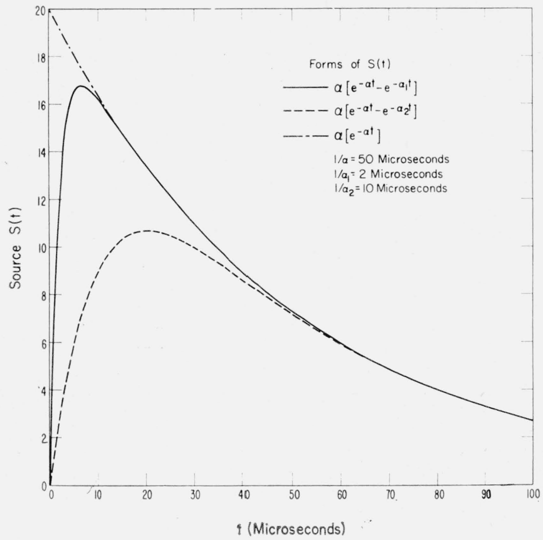

FIGURE 6. *Idealized source functions.*

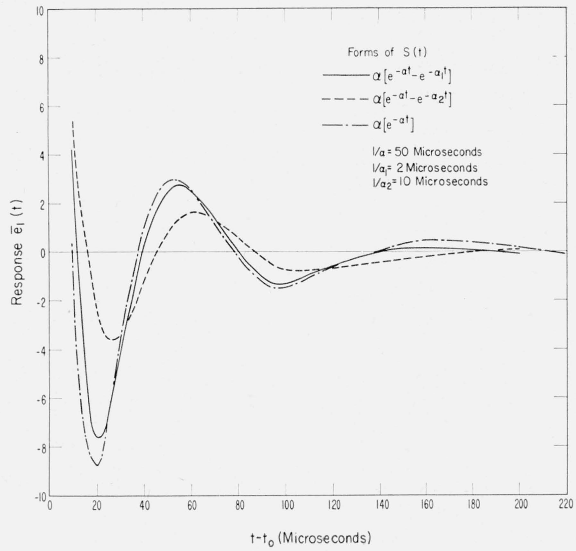

FIGURE 7. *Response of dominant mode for idealized source functions.*

($n=1$, $A\cong 0$, $h=70$ km, $L/H=0.1$) $d=3{,}000$ km; forms of $S(t)$.

again are very similar in appearance to the previous response curves for impulse and single exponential sources. The effect of finite rise time of the source current is to reduce the amplitudes of the peaks in the initial part of the waveform.

It is rather important to note that the quasi-half-periods are not appreciably influenced by the nature of the source pulse. It would therefore seem justified to compare the variation of the quasi-periods with range for the calculated impulse responses and the experimental waveforms of Hepburn.[7] Such a comparison is shown in figure 8. The solid curves are the calculated quasi-half-periods in microseconds for daytime conditions as a function of range. Only the first five half-cycles are shown. The corresponding data from Hepburn's paper is shown by the encircled numbers which indicate the order of the half-cycle. The agreement is quite reasonable.

[7] F. Hepburn, Wave-guide interpretation of atmospheric waveforms, J. Atmospheric and Terres. Phys. **10**, 121 (1957).

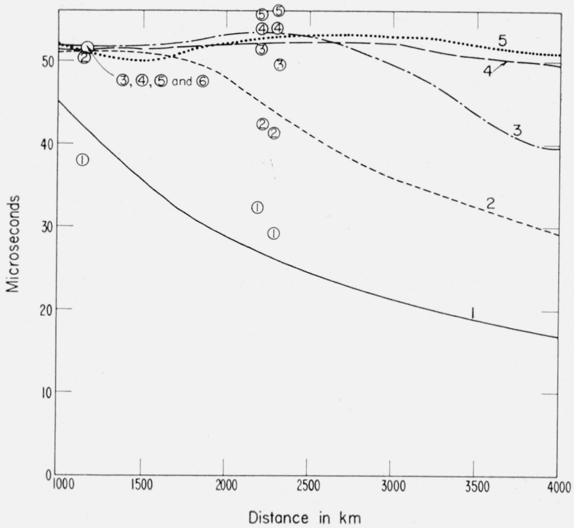

FIGURE 8. *Quasi-half-periods of dominant mode.*

($n=1$, $A \cong 0$, $h=70$ km, $L/H=0.1$),

②From Hepburn's measured waveforms.

