ON THE VIBRATION OF U BARS

By G. H. Keulegan

ABSTRACT

A theoretical study has been made of elongated and short U bars with special reference to their use as vibrators in investigations on elastic hysteresis.

First, an expression for the frequency of the fundamental mode of vibration of the elongated U bar is derived by solving the differential equations of motion of the yoke and of the prongs through the use of an approximation. The physical basis of the approximation is the fact that the bar vibrates with a pitch differing slightly from that of the clamped-free bar of half the length.

Secondly, Ritz's method of approximation is developed for initially curved bars, the development being based on the principle of least action. In this connection a new proof is given of Rayleigh's method for determining the fundamental mode of vibration. As an illustration of Ritz's method, the example of the free-free bar is treated, and the results of the calculation are compared with the known solution of this problem. Next, the method is used to determine the fundamental mode of vibration of the short U bar, a short U bar being defined as one wherein the length of the curved portion, or the yoke, is equal to the sum of the lengths of the two parallel prongs.

Finally, Rayleigh's method is used to determine the static deformation of the short U bar, produced by the application of a single load so applied that the deformation is most nearly that present in the fundamental mode of vibration.

CONTENTS

		Page
I.	Introduction	553
II.	Formation of the equation of motion of a U bar	554
	1. Displacements and stress components in curved bars	554
	2. Equation of motion of a U bar	559
	3. End conditions and conditions of continuity	561
	4. Condition for the invariancy of the center of gravity	561
III.	The frequency of vibration of an elongated U bar	562
	1. Approximate values of the quantities $n_i a$	562
	2. Frequency of vibration in elongated U bars	565
	3. The deformation of the circular yoke	569
IV.	Ritz's method of approximation	571
	1. Application of Ritz's method to curved bars	571
	2. An illustration from a classical problem	575
V.	Rayleigh's method for determining the fundamental mode of vibration	578
VI.	Vibration of a short U bar	580
VII.	Effect of the yoke on vibration	585
VIII.	Reproduction by a single load of deformations due to vibration	586
IX.	Acknowledgment	595

I. INTRODUCTION

Although hysteresis in elastic materials has engaged in recent times the attention of engineers and physicists alike, the information which is available on the subject is very fragmentary. Progress in this field is hindered by the lack of a proper technique for obtaining results quickly. For example, one of the more obvious methods of securing

data is that of making statical measurements. If, however, this method is used to determine the constants relating to hysteresis which are typical of a given material, it is necessary to test a multitude of samples of the material and to average the results, similarly as in determining the fatigue of materials. Accordingly, the method of statical measurements, being thus of a time-consuming nature, is not ideal.

In looking for a more rapid method, the damping of free oscillations readily suggests itself. Here, however, the choice of the vibrator must be made judiciously since two conditions must be fulfilled. First, the damping of the free oscillations must be due mainly to hysteresis, or internal friction. Second, the shape of the vibrator must be such that it can be used in securing statical measurements, with a view to effecting a comparison between statical and dynamical results.

To satisfy the first condition, the method of support must in no way contribute to the damping. When the nodal points of a vibrator executing vibrations of the fundamental type are in close proximity to each other, and the vibrator is suspended at these points, the damping may be assumed to be due principally to internal friction. It suffices, to see the truth of this statement, to point to the fact that during the motion, the distance between the nodal points varies periodically, and the smaller this distance, the less the effect of the supports on the vibration. In making statical measurements, it is advisable always to deal with deformations brought about by the application of a single load. A statical deformation so produced and duplicating as nearly as possible the dynamical deformation is chosen for direct comparison of the hysteresis during dynamical and statical tests. Only for the fundamental mode of vibration can this equivalence be accomplished by properly choosing the point of application of a single load.

The use of bars which are of U shape as possible vibrators for securing data on elastic hysteresis has many advantages. In this paper it is proposed to study the fundamental mode of vibration of two types of U bar, namely, one which is elongated, and one where the curve and the sum of the straight portions are of equal length. In all cases the yoke will be assumed to be of semicircular shape, and the prongs straight and parallel.

