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Comparison of Theoretical and Empirical Relations Between the Shear Modulus and Torsional Resonance Frequencies for Bars of Rectangular Cross Section

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The relations between the modulus of elasticity in shear and the fundamental torsional resonance frequency, mass, and dimensions for bars of rectangular cross section have been evaluated experimentally. The empirical relation was found to be less than the theoretical approximation given by Pickett by an amount increasing to about 1 3/4 percent as the crosssectional width to depth ratio of the bars approached 10.

In addition to the fundamental torsional resonance frequency, the first overtone of the specimens was also determined. The overtone was found not to be an exact multiple of the fundamental; it increased more than 5 percent over double the value of the fundamental as the width to depth ratio increased to 10.

1. Introduction

The exact relation between the modulus of elasticity in shear and the torsional resonance frequency for bars of various cross-sectional shapes is of considerable practical as well as theoretical importance. For certain cross-sectional shapes, including circular and square, the relations for the shape factors involved in determining the shear modulus from the angular deformation have been rigorously developed.[2] Consequently, it is possible to derive an exact expression for the shear modulus as a function of the torsional resonance frequency for these shapes, as is done by Pickett.[3] For a rectangular cross section, however, the situation is not so satisfactory. For this cross section, Roark gives a simplified equation for the shape factor which, he states, involves an approximation resulting in an error not greater than 4 percent. Pickett's shear modulus-torsional frequency equation for this shape, based on Roark's equation, is therefore also approximate, as Pickett notes. Cady[4] also gives an approximate equation relating the shear modulus to the torsional frequency for rectangular bars. It will be shown later that Pickett's and Cady's equations, although different in appearance, lead to essentially the same numerical results.

This lack of a more exact expression for bars of rectangular cross section is unfortunate. This shape of bar can be easily fabricated for most materials and, in addition, lends itself to the experimental excitation of torsional vibration more easily than other simple shapes. Indeed, it is sometimes the only simple shape for which this can be accomplished.

Modern refinements in the sonic method permit the determination of resonance frequencies to a high degree of accuracy (see section 3.3). It would be desirable, then, to develop a relationship between the torsional resonance frequency and the shear modulus that would be comparable in accuracy with the determination of the resonance frequency itself. The main purpose of this paper is to establish such a relationship empirically and to compare this empirical relationship with the approximate theoretical ones given by Pickett and Cady. From here on, since only torsional resonance frequencies are discussed, the term "resonance frequency" always refers to the torsional resonance frequency.

2. Theory

The general form of the relationship between the shear modulus and the resonance frequency is given by Pickett as follows:

$$G=Bmf^2, \tag{1}$$

where m is the mass of the specimen in grams, f is the resonance frequency in cps, and G is the shear modulus in dynes/cm^2 (G is often given in kilobars, where 10^9 dynes/cm^2=1 kilobar). B, in cm^{-1}, is related to the shape in the following manner:

$$B=\frac{4lI_p}{n^2aK}, \tag{2}$$

where[5]

l=length (cm),
a=cross-sectional area,
n=the order of vibration (for the fundamental n=1, first overtone n=2, etc.),
I_p=polar moment of inertia of cross-sectional area,[6]
K=shape factor for same cross section (see footnote 2).

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[2] See, for instance, Raymond J. Roark, Formulas for stress and strain, p. 166, 2d ed. (McGraw-Hill Publishing Co., Inc., New York, N. Y., 1943).
[3] Gerald Pickett, Equations for computing elastic constants from flexural and torsional resonant frequencies of vibration of prisms and cylinders, Am. Soc. Testing Materials, Proc., **45**, 846 (1945).
[4] Walter Guyton Cady, Piezoelectricity, p. 114, 1st ed. (McGraw-Hill Publishing Co., Inc., New York, N. Y., 1946).

