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On the Motion of Two Cylinders in an Ideal Fluid*

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The complex potential of two cylinders moving in an infinite liquid is determined by the method of image doublets, and the solution is expressed as an infinite series in rectangular coordinates. Approximate solutions in finite form are given for various cases. A method for generalizing the solution for the case of more than two cylinders is indicated. Applications to the flow induced by a cylinder moving in the presence of plane boundaries are given and the stream lines are illustrated in certain cases.

1. Introduction

The problem of the motion of two cylinders in any manner with their axes always parallel was apparently first solved by Hicks.[1] The velocity potentials are found in general as definite integrals, which, when the cylinders move as a rigid body, are expressed in finite form as elliptic functions of bipolar coordinates. Greenhill [2] has expressed the solution as an infinite series in bipolar coordinates.

In a problem in wave motion the need arose for a solution of the potential flow for a cylinder moving parallel to a plane boundary. The problem is the same as that of two parallel cylinders of equal radii moving with equal velocities normal to the line of centers. Thus the flow is a special case of the classic results obtained by Hicks and Greenhill. However, it was necessary to have the potential function represented in rectangular coordinates to determine the additional term representing the wave motion. The bipolar coordinates used by Hicks and Greenhill are defined geometrically, and the transformation to rectangular coordinates does not admit a convenient representation. Therefore, the solution was obtained by the method of image doublets and is here generalized to the case of two cylinders moving in any manner with their axes always parallel. The present method may also be applied to the motion of more than two cylinders which was not done in the bipolar system with either Hicks' or Greenhill's analysis.

Only ideal fluids without vorticity are considered in this study.

2. Solution for Two Cylinders

Consider two parallel cylinders C and C' of radii b and b' momentarily at a distance f apart moving normal to their axes in a liquid at rest at infinity. Choosing the coordinate system as in figure 1, the cylinder velocities may be expressed as $Ue^{i\alpha}$ and $U'e^{i\alpha'}$, respectively.

The complex potential $w_0=\phi_0+i\psi_0$ due to C, in the absence of C', is that of a doublet whose axis is in the direction α

$$w_0=Ub^2\frac{e^{i\alpha}}{z}, \tag{1}$$

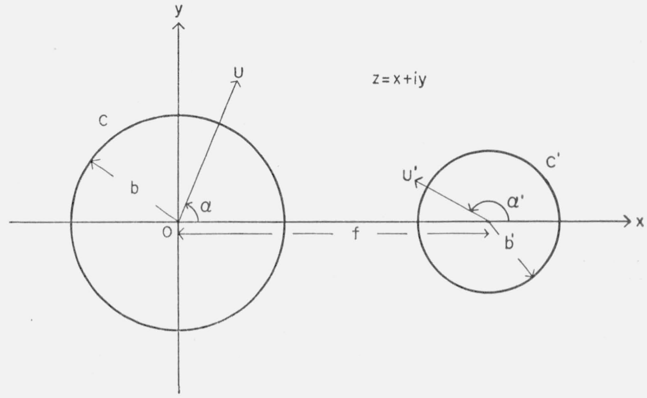

FIGURE 1. *Notation diagram for motion of two cylinders in an infinite liquid.*

where z is the complex coordinate position. The normal velocities introduced on C' by w_0 are cancelled by the image doublet whose axis is in the direction $\pi-\alpha$

$$w_1=-Ub^2\left(\frac{b'}{f}\right)^2\frac{e^{-i\alpha}}{z-(f-f_1)} \tag{2}$$

where

$$f_1=\frac{(b')^2}{f}. \tag{3}$$

The sum w_0+w_1 is the well known expression for the potential of a doublet in the presence of a cylinder (see Milne-Thomson [3]). Similarly, the normal velocities introduced on C by w_1 are cancelled by the second image doublet

$$w_2=Ub^2\left(\frac{b'}{f}\right)^2\left(\frac{b}{f-f_1}\right)^2\frac{e^{i\alpha}}{z-f_2} \tag{4}$$

where

$$f_2=\frac{b^2}{f-f_1}. \tag{5}$$

Continuing, an infinite sequence of image doublets of decreasing strength is obtained. The general expression for w_k is

*This problem was encountered in connection with a wave investigation sponsored by the Office of Naval Research, and the present account is published in order to make more convenient formulas available.