II. FORMATION OF THE EQUATION OF MOTION OF A U BAR

1. DISPLACEMENTS AND STRESS COMPONENTS IN CURVED BARS

Let E_1OE_2 represent the median line of a symmetrically curved bar. (See fig. 1.) The position of a point P on the median line is determined by the distance s of the point from the mid-point O of the curved bar, measured on the median line. s is positive when on the branch OE_1, negative when on the branch OE_2. Choose rectangular axes (x, y) having the mid-point O as origin and the x-direction midway between the prongs. The normal PC to the median line at the point P is drawn, making the angle α with the x-axis. The radius of curvature will be denoted by ρ. Draw the tangent PA making with the x-axis the angle i. The angle i will be spoken of as

the inclination of the bar at P. Expressing x, y as functions of s we have

$$\cos i = \frac{dx}{ds}$$

$$\sin i = \frac{dy}{ds} \tag{1}$$

and

$$\frac{1}{\rho} = -\frac{di}{ds}$$

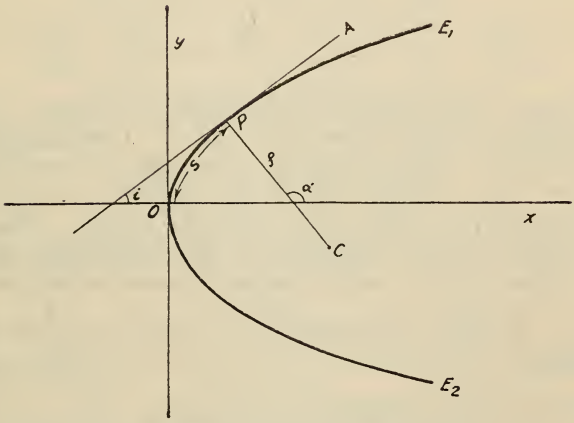

FIGURE 1.—*Median line of a symmetrically curved bar*

Let U and V be the normal and tangential displacements of the point P as it moves to the position P'. (See fig. 2.) U is measured along the normal and is regarded as positive when the displacement

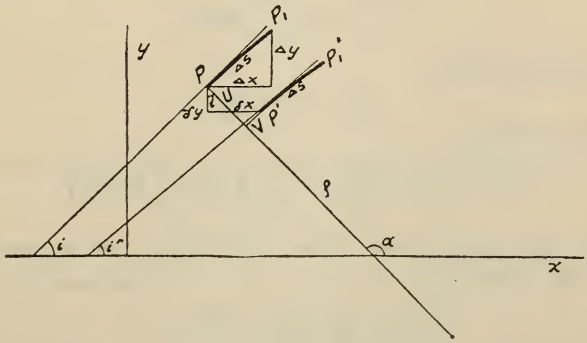

FIGURE 2.—*Displacements of a point on the median line*

is toward the center of curvature (when the normal is behind the tangent). V is measured on the tangent, and is regarded as positive when in the direction of increasing s. It will be assumed temporarily that these displacements are due to body forces and distributed

couples acting on the bar. Denoting by δx and δy the increments in the coordinates of the point P as it moves to the position P', we have

$$\begin{aligned} \delta x &= U \sin i + V \cos i \\ \delta y &= -U \cos i + V \sin i \end{aligned} \tag{2}$$

Differentiating with respect to s

$$\begin{aligned} \frac{d}{ds}\delta x &= \sin i \left(\frac{dU}{ds} + \frac{V}{\rho}\right) + \cos i \left(\frac{dV}{ds} - \frac{U}{\rho}\right) \\ \frac{d}{ds}\delta y &= -\cos i \left(\frac{dU}{ds} + \frac{V}{\rho}\right) + \sin i \left(\frac{dV}{ds} - \frac{U}{\rho}\right) \end{aligned} \tag{3}$$