[5] The cgs system used in this paper eliminates g, the acceleration of gravity from the denominator of the right-hand side of eq (2), making the values for B, and hence G, independent of this factor. Conversion from kilobars to psi's may be accomplished by means of the equation

$$\text{Kilobars}\times 14.5038\times 10^3=\text{psi}.$$

The conversion factor assumes a value of g=980.66 cm/sec^2.
[6] See pages 10 and 68 of reference in footnote 2.

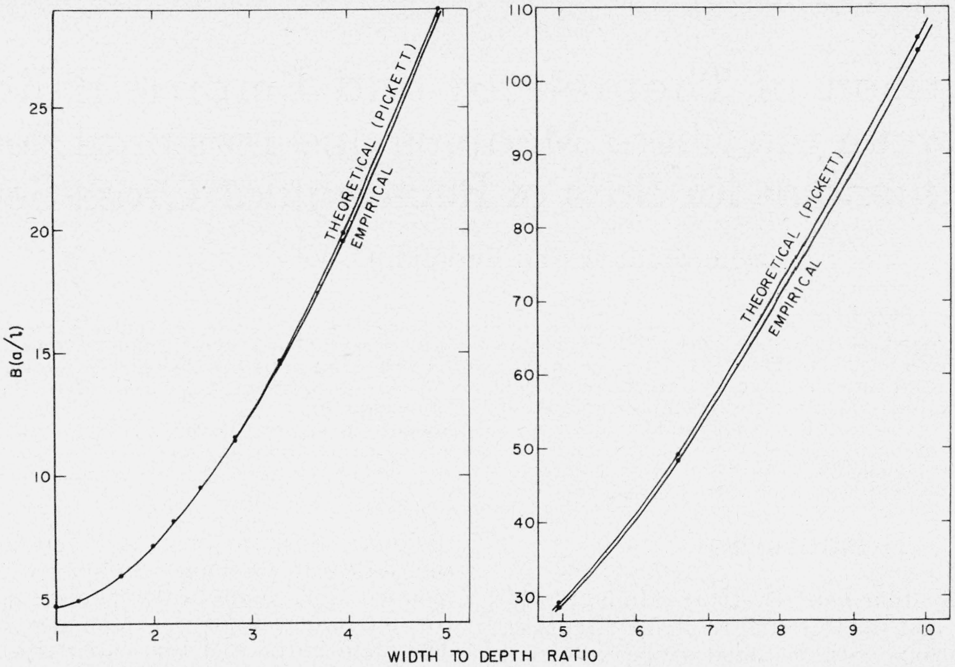

FIGURE 1. *Comparison of theoretical and empirical values of $B(a/l)$.*

The ratio I_p/K, designated as R, is exactly equal to 1.0 for a circular cross section. For a square cross section $R=1.185+$. The value of R for a square cross section is based on Roark's rigorous solution for the shape factor and is accurate to the number of places given. For a rectangular cross section, the following approximate expression for R is given:

$$R=\frac{\left(\frac{w}{d}\right)+\left(\frac{d}{w}\right)}{4\left(\frac{d}{w}\right)-2.52\left(\frac{d}{w}\right)^2+0.21\left(\frac{d}{w}\right)^6}. \tag{3}$$

It should be noted that the approximation given by Roark for K for a rectangular cross section does not reduce to the exact expression for a square cross section. For this reason, the value of $R=1.183$, given by Pickett for a square cross section and obtained by substituting $(w/d)=1$ in eq (3), is lower than the exact value of $R=1.185$ obtained by using Roark's exact expression for the shape factor for a square cross section.

The calculation of the factor, B, in eq (1), has been simplified by means of the following procedure:

If only the fundamental is considered $(n=1)$, eq (2) may be rewritten as

$$B\frac{a}{l}=4R. \tag{2a}$$

R has been calculated for about 30 values of w/d from 1 to 10, using eq (3). Substituting these values of R in eq (2a), corresponding values of $B(a/l)$ have been computed. These are plotted as the upper curve of each section of figure 1. From this graph, it is possible to determine $B(a/l)$ and, consequently, B as a function of w/d, thus eliminating the cumbersome computation of R by eq (3).

