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Random Notes on Matrices

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Three matrix problems are considered in this paper: Bessel functions as limits of determinants, finding all optimal strategies of a matrix game with nonzero value, and conditions for matrices to have equal principal minors.

The three sections of this paper have little relation except that they each concern matrices and each was suggested by the work of a colleague at the National Bureau of Standards.

In the first section we prove that the limit of a certain sequence of determinants is the modified Bessel function $\Lambda_s(x)$.

In the second section we show that a method for computing a pair of optimal strategies for certain square matrix games yields the unique optimal strategy for one player if the optimal strategy for the other player is totally positive. This can result in an appreciable saving in the work of computing all the optimal strategies of an arbitrary matrix game with nonzero value.

The third section contains an amplification of a result which appeared in this journal as well as a necessary and sufficient condition that two matrices have equal corresponding principal minors.

1. *Bessel functions as limits of determinants.* Let (a_{ij}) be an infinite, triple diagonal, symmetric matrix with

$$a_{i,i-1}^2 = a_{11} - a_{ii} = 1 - (2s+1)i^{-1} + i^{-2}p(i^{-1}) \qquad (1.1)$$

for some polynomial $p(t)$ and some real number s which is not a negative integer. Let $D_n(t)$ be the characteristic polynomial of the first n rows and columns of (a_{ij}). We shall prove

THEOREM 1.
$$\lim_{n\to\infty} D_n(a_{11}+1-n^{-2}z^2) = \Lambda_s(z), \qquad |z|<\infty,$$

where $\Lambda_s(z)$ is the Jahnke-Emde modified Bessel function with expansion

$$\Lambda_s(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{4^k k!(s+k)(s+k-1)\ldots(s+1)}, \qquad |z|<\infty.$$

The theorem was suggested in considering a conjecture of J. Todd.

Since (a_{ij}) is triple diagonal and symmetric we can use the first equation in (1.1) to obtain a simple recursion for $D_n(t)$. We have $D_0(t)=1$, $D_1(t)=t-a_{11}$ and

$$D_n(t) = (t-a_{nn})D_{n-1}(t) - (a_{11}-a_{nn})D_{n-2}(t) \qquad n=2, 3, \ldots. \qquad (1.2)$$

Thus

$$D_n(a_{11}+1) = 1.$$

We shall expand $D_n(a_{11}+1-n^{-2}z^2)$ around $a_{11}+1$, and to this end we define

$$D_n^{(k)} = D_n^{(k)}(a_{11}+1).$$

By repeated differentiation of equation (1.2) we get the following recursion for $D_n^{(k)}$: $D_n^{(0)} = D_1^{(1)} = 1$, $D_n^{(k)} = 0$ if $k>n$ and

$$D_n^{(k)} - 2D_{n-1}^{(k)} + D_{n-2}^{(k)} + n^{-1}(2s+1)(D_{n-1}^{(k)} - D_{n-2}^{(k)}) - n^{-2}p(n^{-1})(D_{n-1}^{(k)} - D_{n-2}^{(k)}) = kD_{n-1}^{(k-1)}$$

for $k=1,2,\ldots,n$ and $n=2,3,\ldots.$ This recursion can be solved for $D_n^{(k)}$ yielding

$$D_n^{(k)} = n^{2k}4^{-k}\prod_{j=1}^{k}(j+s)^{-1} + n^{2k-1}d_k + n^{2k-2}q_k(n^{-1})$$

for all n and $k \leq n$, some polynomial $g_k(t)$ of degree $2k-2$, and

$$d_k = 4^k \binom{2k}{k} \frac{\Gamma(k+s+1)\Gamma(s+1)}{\Gamma(2k+2s+1)} \sum_{j=1}^{k} 4^{-2j} \binom{2j}{j} \frac{\Gamma(2j+2s-1)}{\Gamma(j+s)\Gamma(j+s+1)} \frac{(2j+1)(2s+1)+2p(0)}{2j-1}$$

for all k. All that we need to know about d_k is that it is bounded and this is clear from the most crude inequalities. Let d be the bound:

$$|d_k| \leq d, \qquad k=1,2, \ldots .$$

Denote the partial sums of $\Lambda_s(z)$ by

$$\Lambda_{sn}(x) = \sum_{k=0}^{n} \frac{(-1)^k z^{2k}}{4^k \cdot k!(s+k)(s+k-1) \ldots (s+1)}.$$