During flexure the median line remains practically unextended. This establishes a relation between U and V. Consider the points P and P_1, Δs apart. After flexure they occupy the positions P′ and P_1', Δs apart. Let Δx and $\overline{\Delta x}'$ be the projections on the x-axis of this elementary arc Δs, before and after the flexure. The difference $\overline{\Delta x}' - \Delta x$ amounts to the difference in the increments of the x-coordinates of the points P and P_1 as they move to the new positions P' and P_1'. As these increments are δx and $\delta x + \frac{d}{ds}\delta x\ \Delta s$, respectively, we have

$$\overline{\Delta x}' - \Delta x = \frac{d}{ds}\delta x\ \Delta s$$

or

$$\overline{\Delta x}' = \Delta x + \frac{d}{ds}\delta x\ \Delta s$$

Similiarly

$$\overline{\Delta y}' = \Delta y + \frac{d}{ds}\delta y\ \Delta s$$

Since before the flexure

$$\overline{\Delta s}^2 = \Delta x^2 + \Delta y^2$$

and after the flexure

$$\overline{\Delta s}^2 = \left(\Delta x + \frac{d}{ds}\delta x\ \Delta s\right)^2 + \left(\Delta y + \frac{d}{ds}\delta y\ \Delta s\right)^2$$

one obtains by comparison, after neglecting the squares of small quantities, the relation

$$\frac{d}{ds}\delta x \cdot \frac{\Delta x}{\Delta s} + \frac{d}{ds}\delta y \cdot \frac{\Delta y}{\Delta s} = 0$$

which in view of equations (1) and (3) reduces to

$$\frac{dV}{ds} = \frac{U}{\rho} \tag{4}$$

the well-known condition for inextensibility of the median line.

If i and i' are the inclinations at the point P before and after the flexure, the change of inclination, H, is defined to be

$$H=i-i'$$

It also follows, since the change of inclination is small,

$$H=\sin\,(i-i')$$

Expanding

$$H=\sin i\cos i'-\cos i\sin i'$$

since

$$\cos i'=\frac{d}{ds}(x+\delta x)=\cos i+\frac{d}{ds}\delta x$$

and

$$\sin i'=\frac{d}{ds}(y+\delta y)=\sin i+\frac{d}{ds}\delta y$$

it follows that

$$H=\sin i\frac{d}{ds}\delta x-\cos i\frac{d}{ds}\delta y$$

This, in veiw of equations (3), reduces to

$$H=\frac{dU}{ds}+\frac{V}{\rho} \tag{5}$$

which is the expression for the change of inclination in terms of the displacements U and V.

If ρ and ρ' are the curvatures at the point P before and after the flexure, the change of curvature K is defined to be

$$K=\frac{1}{\rho'}-\frac{1}{\rho} \tag{6}$$

As a consequence of the definition of curvature, there is also obtained

$$K=-\frac{d}{ds}(i'-i)$$

or (7)

$$K=\frac{dH}{ds}$$

Therefore, the change of curvature in terms of the displacements is

$$K=\frac{d}{ds}\left(\frac{dU}{ds}+\frac{V}{\rho}\right) \tag{8}$$

Imagine the bar broken at the point P. (Fig. 1.) In order that the deformation of the remaining portion of the bar in the direction of decreasing length s may be unaffected, it will be necessary to apply to the end section at P a bending moment G, a tension T, and a shear N. These are the stress components with which the portion of the bar in the direction of increasing s acts on the normal section at P. Using the Euler-Bernouli assumption; that is, neglecting

antielastic curvature and the effect of the curvature of the bar [1] on the stress distribution in the cross section, one writes

$$G=BK=B\frac{dH}{ds} \tag{9}$$

where B is the flexural rigidity, defined as the product of Young's modulus, E, and the moment of inertia of the section, I. The two remaining stress components T and N are secured as usual in the following manner from the relations of equilibrium obtained for an elementary portion of the bar under body forces.

Let F_n and F_t be the normal and tangential components of the body forces calculated per unit length of the bar. Let M be the distributed couple, applied externally, calculated likewise per unit length of the bar. The condition for equilibrium gives

$$\begin{aligned} &\frac{dN}{ds}+\frac{T}{\rho}+F_n=0\\ &\frac{dT}{ds}-\frac{N}{\rho}+F_t=0\\ &\frac{dG}{ds}+N+M=0 \end{aligned} \tag{10}$$