The equation given by Cady for the relation between resonance frequency and G has the form

$$f=\frac{Adn}{l\sqrt{w^2+d^2}}\sqrt{\frac{G}{\rho}}, \tag{4}$$

where ρ=density in g/cm^3, A is a shape factor different from K, and other symbols have the same significance and units as in previous equations. Then

$$G=\frac{l(w^2+d^2)}{A^2d^3wn^2}mf^2. \tag{4a}$$

Let

$$B=\frac{l(w^2+d^2)}{A^2d^3w}$$

and

$$n=1,$$

then

$$B\frac{a}{l}=\frac{1}{A^2}\left[\left(\frac{w}{d}\right)^2+1\right], \tag{5}$$

where $A^2=3K_1$. Values for K_1 are given by Timoshenko and Goodier.[7] Table 1 gives the values of $B(a/l)$ from Pickett's and Cady's equations for comparison.

TABLE 1. *Values of B (a/l) from Pickett's and Cady's equations*

w/d	$B(a/l)$	
	Pickett	Cady
1.0	4.734	4.742
1.2	4.92	4.90
1.5	5.55	5.53
2.0	7.27	7.28
2.5	9.66	9.71
3.0	12.66	12.67
4.0	20.17	20.17
5.0	29.71	29.78
10.0	107.8	107.9

The values of $B(a/l)$ are seen to be in good agreement. Cady states that his equation loses accuracy for $(w/d)<3$.[8] However, for $w/d=1$, Cady's value of $B(a/l)=4.742$ is identical with the one based on Roark's rigorous expression for the shape factor. This value is believed to be more accurate than the corresponding one from Pickett in table 1.

3. Experimental Approach

3.1. Specimens

Specimens consisted of 12 steel bars of rectangular cross section, all cut from the same stock. All of the specimens were ground to the same length (15.202 cm) and, with one exception, to the same width (3.143 cm), while the depths were varied so that the ratio of width to depth, w/d, ranged from about 10 to 1. The one exception was the specimen of square cross section, which was 3.150 by 3.150 cm. The dimensions of all specimens were uniform to 0.001 cm. Width over depth ratios for all specimens are to be found in column 2 of table 3.

Inasmuch as the evaluation of G required that the mass as well as the dimensions of the specimens be known, the necessary data to calculate the density was available. This served as an internal check on the homogeneity of the specimens and the consistency of the data. The average value of the density of the 12 specimens was found to be 7.814 g/cm^3 with an extreme variation of ± 0.003 and standard deviation of about 0.002.

3.2. Procedure

The fundamental and first overtones of the resonance frequencies of the specimens were determined by a sonic method, taking advantage of certain refinements previously described.[9] These refinements are briefly summarized as follows:

1. Reading the resonance frequency on a frequency counter.
2. Probing with the pickup by hand to establish clearly the mode of vibration at resonance.
3. Supporting the specimens on foam rubber at the nodal points and driving them with a tweeter-type speaker coupled through the air. This method of support and driving provides virtually zero coupling to the driving system, with specimens of the size and mass used here, and results in the truest natural resonance frequencies of the specimens.

An alternate method of driving the seven lighter specimens was also used. In this method, the specimens were suspended from cotton threads, one thread being hung from the pickup and the other thread was hung from a magnetic record-cutting head, which replaced the speaker as the driving unit. The torsional frequencies were obtained by tying the string at opposite edges of the specimen, as shown in the sketch in figure 2. Table 2 lists the fundamental, $f_{n=1}$, and first overtones, $f_{n=2}$, of the resonance frequencies of all the specimens.