[1] W. M. Hicks, On the motion of two cylinders in a fluid, Quart. J. Pure and Appl. Math. **16**, 113 (1879).

[2] A. G. Greenhill, Functional images in Cartesians, Quart. J. Pure and Appl. Math. **18**, 231 (1882).

[3] L. M. Milne-Thomson, Theoretical Hydrodynamics, Article 8.82, p. 221 (MacMillan & Co., Ltd., London, 1938).

$$w_k = \begin{cases} \beta_k \dfrac{e^{i\alpha}}{z-f_k} \text{ for } k \text{ even} \\ \beta_k \dfrac{e^{-i\alpha}}{z-(f-f_k)} \text{ for } k \text{ odd} \end{cases} \tag{6}$$

where $\beta_0 = Ub^2$, $\beta_1 = -Ub^2\left(\frac{b'}{f}\right)^2$,

$$\beta_k = \frac{(bb')^2}{(f-f_{k-2})^2(f-f_{k-1})^2}\beta_k - 2 \tag{7}$$

and

$$f_k = \begin{cases} \dfrac{b^2}{f-f_{k-2}} \text{ for } k \text{ even} \\ \dfrac{(b')^2}{f-f_{k-1}} \text{ for } k \text{ odd} \end{cases} \tag{7a}$$

it being understood $f_0=0$.

Similarly, starting with the potential of C' in the absence of C,

$$w_0' = U'(b')^2 \frac{e^{i\alpha'}}{z-f}, \tag{8}$$

$$w_1' = -U'(b')^2\left(\frac{b}{f}\right)^2 \frac{e^{-i\alpha'}}{z-f_1'} \tag{9}$$

and

$$w_k' = \begin{cases} \beta_k' \dfrac{e^{i\alpha'}}{z-(f-f_k')} \text{ for } k \text{ even} \\ \beta_k' \dfrac{e^{-i\alpha'}}{z-f_k'} \text{ for } k \text{ odd} \end{cases} \tag{10}$$

where

$$\beta_0' = U'(b')^2, \beta_1' = -U'(b')^2 \cdot \left(\frac{b}{f}\right)^2$$

$$\beta_k' = \frac{(bb')^2}{(f-f_{k-2}')^2(f-f_{k-1}')^2}\beta_{k-2}' \tag{11}$$

and

$$f_k' = \begin{cases} \dfrac{(b')^2}{f-f_{k-1}'} \text{ for } k \text{ even} \\ \dfrac{b^2}{f-f_{k-1}'} \text{ for } k \text{ odd} \end{cases} \tag{11a}$$

and again $f_0'=0$.

Thus the complex potential $w=\phi+i\psi$ for the motion of two cylinders is

$$w = \sum_{k=0}^{\infty} (w_k + w_k') \tag{12}$$

where the w_k are obtained from eq (1) through (7) and the w_k' from eq (8) through (11).

3. Approximate Solutions

The series (12) converges rapidly for small values of bb'/f^2 so that for this case one or two terms of the series gives a close approximate solution

$$w \approx w_0 + w_0' + w_1 + w_1' \text{ for } \frac{bb'}{f^2} \text{ small.} \tag{13}$$

For larger values of bb'/f^2 a more accurate solution may be obtained by approximating the remaining terms. For this purpose let

$$\left.\begin{aligned} \eta &= \lim_{k\to\infty} f_k \ (k \text{ odd}) = \lim_{k\to\infty} f_k' \ (k \text{ even}) \\ \eta' &= \lim_{k\to\infty} f_k \ (k \text{ even}) = \lim_{k\to\infty} f_k' \ (k \text{ odd}) \end{aligned}\right\} \tag{14}$$

The conditions

$$(f-\eta')\eta = b^2 \tag{15}$$

$$(f-\eta)\eta' = (b')^2 \tag{16}$$

and $\eta \leq b$, $\eta' \leq b'$ may be used to determine η and η'.