The calculated variation of the quasi-half-period with range and time are also in good qualitative agreement with Norinder's [8] observed waveforms who first pointed out the characteristic transformation of the quasi-half-periods.

The transient waveforms presented in figures 4 to 7 are the dominant ($n=1$) mode only. The same procedure can be used to calculate the higher order mode responses. At ranges exceeding about 2,000 km, these do not contribute appreciably to the total field for the range of times shown on the curves.

The transient response of zero-order mode can also be important for longer times. It corresponds to what has been called the "slow tail" in atmospheric waveforms. Again for the range of time shown in figures 4 to 7 it would not be significant. For the sake of completeness, a short analysis of the transient response of the zero-order mode is presented in section 7.3.

6. Conclusion

The analyses and results presented in this paper should be useful in the interpretation of experimental results for the propagation of pulses to large distances over the surface of the earth. It appears to be desirable to separate the transient analyses into two parts depending on whether the source is a quasi-monochromatic pulse containing a narrow band of frequencies or an impulsive type having many spectral components over a wide frequency range. The parameters describing the propagation of these two classifications of pulses have been presented in graphical form.

7. Appendixes

7.1. Waveguide Modes [9]

The earth is taken to be a homogeneous sphere of conductivity σ and dielectric constant ϵ. The lower edge of the assumed homogeneous ionosphere is taken to be at a height h. The collisional frequency is ν and the plasma frequency is ω_0.

Assuming that the source is a vertical dipole radiating P kilowatts, the vertical electric field in millivolts per meter at the great circle distance d in kilometers, for a time factor exp $(i\omega t)$, is given by the mode sum,

$$E = E_0 W,$$

[8] H. Norinder, The waveform of the electric field in atmospherics, Arkiv for Geophysik **2**, 161 (June 1954).

[9] J. R. Wait, On the mode theory of VLF ionospheric propagation, Geofis. pura e Appl. **37**, 103 (1957).

where

$$W \cong \left[\frac{(d/a)}{\sin (d/a)}\right]^{1/2} \frac{(d/\lambda)^{1/2}}{(h/\lambda)} e^{i\left[\frac{2\pi d}{\lambda}-\frac{\pi}{4}\right]} \sum_{n=0}^{\infty} \delta_n S_n{}^{3/2} e^{-2\pi S_n (d/\lambda) i}, \tag{1.1}$$

$$E_0 = \frac{300\sqrt{P}}{d}, \quad \delta_n = \left[1+\frac{\sin(4\pi C_n h/\lambda)}{(4\pi C_n h/\lambda)}\right]^{-1} \quad \begin{array}{l} \cong \frac{1}{2} \text{ for } n=0 \\ \cong 1 \text{ for } n \neq 0, \end{array}$$

where $S_n = (1-C_n^2)^{1/2}$, and C_n is a solution of

$$R_g(C) R_i(C) \exp(-4\pi i C h/\lambda) = \exp(-i2\pi n). \tag{1.2}$$

$R_g(C)$ is the Fresnel reflection coefficient at the ground for a plane wave whose (complex) angle of incidence is the arc cosine of C, and $R_i(C)$ is the corresponding Fresnel reflection coefficient at the lower edge of the ionosphere.

It follows from the previous analysis (see footnote 9) that

$$R_g(C) = \frac{(KG-i)C - [(K-1)G^2 - iG + C^2G^2]^{1/2}}{(KG-i)C + [(K-1)G^2 - iG + C^2G^2]^{1/2}}, \tag{1.3}$$

where $K = \epsilon/\epsilon_0$, and $G = (\epsilon_0 \omega)/\sigma$, and

$$R_i(C) = \frac{N^2 C - (N^2 - S^2)^{1/2}}{N^2 C + (N^2 - S^2)^{1/2}}, \tag{1.4}$$

where [10]

$$N^2 \cong 1 - i/L = 1 - i(\omega_r/\omega), \quad \omega_r = \omega_0^2/\nu. \tag{1.5}$$