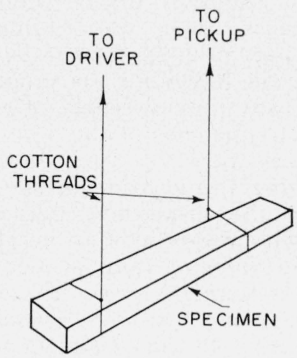

FIGURE 2. *Method of suspending specimen for torsional vibrations.*

TABLE 2. *Fundamental and first overtones of the resonance frequencies of all specimens*

Specimen	Fundamental, $f_{n=1}$ (cps)	First overtone, $f_{n=2}$ (cps)	$\left[\frac{f_{n=2}}{f_{n=1}}-2\right]\times 100$
			%
1	9797	19585	-0.0927
2	9584	19160	$-.084$
3	8725	17465	$+.17$
4	7958	15966	.63
5	7477	15025	.95
6	6920	13939	1.43
7	6294	12709	1.92
8	5589.1	11333	2.77
9	4822.2	9789	3.19
10	3981.4	8115	3.84
11	3069.3	6268	4.23
12	2091.3	4289.1	5.12

3.3. Accuracy

The over-all damping of the entire system, consisting of specimen, driver, pickup, and coupling mechanism, was so small that the accuracy of the resonance frequency measurements approached the accuracy of the frequency counter. This is 1 cycle for the 1-sec gate of the counter, and 0.1 cycle, using

[7] S. Timoshenko and J. N. Goodier, Theory of elasticity, p. 277, 2d ed. (McGraw-Hill Publishing Co., Inc., New York, N. Y., 1951).
[8] Both Roark's and Cady's equations appear to be approximations based on a power-series expansion given by Timoshenko and Goodier, p. 278, eq (160).
[9] Sam Spinner, Elastic moduli of glasses by a dynamic method. J. Am. Ceram. Soc. **37**, 229 (1954).

the 10-sec gate. Generally, the 1-sec gate was sufficient for the specimens of low value of w/d, i. e., those having a high resonance frequency, whereas the 10-sec gate was used for specimens having high values of w/d (low resonance frequency). An additional small source of error is introduced by the coupling mechanism. For those specimens vibrated by the suspension method, it was found that although the resonance frequencies were reproducible up to the maximum accuracy of the counter for any particular given position of the threads on the specimen, nevertheless, the fundamental resonance frequencies were found to vary by about 2 cycles, depending on whether the threads were placed near or far from the nodes; the highest values resulted when the threads were nearest the nodes.[10] The frequencies obtained by driving through air (given in table 2) were usually intermediate between the highest and lowest values for the suspended specimens. In general, the bulkier the specimens, the less the measured resonance frequencies were affected by the mode of coupling or the position of the supports. The accuracy of the fundamental resonance frequencies, then, was estimated to range from about 1 part in 2,000 for the flattest specimen to about 1 part in 10,000 for the square specimen.

To evaluate any possible effect of air damping on the resonance frequencies of the specimens, some of the flatter bars were vibrated, both in air and, without changing the position of the fibers, also in vacuum, using the suspension method. The resonance frequencies were found to be about 1 part in 6,000 higher in vacuum than in air. Because this variation is less than the error in measurement for the flatter specimens, it was not considered sufficient to justify vibrating all the specimens in this manner.

[10] The entire problem of these small variations in measured resonance frequencies with differences in suspension position requires further study. For lighter and smaller specimens than those used in this investigation, the suspension method yields even more reliable results than does the method of air coupling. Also, for the most accurate results, specimens should be vibrated in vacuum (see next paragraph of text). The suspension method is easily adapted to this. For flexural vibrations, in contrast with torsional, the highest frequencies are obtained when the supports are placed furthest from the nodes. In all cases it is believed that the truest resonance frequencies are obtained when the driving and pickup fibers are nearest the nodes.