Now for k greater than some number n replace f_k by η' for k even, by η for k odd, and replace f_k' by η for k even, by η' for k odd giving

$$w_k \approx \frac{(bb')^2}{(f-\eta)^2(f-\eta')^2} w_{k-2} = \beta w_{k-2} \tag{17}$$

and

$$w_k' \approx \beta w_{k-2}' \tag{18}$$

for k sufficiently large. Thus the solution from (12) would be

$$w \approx \sum_{k=0}^{n} (w_k + w_k') + \{w_{n-1} + w_{n-1}' + w_n + w_n'\} \frac{\beta}{1-\beta} \tag{19}$$

because

$$\sum_{k=1}^{\infty} \beta^k = \frac{\beta}{1-\beta}, \tag{20}$$

An accurate solution for any value of bb'/f^2 can be obtained from eq (19) by an appropriate choice of n. It was found in the computations for the examples to follow that for $bb'/f^2 = 1/16$, $n=1$ gave an accurate result, and for the extreme case $bb'/f^2 = 1/4$ an accurate solution was obtained for $n=3$.

Again when bb'/f^2 is sufficiently small η and η' are negligible so that eq (19) reduces to

$$w \approx \{w_0 + w_1 + w_0' + w_1'\} \frac{1}{1-\left(\frac{bb'}{f^2}\right)^2} \tag{21}$$

where w_0, w_1, w_0', and w_1' are given by eq (1), (2), (8), and (9) after setting f_1 and f_1' equal to zero.

4. More Than Two Cylinders

When more than two cylinders are moving in an infinite liquid the method of solution is completely analogous to that given in section 2.

If n cylinders are moving in a liquid choose any one C_R as a reference and introduce an appropriate coordinate system. The complex potential of C_R in the absence of the other cylinders will be that of a doublet $w_{R,0}$. The image of $w_{R,0}$ is taken in each of the remaining $n-1$ cylinders in the same manner as before. Each of these $n-1$ image doublets will have an image in C_R and in each of the other cylinders as well. The process is continued in this manner with each doublet introducing $n-1$ image doublets at each step. The same process is applied starting with each of the $n-1$ remaining cylinders. Again the potential of the system will be the sum of the potentials of all the doublets. An application of this method will be given in section 6.

5. Cylinder Near a Plane Boundary

If $b^1=b$, $U'=U$, $\alpha'=\pi-\alpha$ the plane equidistant between the cylinder axes is a streamline and the solution represents the flow induced by a cylinder moving in the presence of a plane boundary. Figure 2 shows the streamlines for a cylinder moving toward a wall, $\alpha=0$.

The equations can be simplified when the cylinder moves parallel to the wall, $\alpha=\pi/2$, with its circumference touching the wall, $b/f=1/2$. In this case the complete solution is

$$w=iUb\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}\left\{\frac{1}{\frac{z}{b}-\frac{k}{k+1}}+\frac{1}{\frac{z}{b}-\frac{k+2}{k+1}}\right\} \qquad (22)$$

An approximate solution is obtained by taking n terms of the series and in the remaining terms replacing $k/(k+1)$ and $(k+2)/(k+1)$ by 1. Thus

$$w\approx iUb\sum_{k=0}^{n}\frac{1}{(k+1)^2}\left\{\frac{1}{\frac{z}{b}-\frac{k}{k+1}}+\frac{1}{\frac{z}{b}-\frac{k+2}{k+1}}\right\}$$

$$+iUb\frac{2}{\frac{z}{b}-1}\left\{\frac{\pi^2}{6}-\sum_{k=0}^{n}\frac{1}{(k+1)^2}\right\}. \qquad (23)$$

The streamlines are shown in figure 3a. The computations were made with $n=3$. The streamlines for figure 3b were obtained by superposing a velocity $-U$ on the cylinder and the liquid.