Then we have

$$D_n(a_{11}+1-n^{-2}z^2) - \Lambda_{sn}(z) = \sum_{k=0}^{n} \frac{(-1)^k z^{2k}}{k!} \left\{ \frac{D_n^{(k)}}{n^{2k}} - \frac{1}{4^k(s+k)(s+k-1) \ldots (s+1)} \right\}$$

$$= \sum_{k=1}^{n} \frac{(-1)^k z^{2k}}{k!} \left(\frac{d_k}{n} + o(n^{-1}) \right),$$

so that

$$D_n(a_{11}+1-n^{-2}z^2) - \Lambda_{sn}(z) \leq n^{-1} d e^{|z|^2} + o(1), \qquad |z| < \infty .$$

Since $\Lambda_{sn}(z)$ tends to $\Lambda_s(z)$ for all $|z| < \infty$ as n gets large the theorem is proved.

Note that

$$\Lambda_{-1/2}(z) = \cos z,$$

$$\Lambda_{1/2}(z) = z^{-1} \sin z,$$

and in general

$$\Lambda_s(z) = \Gamma(s+1)(z/2)^{-s} J_s(z),$$

where $J_s(z)$ is the Bessel function of order s. Thus our theorem can be interpreted as meaning that if $z_1, z_2, \ldots, z_n$ are the n smallest roots of $J_s(z)$ then the roots of $D_n(t)$ are approximated by

$$a_{11}+1-n^{-2}z_i^2 \qquad i=1,2, \ldots, n,$$

with the best approximations being for the smaller z_i.

2. *On finding all optimal strategies of a matrix game with nonzero value.* Let B be a payoff matrix with nonzero value. Let K_1 and K_2 be the extremal sets of the optimal strategy spaces of the row player and the column player, respectively.

A result due to L. S. Shapley and R. N. Snow [1] states

THEOREM. Let x and y be optimal strategies for the row player and the column player respectively. A necessary and sufficient condition that $x \epsilon K_1$ and $y \epsilon K_2$ is that there exist a nonsingular submatrix A of B of order n such that, if $\bar{e}$ is the 1 by n vector of all ones,

$$\begin{aligned} v &= [\bar{e} A^{-1} \bar{e}^T]^{-1} \\ \bar{x} &= v \bar{e} A^{-1} \\ \bar{y} &= v \bar{e} (A^T)^{-1} \end{aligned} \tag{2.1}$$

where v is the value of the game, $\bar{x}$ is the vector obtained from x by deleting the elements corresponding to the rows deleted to obtain A from B, and $\bar{y}$ is the vector obtained from y by deleting the elements corresponding to the columns deleted to obtain A from B.

We shall prove that if either $\bar{x} > 0$ (that is every element in $\bar{x}$ is positive) or $\bar{y} > 0$ then the other represents the unique optimal strategy for A for the corresponding player. In other words if, say, $\bar{x}>0$ then no element of K_2 other than y can be obtained from any submatrix of A.

This result can be used to reduce the number of submatrices of B that have to be inspected to exhaust K_1 or K_2.

Our result takes the following form:

THEOREM 2. Let A be a nonsingular payoff matrix of order n such that the nonzero row and column sums of A^{-1} are all of the same sign. If none of the row (column) sums of A^{-1} are zero then the row (column) player has a unique optimal strategy.