3.4. Calculations

Comparison of the experimental results will first be made with the curve based on Pickett's equation. The empirical (lower) curve shown in figure 2 was obtained in the following manner: For some particular value of w/d say $(w/d)=1$,

$$G=B_1'\frac{l_1}{a_1}m_1f_1^2,$$

where $B'=B(a/l)$ and the subscripts indicate particular values for $w/d=1$. For any other value of w/d, the specimen will have the same value of G, and all the terms on the right-hand side have the subscript 2 thus,

$$G=B_2'\frac{l_2}{a_2}m_2f_2^2.$$

Therefore,

$$\frac{B_2'}{B_1'}=\frac{m_1}{m_2}\cdot\frac{l_1}{l_2}\cdot\frac{a_2}{a_1}\left(\frac{f_1}{f_2}\right)^2. \tag{6}$$

All the members on the right-hand side are known and B_2'/B_1' may be evaluated. Values of this ratio are given in column 3 of table 3.

In order to obtain $B(a/l)$ (experimental) directly as a function of w/d, it is first necessary to select a reliable base value of $B(a/l)$ for some particular value of w/d. It has already been shown that the value of $B(a/l)=4.742$ for $w/d=1$ is believed most accurate. This number then, is selected as the base value for $w/d=1$. Values of $B(a/l)$ (experimental) for higher w/d ratios are then obtained by multiplying this base value by the B_2'/B_2', ratio for the corresponding value of w/d. These are given in column 5 of table 3.

It should be noted that the choice of a "base" value for $B(a/l)$ (experimental) is somewhat arbitrary. Should it develop that some other value of $B(a/l)$ either at $w/d=1$, or at some other w/d ratio, is more

TABLE 3. *Theoretical and empirical data for rectangular bars of varying w/d ratios*

[Data rounded off to final figure. Calculations are on basis of more significant figures]

Specimen	$\frac{w}{d}$	$\frac{B_2'}{B_1'}$	$B(a/l)$ Theoretical	$B(a/l)$ Experimental	$\left[\frac{B\frac{a}{l}\text{ (theor.)}}{B\frac{a}{l}\text{ (exp.)}}-1\right]\times 100$	A^2
					%	
1	1	1	4.734	4.742	−0.17	0.4218
2	1.237	1.045	4.973	4.955	+.36	.51065
3	1.650	1.261	5.979	5.981	−.03	.6222
4	1.980	1.516	7.198	7.186	+.17	.6847
5	2.198	1.7175	8.162	8.144	.23	.7160
6	2.473	2.005	9.544	9.505	.40	.7489
7	2.827	2.424	11.563	11.495	.59	.7822
8	3.298	3.074	14.681	14.577	.71	.8148
9	3.957	4.130	19.815	19.583	1.18	.8509
10	4.938	6.060	29.096	28.734	1.26	.8834
11	6.586	10.198	49.068	48.352	1.48	.9177
12	9.908	21.959	105.90	104.12	1.71	.9524

accurate, then a readjustment could easily be made in column 5, using the more basic data from column 3.

The data in column 5 was obtained without assuming any particular value for G for the specimens, but merely that all the specimens had the same value of G. An alternative method for calculating the data in column 5 may be used if, in addition to assuming that the value of G of all the specimens is the same, a definite value of G is derived for some particular value of w/d. Thus, using $B(a/l)=4.742$ for $w/d=1$, and substituting in eq (1) and (2), leads to a value of 822.1 kilobars for G. Then, resubstituting this value in the same two equations, one obtains the following expression, from which $B(a/l)$ (experimental) for the corresponding w/d values may be calculated,

$$B\frac{a}{l}\,(\text{experimental})=\frac{822.1a}{lmf^2}. \tag{7}$$

Comparison between theoretical and experimental curves can now also be made in terms of A^2, used in Cady's equations. A^2 is evaluated empirically by substituting $B(a/l)$ (experimental) and the corresponding value of w/d in eq (5). The resulting values of A^2 are given in column 7 of table 3 and are plotted as a function of w/d in the upper curve of figure 3. The lower curve in figure 3 is a plot of the factor $3K_2(=A^2$ "theoretical") against the selected values of w/d from Timoshenko and Goodier (given in table 1).