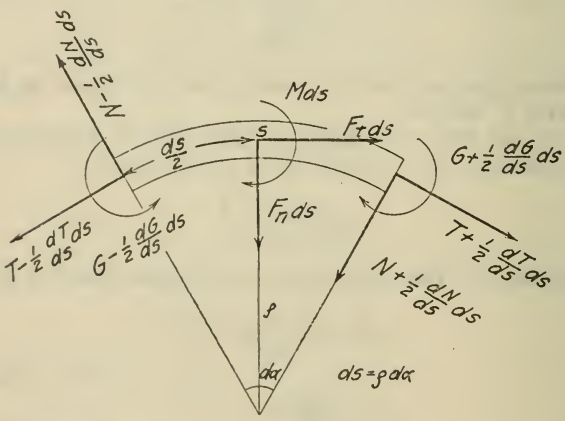

FIGURE 3.—*Stress components and body forces acting on an elementary portion of a curved bar*

These equations result readily from the normal and tangential resolution of the forces and couples, as shown in Figure 3.

For bars in vibration the body forces are taken to be the reverse of the effective forces. Accordingly

$$\begin{aligned} F_n&=-\omega\gamma\frac{d^2U}{dt^2}\\ F_t&=-\omega\gamma\frac{d^2V}{dt^2} \end{aligned} \tag{11}$$

[1] A discussion, dealing with the stress distribution in curved bars, will be found in Applied Elasticity, by Timoshenko and Lessels, Chap. IX. Westinghouse Technical Night School Press, East Pittsburgh, Pa., 1925.

where ω is the area of normal section and γ the density of the bar material. It is customary in problems of vibration to discard the rotational inertia. This is equivalent to supposing that M is negligible in comparison with N. Thus, from the third equation in (11), assuming that the bar is uniform in flexural rigidity, the value of the shear is

$$N = -B\frac{d^2H}{ds^2} \tag{12}$$

And from the first equation in (10), the expression for the tension is

$$T = B\rho\frac{d^3H}{d\ s^3} - F_n\rho \tag{13}$$

2. EQUATION OF MOTION OF A U BAR

The equation of motion of a bar of variable curvature in terms of the displacement follows directly from the second equation in (10), using equations (11), (12), and (13), and is, in fact,

$$\frac{Bd}{ds}\left(\rho\frac{d^3H}{ds_3}\right) + \frac{B}{\rho}\frac{d^2H}{ds^2} - \omega\gamma\frac{d^2V}{dt^2} + \omega\gamma\frac{d}{ds}\left(\rho\frac{d^2U}{dt^2}\right) = 0 \tag{14}$$

Set

$$H = \eta \cos kt$$
$$U = u \cos kt$$
$$V = v \cos kt$$

and

$$\mu^4 = \frac{\omega\gamma k^2}{B} \tag{15}$$

in equation (14), and there is obtained

$$\frac{d}{ds}\left(\rho\frac{d^3\eta}{ds^3}\right) + \frac{1}{\rho}\frac{d^2\eta}{ds^2} + \mu^4\left(v - \frac{d}{ds}[\rho u]\right) = 0 \tag{16}$$

with the auxiliary relations

$$\frac{dv}{ds} = \frac{u}{\rho} \qquad \eta = \frac{du}{ds} + \frac{v}{\rho} \tag{17}$$

Equation (15) defines the frequency of vibration, $\frac{k}{2\pi}$, in terms of μ and the constants of the bar B, the flexural rigidity, γ, the density, and ω the cross-sectional area. When μ is multiplied by a linear dimension, the product gives a dimensional number. We take s_1 the half length of the bar for this linear dimension for bars curved symmetrically in a plane. For bars with one end clamped and the other end free, this dimension is s_1 the total length of the bar. In subsequent pages μs_1 will be occurring constantly, and will be referred to as the frequency characteristic.