We shall say that 1 by n vectors $\bar{x}_0$ and $\bar{y}_0$ represent optimal strategies for the row player and the column player of A respectively if

$$\bar{x}_0 \geq 0, \qquad \bar{y}_0 \geq 0, \qquad \bar{x}_0\bar{e}^T = \bar{y}_0\bar{e}^T = 1$$

and for some number v_0, the value of the game,

$$\bar{x}_0 A \geq v_0\bar{e} \geq \bar{y}_0 A^T.$$

Let v, $\bar{x}$ and $\bar{y}$ be as in (2.1). Then $\bar{x}\bar{e}^T = \bar{y}\bar{e}^T = 1$ and

$$\bar{x}A = v\bar{e} = \bar{y}A^T.$$

By hypothesis $\bar{x} \geq 0$ and $\bar{y} \geq 0$ so that $\bar{x}$ and $\bar{y}$ represent optimal strategies and v is the value of the game.

We shall assume that $\bar{x}>0$ and prove that $\bar{y}$ represents the unique optimal strategy for the column player. The proof for the other case follows in exactly the same way so this will prove Theorem 2.

Suppose $\bar{y}_0$ is a non-negative 1 by n vector such that $\bar{y}_0\bar{e}^T = 1$ and

$$v\bar{e} \geq \bar{y}_0 A^T. \tag{2.2}$$

Since $\bar{x}>0$ we can find a 1 by n vector $\bar{u}$ such that the matrix

$$X = \bar{x}^T\bar{u} - (A^T)^{-1}$$

is non-negative. Then (2.2) implies

$$v\bar{e}X \geq \bar{y}_0 A^T X.$$

Computing these two vectors we get

$$v\bar{e}X = v\bar{e}\bar{x}^T\bar{u} - v\bar{e}(A^T)^{-1} = v\bar{u} - \bar{y}$$

and

$$\bar{y}_0 A^T X = \bar{y}_0 A^T \bar{x}^T \bar{u} - \bar{y}_0 A^T (A^T)^{-1} = v\bar{y}_0\bar{e}^T\bar{u} - \bar{y}_0 = v\bar{u} - \bar{y}_0.$$

Therefore we have $\bar{y}_0 \geq \bar{y}$. Since the sum of the elements of these two vectors are equal we must have $\bar{y}_0 = \bar{y}$, as desired.

3. *Matrices with equal principal minors.* In a previous paper in this journal [2] we stated a result which is a special case of the following

THEOREM 3. Let (a_{ij}) be a matrix of order n with complex elements and the properties

$$a_{ij}a_{ji} \text{ is real and non-negative}, \qquad i, j = 1, 2, \ldots, n, \tag{3.1}$$

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} = \overline{a_{i_2 i_1} a_{i_3 i_2} \cdots a_{i_1 i_k}}, \tag{3.2}$$

for all $k=1,2, \ldots, n$ and $i_j = 1,2, \ldots, n$. Then (a_{ij}) has the same principle minors as the Hermitian matrix (b_{ij}) with

$$b_{ij} = (\operatorname{sgn} \mathscr{R} a_{ij})(a_{ij}\bar{a}_{ji})^{1/2} \tag{3.3}$$

and so has only real characteristic roots.

The symbol $\bar{x}$ means the complex conjugate of x, and the symbol $x^{1/2}$ means that square root of x with non-negative real part.

We shall prove this theorem by showing that

$$b_{i_1 i_2} b_{i_2 i_3} \ . \ . \ . \ b_{i_k i_1} = a_{i_1 i_2} a_{i_2 i_3} \ . \ . \ . \ a_{i_k i_1} \tag{3.4}$$

for all $k=1,2, \ . \ . \ . \ ,n$ and $i_j=1,2, \ . \ . \ . \ ,n$ and the result will follow as in the previous paper.