The complex values of $S_n (n=0,1,2, \ldots)$ satisfying eq (2) have been obtained from an automatic computor by using a program devised by H. H. Howe. When L is small compared with unity, it was shown previously (see footnote 9) that

$$S_n \cong \left[1-\left(\frac{\lambda n}{2h}\right)^2\right]^{1/2} - \frac{\lambda}{4\pi h} e^{i3\pi/4} (G^{1/2} + L^{1/2}) \frac{\bar{\epsilon}_n}{\left[1-\left(\frac{\lambda n}{2h}\right)^2\right]^{1/2}}, \tag{1.6}$$

where

$$\bar{\epsilon}_0 = 1, \quad \bar{\epsilon}_n = 2 (n \neq 0). \tag{1.7}$$

The above formula for S_n is particularly suitable for very low frequencies, in which case only the zero mode is of any consequence.

The exponential term inside the summation of eq (1.1) determines, in the main, the propagation characteristics of the modes. It can be rewritten as follows:

$$\exp[-i2\pi S_n (d/\lambda)] = \exp\left[-\frac{2\pi d}{h} u_n\right] \exp\left[\frac{-i2\pi d}{\lambda} s_n\right], \tag{1.8}$$

where $u_n = -Im S_n (h/\lambda)$, and $s_n = Re S_n$.

7.2. Transient Response of Ideal Waveguides

Most of the complications in the waveguide mode theory of VLF ionospheric propagation are due to the finite conductivity of the bounding walls. The characteristics of the modes are obtained only from a numerical iteration procedure. When the losses in the walls are negligible, it is possible to derive somewhat simpler forms for the transient response and associated parameters.

Assuming that both walls of the waveguide are of perfect conductivity, the phase velocity is given by

$$\frac{v_p}{c} = \frac{1}{s_n} = \left[1-\left(\frac{n}{2H}\right)^2\right]^{-1/2} \tag{2.1}$$

[10] These expressions for the effective refractive index which tacitly neglect the earth's magnetic field are valid for VLF and small values of C.

and the group velocity by

$$\frac{v_g}{c}=s_n=\left[1-\left(\frac{n}{2H}\right)^2\right]^{1/2}. \tag{2.2}$$

Furthermore,

$$F(\omega)=F(H)=\frac{2\pi H}{h}\left(1-\left(\frac{n}{2H}\right)^2\right)^{1/2}d-\frac{2\pi ct}{h}H, \tag{2.3}$$

and, consequently, the saddle point is obtained from

$$\frac{\partial F(H)}{\partial H}=0, \tag{2.4}$$

which is satisfied by

$$H=H_s=\frac{n\,ct}{2X}, \tag{2.5}$$

where

$$X=[(ct)^2-d^2]^{1/2}. \tag{2.6}$$

Consequently,

$$F(H_s)=\frac{-\pi nX}{h}, \tag{2.7}$$

$$[\overline{T}_b]^2=\left[\frac{\partial^2}{\partial H^2}s_nH\right]_{H=H_s}=\left(\frac{2}{n}\right)\frac{X^3}{d^3}, \tag{2.8}$$

and

$$\bar{s}_n=s_n(H_s)=d/ct. \tag{2.9}$$

Inserting these values into the saddle-point formula given by eq (32) leads to

$$e_n(t)=\left(\frac{n}{2}\right)\frac{d^3}{X^2(ct)}\cos\left(\frac{\pi nX}{h}\right), \tag{2.10}$$

for $n=1, 2, 3. \ . \ . \ .$

The preceding formula is the waveform for the vertical electric field at range d on an ideally perfectly conducting earth with a perfectly conducting ionosphere at height h. It is valid for large distances such that $d>>h$ and X.

If the upper boundary was a perfect magnetic conductor (phase shift of 180° on reflection rather than 0°), the preceding formulas are modified by replacing n by $n-\frac{1}{2}$.