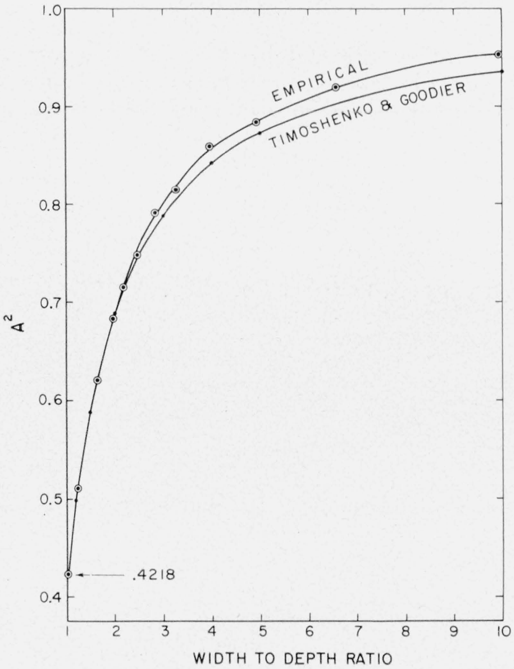

FIGURE 3. *Comparison of empirically determined shape factor, A^2, with that obtained by using Timoshenko and Goodier's values of K_1, where $A^2=3K_1$.*

4. Results and Discussion

The empirical curves of figures 1 and 3, along with their appropriate equations, should yield the same values for the shear modulus. Also, since the theoretical curves based on Pickett's and Cady's equations yield similar results, the percentage difference between the theoretical and empirical curves should be of the same magnitude in both cases. These observations are shown to hold in figures 1 and 3. It is also noted that in figure 1, at high values of w/d, a given change in abscissa is associated with a relatively large change in ordinate, whereas in figure 3 this occurs at low values of w/d. Therefore, for graphs of comparable size, the empirical curve in figure 1 would yield more precise results at low values of w/d, and conversely figure 3 would yield the more precise results at high w/d ratios.

Column 6 of table 3 gives the percentage by which $B(a/l)$ (theoretical) from Pickett is larger than $B(a/l)$ (experimental). The values of $B(a/l)$ (theoretical) are given to the same number of significant figures as $B(a/l)$ (experimental) for comparison. Figure 4 illustrates this variation between the theoretical and experimental curves. It is seen that, using the fundamental mode of vibration, Roark's estimate of an error "not greater than 4 percent" is quite justified—even conservative, the actual deviation being less than half Roark's figure up to $w/d=10$. Furthermore, the appearance of the curves in figures 1 and 3 suggests that at higher values of w/d the deviation will not increase significantly above that already observed.

The empirical curves of figures 1 and 2, and the data on which they are based, even though obtained by using steel specimens, do not lose generality and are applicable to any elastic solid. Some random checks comparing the theoretical and empirical curves of figures 1 and 3 were made by using glass specimens.

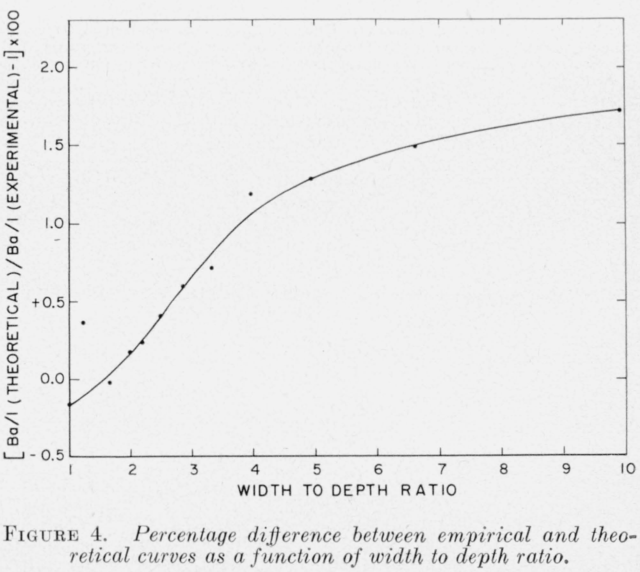

FIGURE 4. *Percentage difference between empirical and theoretical curves as a function of width to depth ratio.*