From (3.1) we see that either $a_{ij}a_{ji}=0$ or

$$a_{ji}=c_{ij}\overline{a_{ij}}$$

for some positive real c_{ij}. These c_{ij} satisfy

$$c_{ij}=c_{ji}^{-1}$$

and, from equation (3.2),

$$c_{i_1 i_2} c_{i_2 i_3} \ . \ . \ . \ c_{i_k i_1}=1 \tag{3.5}$$

whenever $a_{i_1 i_2} a_{i_2 i_3} \ . \ . \ . \ a_{i_k i_1}$ does not vanish.

Therefore, either $b_{ij}=b_{ji}=0$ or,

$$b_{ij}=c_{ij}^{1/2}a_{ij}, \tag{3.6}$$

so that

$$\overline{b_{ij}}=c_{ij}^{1/2}\overline{a}_{ij}=c_{ij}^{-1/2}a_{ji}=c_{ji}^{1/2}a_{ji}=b_{ji}$$

as desired to prove that (b_{ij}) is Hermitian.

Choose an arbitrary ordered subset $i_1,i_2, \ . \ . \ . \ ,i_k$ of $\{1,2, \ . \ . \ . \ ,n\}$. Let $b=b_{i_1 i_2} b_{i_2 i_3} \ . \ . \ . \ b_{i_k i_1}$, $a=a_{i_1 i_2} a_{i_2 i_3} \ . \ . \ . \ a_{i_k i_1}$, $a'=a_{i_2 i_1} a_{i_3 i_2} \ . \ . \ . \ a_{i_1 i_k}$.

If b vanishes then, by (3.3), either a or a' vanishes. By (3.2) this means that a vanishes, so $b=a$.

If b does not vanish we have

$$b=c_{i_1 i_2}^{1/2} c_{i_2 i_3}^{1/2} \ . \ . \ . \ c_{i_k i_1}^{1/2} a_{i_1 i_2} a_{i_2 i_3} \ . \ . \ . \ a_{i_k i_1}=a$$

by combining equations (3.5) and (3.6).

Therefore $b=a$ in general and we have proved equation (3.3) and so the theorem.

We are naturally led from this theorem to ask for a necessary and sufficient condition that two matrices have the same principal minors. We shall find such a condition based on a little known expansion of the determinant of a matrix.

Let $A=(a_{ij})$ be a matrix of order n. Choose any set of k distinct indices $\{i_1,i_2, \ . \ . \ . \ ,i_k\}$ with $1 \leq i_j \leq n$. Then define the symbols

$$\{i_1\}_A=a_{i_1 i_1}$$

if $k=1$, and (3.7)

$$\{i_1,i_2, \ . \ . \ . \ ,i_k\}_A=\sum_{\sigma} a_{i_1 i_{\sigma(1)}} a_{i_{\sigma(1)} i_{\sigma^2(1)}} \ . \ . \ . \ a_{i_{\sigma^{k-1}(1)} i_1}$$

if $k=2,3, \ . \ . \ . \ ,n$, where the sum is taken over all full cycles σ on $\{1,2, \ . \ . \ . \ ,k\}$.

Given an arbitrary partition of $\{1,2, \ . \ . \ . \ ,n\}$

$$\tau=\{i_{11},i_{12}, \ . \ . \ . \ ,i_{1k_1}\}\{i_{21},i_{22}, \ . \ . \ . \ ,i_{2k_2}\} \ . \ . \ . \ \{i_{m1},{}_{m2}, \ . \ . \ . \ ,i_{mk_m}\}$$

define the symbol

$$\tau_A=(-1)^m\{i_{11},i_{12}, \ . \ . \ . \ ,i_{1k_1}\}_A\{i_{21},i_{22}, \ . \ . \ . \ ,i_{2k_2}\}_A \ . \ . \ . \ \{i_{m1},i_{m2}, \ . \ . \ . \ ,i_{mk_m}\}_A. \tag{3.8}$$

Then the determinant of A may be expanded in

$$\det A=(-1)^n \sum \tau_A \tag{3.9}$$

where the sum is taken over all partitions τ of $\{1, 2, \ldots, n\}$.