When the bar is of uniform curvature, there is but one fundamental equation of the type of (16) to consider. Where there are different curvatures, as in a U bar, the differential equation (16) assumes

different forms for different portions of the bar. In the present case, it is assumed that the yoke is a semicircle of radius a and of length $2s_o$. The prongs are straight and parallel to each other. The total length of the U bar is $2s_1$. (See fig. 4.) Accordingly, the distribution of curvature is

$$\frac{1}{\rho}=\frac{1}{a}, \text{ when } 0\leq s\leq s_0$$

$$\frac{1}{\rho}=0, \text{ when } s_o\leq s\leq s_1$$

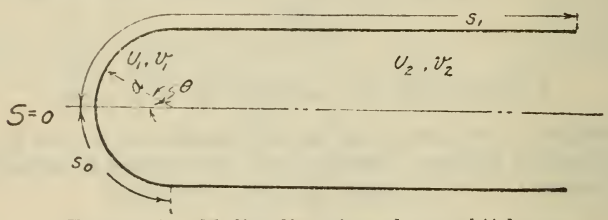

FIGURE 4.—*Median line of an elongated* U *bar*

Denoting the longitudinal and the normal displacements in the yoke by v_1 and u_1, equation (16) yields, by putting $\rho=a$

$$a^6\frac{d^6v_1}{ds^6}+2a^4\frac{d^4v_1}{ds^4}+(1-\mu^4a^4)a^2\frac{d^2v_1}{ds^2}+\mu^4a^4v_1=0 \tag{18}$$

Hence, if $\pm in_1$, $\pm in_2$, and $\pm in_3$ are the roots of the equation

$$a^6x^6+2a^4x^4+(1-a^4\mu^4)\,a^2x^2+a^4\mu^4=0 \tag{19}$$

the displacements v_1 and u_1 are given by

$$\begin{aligned} v_1&=\sum_{i=1}^{3}A_i\cos n_is+\Sigma\, B_i\sin n_1s\\ u_1&=a\frac{dv_1}{ds}\end{aligned} \tag{20}$$

Denoting the longitudinal and the normal displacements in the prongs by v_2 and u_2 there results from equation (16), by putting $\rho=\infty$

$$\frac{d^4u_2}{ds^4}-\mu^4u_2=0 \tag{21}$$

Accordingly

$$v_2=c_1a \text{ constant}$$

and

$$\begin{aligned} u_2=&D(\cos\mu s+\cosh\mu s)+E(\cos\mu s-\cosh\mu s)\\ &+F(\sin\mu s+\sinh\mu s)+G(\sin\mu s-\sinh\mu s)\end{aligned} \tag{22}$$

It is of interest to note that equation (21) also follows directly from (13). Since T is a finite quantity, the multiplier of ρ must approach the limit zero as ρ becomes infinite.

3. END CONDITIONS AND CONDITIONS OF CONTINUITY

The application of equation (20) to the vibration of a circular arc and of equations (22) to the vibrations of a straight bar are well known. Here the two sets are to be taken together, as the U bar considered here is partly circular and partly straight. This means merely that the constants of integration in (20) and (22) bear definite relations to one another. There are altogether 11 constants to be considered. All the A's drop out when the modes of vibration are symmetrical with respect to the axis of the U bar. The fundamental mode of vibration, which is of practical interest, is of this type. The U bar having free ends,

$$\begin{aligned} \frac{d^2u_2}{ds^2}&=0, \text{ when } s=s_1 \\ \frac{d^3u_2}{ds^3}&=0, \text{ when } s=s_1 \end{aligned} \tag{23}$$

Since the displacements, the change of inclination, the bending moment, and the shear are continuous

$$\begin{aligned} v_1&=v_2 \\ u_1&=u_2 \\ \frac{du_1}{ds}+\frac{v_1}{a}&=\frac{du_2}{ds} \\ \frac{d}{ds}\left(\frac{du_1}{ds}+\frac{v_1}{a}\right)&=\frac{d^2u_2}{ds^2} \end{aligned} \tag{24}$$

and

$$\frac{d^2}{ds^2}\left(\frac{du_1}{ds}+\frac{v_1}{a}\right)=\frac{d^3u_2}{ds^3}$$

all when $s=s_o$.