7.3. Transient Response of the Zero-Order Mode

The saddle-point method is valid only if the function $\exp[-iF(\omega)]$ is rapidly varying compared to other factors in the integrand of the inverse Fourier integral. For mode zero this is not so, but in this case it is known (see footnote 9) that

$$\left[\frac{2\pi u_0}{h}+is_0\frac{\omega}{c}\right]d\simeq\left(\frac{1}{\alpha_g}+\frac{1}{\alpha_i}\right)\frac{d}{c}(i\omega)^{1/2}, \tag{3.1}$$

where

$$\alpha_g\simeq\sqrt{\sigma_g\mu}\cdot 2h \qquad \text{and} \qquad \alpha_i\simeq\sqrt{\sigma_i\mu}\cdot 2h.$$

Therefore, for the impulse current source,

$$E(\omega)\simeq\frac{E_1}{2}\left[\frac{d/a}{\sin d/a}\right]^{1/2}\frac{(2\pi cd)^{1/2}}{h}(i\omega)^{1/2}\exp\left[-\frac{d}{c}\left(\frac{1}{\alpha_e}+\frac{1}{\alpha_i}\right)(i\omega)^{1/2}\right] \tag{3.2}$$

The transient response $e_0(t)$ of the zero-order mode is thus given by

$$e(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} E(\omega)e^{i\omega t'}d\omega,$$

where $t'=t-t_0$. This can be evaluated to yield

$$e(t)=E_1\left[\frac{d/a}{\sin d/a}\right]^{1/2}(2\pi)^{1/2}\frac{c^2}{h^2}e_0(t)\cdot\tfrac{1}{2}, \tag{3.3}$$

where

$$e_0(t)=\frac{4}{\sqrt{\pi}}\left(\frac{h}{d}\right)^3\left(\frac{h}{c}\right)\left(\frac{d}{c}\right)^{1/2}\left(\frac{\sigma}{\epsilon}\right)^{3/2}P(\beta/t'), \tag{3.4}$$

and

$$P(\beta/t')=\left(\frac{\beta}{2t'}-1\right)\left(\frac{\beta}{t'}\right)^{3/2}e^{-\beta/4t'}, \tag{3.5}$$

with

$$\beta^{1/2}=\left(\frac{d}{2h}\right)\left(\frac{\epsilon_0}{\sigma}\right)^{1/2} \quad \text{and} \quad \frac{1}{\sqrt{\sigma}}=\frac{1}{\sqrt{\sigma_i}}+\frac{1}{\sqrt{\sigma_g}}.$$

The transient response is thus proportional to the characteristic function $P(\beta/t')$. In view of the combination in which the ionospheric conductivity σ_i and the ground conductivity σ_g occur, and because $\sigma_g>>\sigma_i$, it follows that $\sigma\cong\sigma_i$. The transient response of the zero-order mode does then not depend to any extent on σ_g. The function $P(\beta/t)$ is plotted in figure 9 as a function of T, where $T=t'/\beta$. The multiple scale shown at the bottom of the figure is to facilitate the conversion of the parameter T to actual time in microseconds.

The transient response of the zero-order mode has been studied extensively by Schumann.[11]

[11] W. O. Schumann, Uber die Oberfelder bei Ausbreitung langer elektrischer Wellen um die Erde und die Signale des Blitzes, Nuovo cimento: **9,** 1 (December 1952).

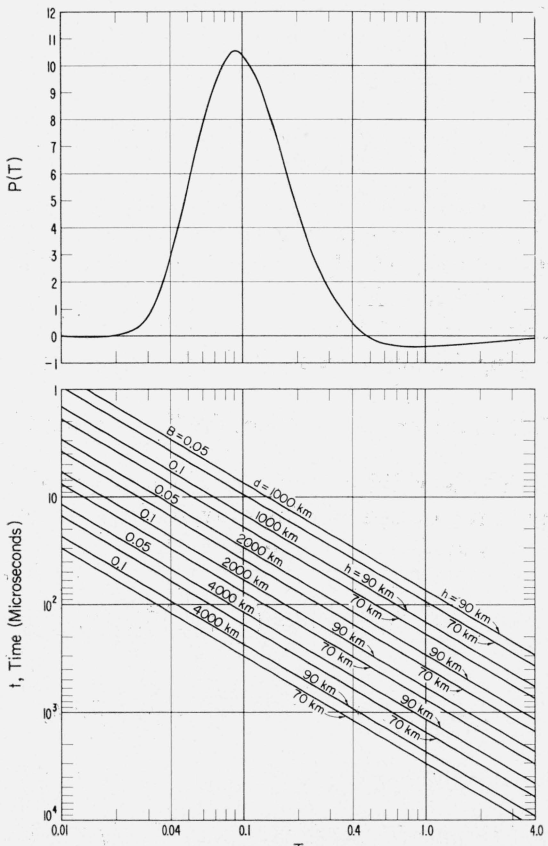

FIGURE 9. *Zero-mode transient response for impulse source.*

7.4. Justification of the Saddle-Point Method

It is worth while to examine the transient calculation from a slightly different viewpoint in order to give further justification to the use of the saddle-point method of integration. When the earth and the ionosphere are represented by perfectly conducting planes with constant separation h, the magnetic field on the earth of a vertical electric dipole on the earth is a sum of modes of the type