To prove that this expansion is valid we need only show that the generic term in the usual expansion of the determinant appears in this expansion, that whenever it does appear it does so with the proper sign, and that there are exactly $n!$ such terms.

The generic term in the expansion of the determinant can be denoted by

$$a=(-1)^{\rho} a_{1\rho(1)} a_{2\rho(2)} \ldots a_{n\rho(n)}$$

where ρ is a permutation of degree n and $(-1)^{\rho}$ is $+1$ or -1 as ρ is even or odd respectively.

Write ρ in its unique decomposition into cycles

$$\rho=(i_{11}i_{12} \ldots i_{1k_1})(i_{21}i_{22} \ldots i_{2k_2}) \ldots (i_{m1}i_{m2} \ldots i_{mk_m}).$$

Then the only τ_A in which a can appear is the one described in (3.8). Since $(-1)^{\rho}=(-1)^{n-m}$ the sign is always right. Finally it is clear that a appears at least once in τ_A.

Now, consider the partition $k_1+k_2+\ldots+k_m$ of n. Let s_i be the number of k_j equal to i. Then there are

$$\frac{n!}{s_1!s_2! \ldots s_n!k_1!k_2! \ldots k_m!}$$

partitions τ of $\{1,2, \ldots ,n\}$ with subsets of order $k_1,k_2, \ldots , k_m$. By (3.7) a subset with k_i elements contributes $(k_i-1)!$ summands to τ_A. Thus the total number of summands on the r.h.s. of equation (3.9) is

$$\sum \frac{n!}{s_1!s_2! \ldots s_n!} \frac{(k_1-1)!(k_2-1)! \ldots (k_m-1)!}{k_1!k_2! \ldots k_m!}=\sum \frac{n!}{s_1!s_2! \ldots s_n!1^{s_1}2^{s_2} \ldots n^{s_n}}=n!$$

as desired, the sums being taken over all partitions of n.

Suppose $B=(b_{ij})$ is a matrix of order n and

$$\{i_1,i_2, \ldots , i_k\}_A=\{i_1,i_2, \ldots , i_k\}_B \tag{3.10}$$

for all subsets $\{i_1,i_2, \ldots, i_k\}$ of $\{1,2, \ldots, n\}$. By (3.8) and the expansion (3.9) this implies that the corresponding principle minors of A and B are equal. We shall prove the converse.

Suppose A and B have equal corresponding principal minors. Then (3.10) holds for $k=1$. Suppose it holds for $k=1,2, \ldots, m-1$. Consider the expansion as in (3.9) of the minor with row and column indices $i_1, i_2, \ldots, i_m$. For every τ except $\tau'=\{i_1,i_2, \ldots, i_m\}$ we are summing products of equal sums $\{i_1', i_2', \ldots, i_k'\}_A=\{i_1',i_2', \ldots, i_k'\}_B$ with $k\leq m-1$. Therefore the equality of the minors implies $\tau_A'=\tau_B'$ which completes the proof by induction. We have proved

THEOREM 4. Let A and B be matrices of order n. They have equal corresponding principal minors if and only if equation (3.10) holds for all subsets $\{i_1,i_2, \ldots, i_k\}$ of $\{1,2, \ldots, n\}$.

We remark that theorem 4 (and thus theorem 3 as well) also follows from a result noted by A. Ostrowski (footnote 13 in [3]), which is closely related to the expansion (3.8).

References

[1] L. S. Shapley and R. N. Snow, Basic solutions of discrete games, Contributions to the Theory of Games, vol. I, Annals of Mathematics Study No. 24 (Princeton University Press, 1954).

[2] K. Goldberg, A matrix with real characteristic roots, J. Research NBS **56**, 87 (1956) RP2652.

[3] A. Ostrowski, Determinanten mit überwiegender Hauptdiagonale und die absolute Konvergenz von linearen Iterationsprozessen, Commentarii Mathematici Helvetici **30**, 175 (1956).

WASHINGTON, October 18, 1957.