4. CONDITION FOR THE INVARIANCY OF THE CENTER OF GRAVITY

One more condition is needed. It is obtained under the supposition that the center of gravity of the whole U bar experiences no displacement during the vibration. This is necessarily true, as no external forces are acting. Since the mode of vibration to be considered (the fundamental) is one of symmetry with respect to the axis of the U bar (that is, the x-axis), the y coordinate of the center of gravity is always zero. The x coordinate, however, since the bar is of uniform cross section and of uniform density, is given by

$$\bar{x}=\frac{1}{s_1}\int_0^{s_1} x\,ds$$

when there is no flexure. On the other hand, when flexure exists, as during vibration,

$$\bar{x}=\frac{1}{s_1}\int_0^{s_1} (x+\delta x)\,ds$$

Thus the condition for the invariancy of the center of gravity resolves into

$$\int_0^{s_1} \delta x ds = 0 \tag{25}$$

From equation (2)

$$\delta x = u_1 \sin i + v_1 \cos i \quad \text{when } o \leq s \leq s_0$$

$$\delta x = v_2 \quad \text{when } s_0 \leq s \leq s_1$$

Substituting these values of the increment αx in equation (25), there results

$$\int_0^{s_0} (u_1 \sin i + v_1 \cos i)\, ds + \int_{s_0}^{s_1} v_2 ds = 0$$

Since in the curved portion we have

$$\frac{di}{ds} = -\frac{1}{a}$$

$$u_1 = -\frac{dv_1}{di}$$

and

$$v_2 = v_1 (s_o)$$

the above equation may be written also as

$$a \int_{\frac{\pi}{2}}^{0} \left(-\sin i \frac{dv_1}{di} + v_1 \cos i\right) di + v_1 (s_o) \int_{s_0}^{s_1} ds = 0$$

or

$$a \int_{\frac{\pi}{2}}^{0} \frac{d}{di} (v_1 \sin i)\, di + 2a \int_{\frac{\pi}{2}}^{0} v_1 \cos i\, di + v_1 (s_o) \int_{s_0}^{s_1} ds = 0$$

The first integral vanishes; for v is equal zero when $i = \frac{\pi}{2}$. Thus the condition for the invariancy of the center of gravity becomes, replacing i by $\frac{\pi}{2} - \theta$

$$v_1 (s_o) = -\frac{2a}{s_1 - s_o} \int_0^{\frac{\pi}{2}} v_1 \sin \theta d\theta \tag{26}$$

where $\theta = s/a$.

III. THE FREQUENCY OF VIBRATION OF AN ELONGATED U BAR

1. APPROXIMATE VALUES OF THE QUANTITIES $n_i a$

The expressions of the tangential and radial displacements, as given in equations (20) involve the quantities $n_i a$ when s is replaced by νa, a being the radius of the circular yoke. The quantities $n_i a$, however, depend on the frequency characteristic μs_1. It, therefore, becomes necessary to express them as explicit functions of μa.

Setting

$$a^2\mu^2=\phi^2$$

and

$$a^2x^2=y \tag{27}$$

in equation (19), the resultant cubic is

$$y^3+2y^2+y+\phi^4(1-y)=0 \tag{28}$$

with the roots y_1, y_2, y_3, say. The roots of equation (19) being $\pm in_1$, $\pm in_2$, and $\pm in_3$, it follows that

$$\begin{aligned} y_1&=-n_1^2a^2\\ y_2&=-n_2^2a^2\\ y_3&=-n_3^2a^2 \end{aligned} \tag{29}$$

If the bar is sufficiently long the quantities n_ia can be developed into series of ascending powers of μ^2a^2, or θ^2, since μ^2a^2 is a small quantity. The smallness of μ^2a^2 follows from the fact that the frequencies of vibration of an elongated U-bar of length $2s_1$ and of a clamped-free bar of length s_1 are nearly equal. Now, the frequency characteristic of a clamped-free bar satisfies the equation [2]

$$\cosh \mu s_1 \cos \mu s_1=-1$$

To obtain the frequency characteristic for the fundamental we take the smallest root; that is, $\mu s_1=1.8751$. Hence, for the elongated U bar, we put, since its fundamental frequency is nearly the same, $\mu s_1=1.8751+x$, where x is a small quantity. Hence

$$\mu a=(1.8751+x)\frac{a}{s_1}$$

and, therefore, can be made as small as it is desired by taking a small enough while s_1 is kept constant.