$$H_n(\omega) = i\omega \frac{\partial \pi_n}{\partial d}, \tag{4.1}$$

where π_n is a z-directed Hertz vector given by

$$\pi_n = \frac{2i\pi^2}{h} e^{i\omega t} H_0^{(2)}\left[kd\left(1-\frac{\pi^2 n^2}{k^2 h^2}\right)^{1/2}\right] \tag{4.2}$$

apart from a constant factor. $H_0^{(2)}$ is the Hankel function of the second kind for an impulsive current source (i. e., current moment proportional to $\delta(t)$), the magnetic field of a mode is then given by

$$h_n(t) = \frac{\pi}{h}\frac{\partial}{\partial d}\int_{-\infty}^{+\infty} e^{i\omega t} H_0^{(2)}\left[\left(\omega^2 - \frac{\pi^2 n^2 c^2}{h^2}\right)^{1/2} \frac{d}{c}\right] d\omega. \tag{4.3}$$

This integral can be evaluated [12, 13] to yield

$$h_n(t) = \frac{\pi}{h}\frac{\partial}{\partial d}\frac{c}{X}\left[\cos\frac{\pi n X}{h} - \frac{2}{\pi}\sin\frac{\pi n X}{h}\log_e\frac{ct+X}{d}\right], \tag{4.4}$$

where $X=[(ct)^2-d^2]^{1/2}$. When $X << ct$ or d and $d >> h$, the term containing the logarithm is negligible, and consequently

$$h_n(t) \simeq \frac{\pi c d}{h X^3}\left\{\cos\left(\frac{\pi n X}{h}\right) + \frac{\pi n X}{h}\sin\left(\frac{\pi n X}{h}\right)\right\}. \tag{4.5}$$

This latter form could have been obtained directly from eq (4.3), using the saddle-point procedure adopted in the main body of the paper.

7.5. Symbols

ω = (angular) frequency,
d = distance from source to observer measured along the surface of the earth,
h = separation between the earth and the lower edge of the ionosphere,
λ = wavelength in free space,
u_n = attenuation factor of the waveguide mode of order n,
s_n = phase factor for a mode of order n,
c = velocity of light in free space,
v_p = phase velocity of a mode,
v_g = group velocity of a mode,
$H = h/\lambda$,
$E_0(t)$ = form of undistorted signal,
$A(t)$ = envelope of undistorted signal,
ω_0 = carrier frequency of undistorted signal,
T = duration or rectangular envelope,
$G(\omega)$ = Fourier or frequency spectrum of $E_0(t)$,
$E(t)$ = form of signal after propagation through dispersive medium,

[12] N. W. McLachlan and P. Humbert, Memorial des sciences mathematiques fascicule **100**, 34. Paris (1941).
[12] K G. Budden, The propagation of a radio atmospheric, **42**, 1 (January 1951).

$\phi(\omega)=s_n\dfrac{2\pi d}{\lambda}$ and

$\alpha(\omega)=u_n\dfrac{2\pi d}{h}$ are phase and attenuation factors of a mode as a function of frequency for a range d,

t' = an integration variable in eq (8) to (15),

$\Omega=\omega-\omega_0$, an expansion parameter,

$\phi'(\omega_0)=[d\phi(\omega)/d\omega]_{\omega=\omega_0}$,

$\phi''(\omega_0)=[d^2\phi(\omega)/d\omega^2]_{\omega=\omega_0}$,

u_1 and u_2 = limits of integration in eq (19),

θ = the time measured from the instant $-\dfrac{T}{2}+\phi'(\omega_0)$,

t_b = time from $\theta=0$ to the point where the envelope approaches within 5 percent of unity,