The streamlines for liquid flowing around a cylinder parallel to a plane wall are shown in figure 4. Equation (19) was used with $n=1$, $\alpha=\pi/2$, $b/f=1/4$.

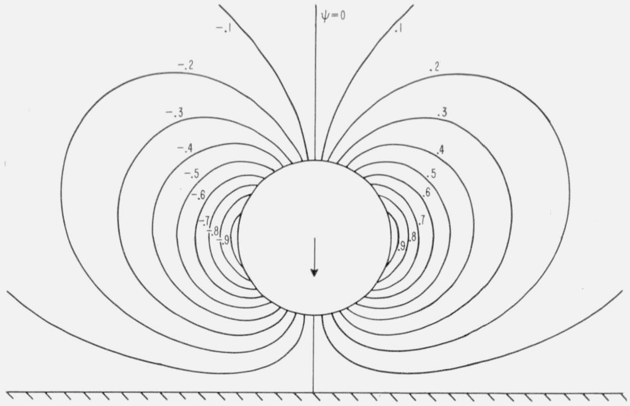

FIGURE 2. *Streamlines for cylinder moving toward a plane boundary.*

$U=1$, $b=1$.

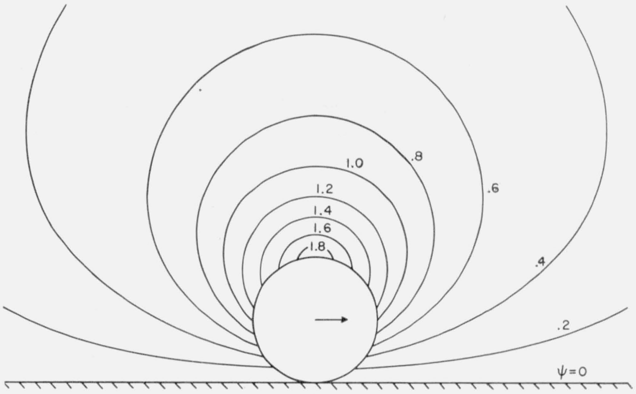

FIGURE 3a. *Streamlines for cylinder moving parallel to a plane boundary when the cylinder touches the boundary.*

$U=1$, $b=1$.

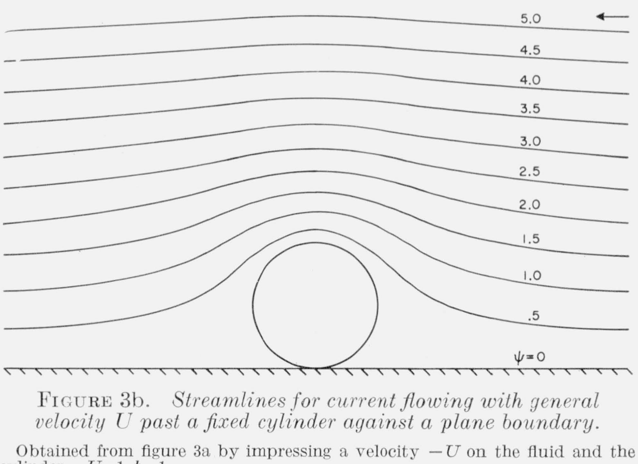

FIGURE 3b. *Streamlines for current flowing with general velocity U past a fixed cylinder against a plane boundary.*

Obtained from figure 3a by impressing a velocity $-U$ on the fluid and the cylinder. $U=1$, $b=1$.

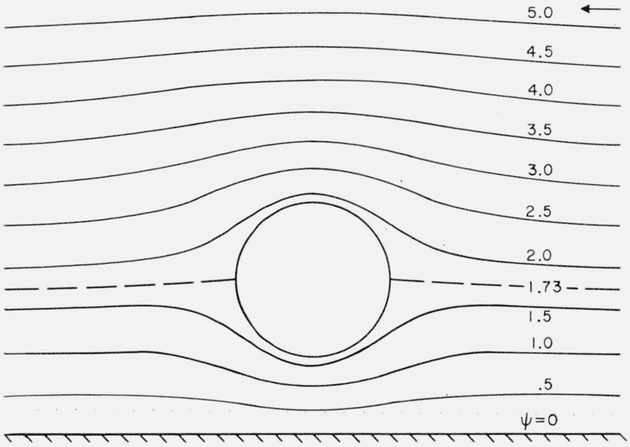

FIGURE 4. *Streamlines for current flowing with general velocity U past a fixed cylinder near a plane boundary.*

$U=1$, $b=1$.