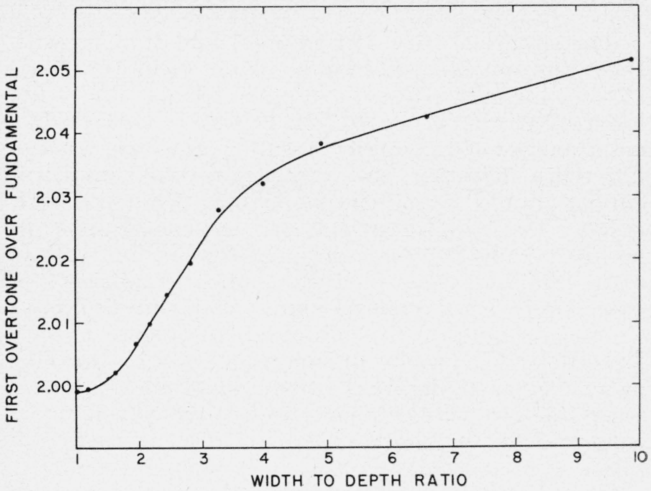

FIGURE 5. *Ratio of first overtone of torsional vibration over fundamental overtone as a function of width to depth ratio.*

Length 15.202 cm.

Results showed that for the same glass, a more nearly consistent value of the shear modulus was obtained for different values of w/d when the empirical curves were used.

It is recalled that, in addition to the fundamental, the first overtones of the resonance frequencies were also determined. Column 4, table 2, gives the percentage variation of the first overtone, $f_{n=2'}$ from the exact double of the fundamental, $f_{n=1'}$, and figure 5 shows the same data graphically as a function of w/d.

Giebe and Blechschmidt [11] resonated quartz bars at the fundamental and higher overtones of the torsional resonance frequency. They also found that the harmonic law did not hold. Their results for the first overtone are in general agreement with those obtained here in that the deviation from the exact double of the fundamental was usually positive and the amount of deviation tended to increase as the w/d ratio increased.

[11] E. Giebe and E. Blechschmidt, Uber Drillungsschwingungen von Quarzstäben und ihre Benutzung für Frequenznormale, Hochfrequentztechnik und Electroakustik—Jahrbuch der drahtlosen Telegraphie und Telephonie 56 (3) 65–87 (1940).

Equations (2) and (4a) show that the overtones of vibration are taken to be whole-number multiples of the fundamental. Use of the term n^2 in the denominator is merely a device for finding the fundamental frequency on the assumption that the overtones are whole-number multiples of the fundamental. The present investigation of the overtones is obviously far from complete, inasmuch as it was determined only for a single length and for one overtone. However, there is already sufficient evidence to show that the belief that the overtone is a whole-number multiple of the fundamental is unjustified. At a w/d ratio of 10, the first overtone is more than 5 percent higher than the exact double of the fundamental. If this were overlooked, it would lead to an error of about the same amount in the calculation of the shear modulus.

5. Summary

1. Two empirical curves have been presented which can be used to determine the shear modulus of bars of rectangular cross section from their resonance frequencies.

2. These empirical curves show a small but significant difference from their corresponding theoretical approximations. This difference increases to about 1¾ percent for a specimen having a cross-sectional width to depth ratio of 10.

3. The resonance frequencies of the first overtone of vibration of a rectangular bar is found to deviate from the harmonic law by more than 5 percent as the ratio of width to depth reaches 10.

The authors are indebted to W. Capps of the Bureau for his assistance in performing many of the calculations involved in preparing table 3.

WASHINGTON, October 28, 1957.