$T_b=\dfrac{c}{2\sqrt{2dh}}\,t_b$, a dimensionless build-up parameter,

$\epsilon_0=8.854\times10^{-12}$, permittivity of free space,

σ_g = conductivity of the ground,

$G=\epsilon_0\omega/\sigma_g$,

$A=G/H=1/60\sigma_g h$,

ω_0 = (angular) plasma frequency,

ν = (angular) collisional frequency,

$\omega_r=\omega_0^2/\nu$,

σ_i = effective ionospheric conductivity,

$L=\omega/\omega_r$,

$B=L/H=1/60\sigma_i h$,

$\bar{e}_0(t)$ = the radiated electric field for an ideal flat perfectly conducting ground plane,

$\overline{E_0}(\omega)$ = the frequency spectrum of $\bar{e}_0(t)$,

$j(t)$ = instantaneous average current of the source dipole,

$h(t)$ = instantaneous height of the source dipole,

$\delta(t)$ = unit impulse or Dirac function,

$\mu=4\pi\times10^{-7}$, permeability of free-space,

$t_0=d/c$,

$t'=t-t_0$,

$e(t)$ = resultant field of (transient) dipole sources,

$E(\omega)$ = the frequency spectrum of $e(t)$,

a = radius of the earth,

$\delta_n\cong1$ for $n\neq0$,

$\cong\frac{1}{2}$ for $n=0$ (see eq (1) for more accurate definition),

$F(\omega)=\dfrac{\omega}{c}\,s_n d-\omega t$,

ω_s = the saddle point of $F(\omega)$,

E_1 = constant of proportionality in eq (9b),

$\delta'(t)$ = doublet impulse function,

$T_b(H_s)=T_b$, value of T_b at the saddle point $H=H_s=\omega_s h/2\pi c$,

C_n = cosine of the (complex) angle of incidence of the mode of order n at the bounding walls in the waveguide,

$S_n=(1-C_n^2)^{1/2}$,

ϵ = dielectric constant of the ground,

$K=\epsilon/\epsilon_0$,

$N^2=1-i/L=1-i\omega_r/\omega$,

$\bar{\epsilon}_n=2$ for $n\neq0$,

$\bar{\epsilon}_0=1$,

$X=[(ct)^2-d^2]^{1/2}$,

$\beta^{1/2}=\left(\dfrac{d}{2h}\right)\left[\left(\dfrac{\epsilon_0}{\sigma_i}\right)^{1/2}+\left(\dfrac{\epsilon_0}{\sigma_g}\right)^{1/2}\right]$,

$P(x)=\left(\dfrac{x}{2}-1\right)(x)^{3/2}e^{-\frac{x}{4}}$,

$H_n(\omega)$ = magnetic field of the nth mode of a dipole source in a parallel plate waveguide,

π_n = the corresponding z-directed Hertz vector.

8. Addendum

Because of certain normalizations, the distance d between transmitter and receiver is measured in terms of the great-circle distance at height $h/2$. The phase and group velocities shown in figures 2 and 3 should then be multiplied by $(1-h/2a)\simeq 0.995$, if referred to the great-circle distance on the earth's surface. In other places in the paper, this correction is negligible and can be ignored.

It should be noted that the correction factor $(1-h/2a)$ can vary because of changes in height of the ionosphere and changes in the effective earth radius a. For this reason the velocities were presented in figures 2 and 3 in uncorrected form.

There is a further modification to the attenuation factors when a more recondite mode equation is employed.[14] It appears that the attenuation factors are increased by as much as 50 percent for $H>4$ when $n=1$ and $A=0$ but less for other values of n and A. This will not change the appearance of the waveform shapes for the range of time indicated in figures 5 to 9.

The numerical results in section 3 were obtained by H. H. Howe and A. M. Conda, and those in section 5 were obtained by W. E. Mientka. Many helpful suggestions were received from K. G. Budden, A. G. Jean, E. A. Lewis, K. A. Norton, and W. L. Taylor in the course of this work.

[14] This equation is

$$R_g(C)R_i(C)\exp\left[-4\pi iCH\left(1+\frac{S^2h}{C^2a}\right)^{1/2}\right]=\exp(-i2\pi n),$$

which reduces to eq (1.2) for $[ReC]^2>>h/a$.

BOULDER, COLO., December 18, 1957.