The series for n_ia will now be derived. The reduced cubic

$$z^3+pz+q=0 \tag{30}$$

follows, when $y=z-\frac{2}{3}$ is set in equation (28). The explicit expressions for p and q are

$$\begin{aligned} p&=-1/3(1+3\phi^4)\\ q&=-\frac{2+45\phi^4}{27} \end{aligned} \tag{31}$$

The reduced cubic will have three distinct roots, all real, when the discriminant is negative. The discriminant is negative as long as the quantity ξ^2, defined as the ratio

$$\xi^2=\frac{1}{4}\,\frac{(-2+45\phi^4)^2}{(1+3\phi^4)^3} \tag{32}$$

[2] Rayleigh. The Theory of Sound, 1, 2d ed., p. 278.

is smaller than unity. Accordingly, when $3\phi^4 < 0.34$ we may adopt the trigonometric form of solution, and pass over the roots z_1, z_2, z_3, to the roots y_1, y_2, y_3, writing

$$
\begin{aligned}
y_1 &= -\frac{2}{3} - \frac{2}{3}\sqrt{1+3\phi^4}\,\cos\frac{\psi}{3} \\
y_2 &= -\frac{2}{3} - \frac{2}{3}\sqrt{1+3\phi^4}\left(\frac{1}{2}\cos\frac{\psi}{3} - \frac{\sqrt{3}}{2}\sin\frac{\psi}{3}\right) \qquad (33) \\
y_3 &= -\frac{2}{3} - \frac{2}{3}\sqrt{1+3\phi^4}\left(\frac{1}{2}\cos\frac{\psi}{3} - \frac{\sqrt{3}}{2}\sin\frac{\psi}{3}\right)
\end{aligned}
$$

where $\cos\psi = \xi$.

The trigonometric expressions in equation (33) require that $3\phi^4$ shall be smaller than unity if the quantities y_1, y_2, y_3 are to be developed into series of ascending powers of ϕ^2 or $\mu^2 a^2$. With the criterion of convergence thus established $[3\phi^4 < 0.34]$ we shall proceed as explained below, instead of working with the above trigonometric expressions, to obtain the desired series. Setting separately in the original cubic equation (28)

$$y = \sum_{n=1}^{n=\infty} b_n \phi^2$$

$$y = 1 + \sum_{n=1}^{n=\infty} a_n \phi^2$$

and then equating to zero in the separate series thus obtained the coefficients of like powers of ϕ, we have

$$
\begin{aligned}
y_1 &= -n_1^2 a^2 = -\phi^4 - 3\phi^8 - \ \ldots \\
y_2 &= -n_2^2 a^2 = -1 - \sqrt{2}\,\phi^2 + \frac{1}{2}\phi^4 - \frac{9\sqrt{2}}{16}\phi^6 + \frac{3}{2}\phi^8 - \ \ldots \qquad (34) \\
y_3 &= -n_3^2 a^2 = -1 + \sqrt{2}\,\phi^2 + \frac{1}{2}\phi^4 + \frac{9}{16}\sqrt{2}\,\phi^6 + \frac{3}{2}\phi^8 + \ \ldots
\end{aligned}
$$

It is now apparent that the treatment of the vibration of the U bar by the method of differential equations is exceedingly complicated. Unless the treatment deals with long bars, the method must be discarded. With a view to introducing an approximation, it will be required that $\pi\phi^4$ be negligible in comparison with unity. Mathematically,

$$A(1+g\phi^4) = A$$

where (35)

$$g \lesssim \pi$$

The corresponding approximations for nia, therefore, follow:

$$
\begin{aligned}
n_1 a &= \phi^2 \\
n_2 a &= 1 - \frac{\sqrt{2}}{2}\phi^2 - \frac{1}{2}\phi^4 \qquad (36) \\
n_3 a &= 1 + \frac{\sqrt{2}}{3}\phi^2 - \frac{1}{2}\phi^4
\end{aligned}
$$

2. THE FREQUENCY OF VIBRATION OF AN ELONGATED U BAR

As stated before, v_1, u_1, and v_2, u_2 are the tangential and normal displacements in the yoke and in the prong, respectively. For the fundamental mode of vibration

$$\begin{aligned} v_1 &= B_1 \sin n_1 s + B_2 \sin n_2 s + B_3 \sin n_3 s \\ u_1 &= n_1 a B_1 \cos n_1 s + n_2 a B_2 \cos n_2 s + n_3 a B_3 \cos n_3 s \end{aligned} \tag{37}$$