6. Cylinder Near Two Plane Boundaries

If four cylinders of equal radii have symmetric positions and velocities with respect to two perpendicular planes as indicated in figure 5, then each plane is a streamline and the solution represents the flow induced by a cylinder moving in the presence of two perpendicular plane boundaries. Considering the case $f=4a$, $g=4a$, $\alpha=0$, an approximate solution is obtained by the method of section 4 stopping after the first set of images. The streamlines are shown in figure 6.

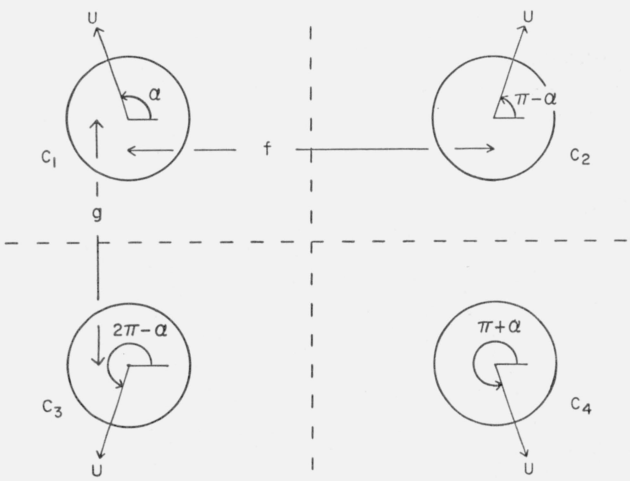

FIGURE 5. *Notation diagram for motion of a cylinder near two perpendicular plane boundaries.*

The dotted lines indicate the streamlines that can be considered as solid boundaries.

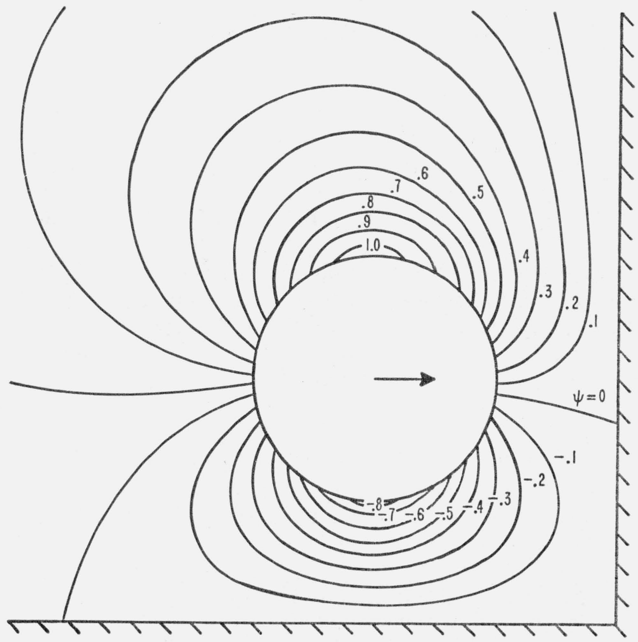

FIGURE 6. *Streamlines for a cylinder moving toward one of two perpendicular plane boundaries.*

$U=1$, $b=1$.

7. Cylinders Moving as a Rigid Body

If the velocity, $Ue^{i\alpha}$, is the same for each cylinder, they move as a rigid body. The previous solution for a cylinder moving parallel to a plane boundary is one example of this case. Another example is given in figure 7 where $b'=b=f/2$, $\alpha'=\alpha=0$. The solution simplifies to

$$w=Ub\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}\left\{\frac{1}{\frac{z}{b}-\frac{k}{k+1}}+\frac{1}{\frac{z}{b}-\frac{k+2}{k+1}}\right\}. \tag{24}$$

The approximate solution obtained by the method of section 5 is

$$w\approx Ub\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)^2}\left\{\frac{1}{\frac{z}{b}-\frac{k}{k+1}}+\frac{1}{\frac{z}{b}-\frac{k+2}{k+1}}\right\}$$
$$+Ub\frac{2}{\frac{z}{b}-1}\left\{\frac{\pi^2}{12}-\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)^2}\right\}. \tag{25}$$

The computations were made with $n=3$.

Another example is given in figures 8a and 8b where $b'=b=f/4$, $\alpha'=\alpha=0$. The solution was obtained using eq (19) with $n=1$.

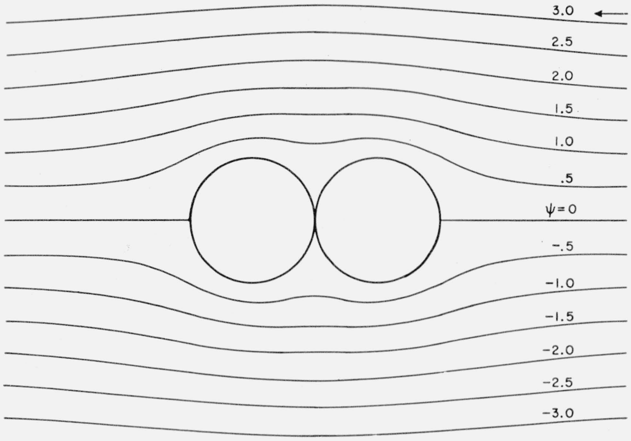

FIGURE 7. *Streamlines for current flowing with general velocity U past fixed cylinders.*
$U=1, b=1.$

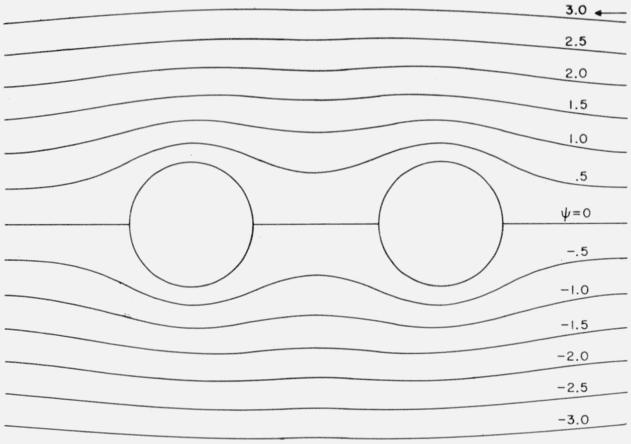

FIGURE 8a. *Streamlines for two cylinders moving as a rigid body.*
$U=1, b=1.$

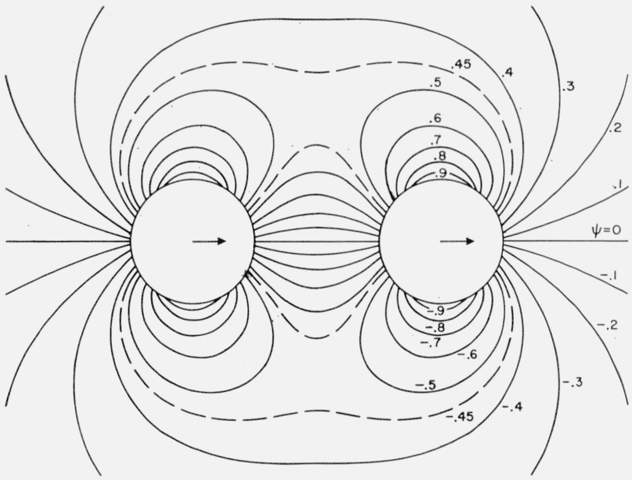

FIGURE 8b. *Streamlines for current flowing with general velocity U past fixed cylinders.*
$U=1, b=1.$

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