The conditions of continuity, equations (24) require that

$$\begin{aligned} &\sum_{i=1}^{3} n_i a\ B_i \cos n_i s_0 = C_0 \phi^2 \\ &\sum_{i=1}^{3} (1 - n_i^2 a^2)\ B_i \sin n_i s_0 = C_1 \phi^2 \\ &\sum_{i=1}^{3} n_i a\ (1 - n_i^2 a^2)\ B_i \cos n_i s_0 = C_2 \phi^2 \\ &\sum_{i=1}^{3} n_i^2 a^2\ (1 n_i^2 a^2)\ B_i \sin n_i s_0 = -C_3 \phi^2 \end{aligned} \tag{38}$$

where

$$\begin{aligned} C_0 \phi^2 &= (u_2) s_0 \\ C_1 \phi^2 &= a \left(\frac{du^2}{ds} \right)_{s_0} \\ C^2 \phi^2 &= a^2 \left(\frac{d^2 u_2}{ds^2} \right)_{s_0} \\ C_3 \phi^3 &= a^3 \left(\frac{d^3 u_2}{ds^3} \right)_{s_0} \end{aligned} \tag{39}$$

On forming the approximations

$$\cos \mu s_0 = 1 - \frac{\pi^2}{8} \phi^2 \qquad \sin \mu s_0 = \frac{\pi}{2} \phi$$

$$\cos h\mu s_0 = 1 + \frac{\pi^2}{8} \phi \qquad {}^2\sin h\mu s_0 = \frac{\pi}{2} \phi$$

on the basis of equation (35), and setting them in equations (39), there result

$$\begin{aligned} C_0 &= 2d - \frac{\pi^2}{4} E + \left(\pi f - \frac{\pi^3}{24} G \right) \phi \\ C_1 &= -E + \left(2f - \frac{\pi^2}{4} G \right) \phi \\ C_2 &= -2E - \pi G \phi \\ C_3 &= -2G + \left(\pi d - \frac{\pi^3}{24} E \right) \phi^3 \end{aligned} \tag{40}$$

where

$$D=d\phi^2$$
$$F=f\phi^2 \tag{41}$$

In the left-hand members of equations (38) occur various products of $n_i a$ and trigonometric functions of $n_i s_0$. Quantities of the type $n_i a\ (1-n_i^2 a^2)$ are to be evaluated on the basis of the approximate values of $n_i a$ given in equations (36). As regards the trigonometric functions, we put immediately, since $n_i s_0=\frac{\pi}{2}n_i a$, the approximations

$$\sin n_1 s_0=\frac{\pi}{2}\phi^2 \qquad \cos n_1 s_0=1$$

$$\sin n_2 s_0=1 \qquad \cos n_2 s_0=\frac{\pi}{4}\sqrt{2}\phi^2$$

$$\sin n_3 s_0=1 \qquad \cos n_3 s_0=-\frac{\pi}{4}\sqrt{2}\phi^2$$

Thus, the various equations (38), now become

$$B_1+\frac{\pi}{4}\sqrt{2}(B_2-B_3)-\frac{\pi}{4}(B_2+B_3)\phi^2=C_0$$

$$\frac{\pi}{2}B_1+\sqrt{2}(B_2-B_3)+\frac{1}{2}(B_2+B_3)\phi^2=C_1 \tag{42}$$

$$B_1-\frac{\pi}{8}\sqrt{2}(B_2-B_3)\phi^4+\frac{\pi}{2}(B_2+B_3)\phi^2=C_2$$

$$\frac{\pi}{2}B_1\phi^4+\sqrt{2}(B_2-B_3)-\frac{3}{2}(B_2+B_3)\phi^2-C_3\phi$$

The condition of the constancy of the center of gravity, equation (26), yields, putting $s_1-s_0=qa$

$$\sum_{i=1}^{3}B_i\left(q\sin n_i s_0+\frac{2\cos n_i s_0}{1-n_i^2a^2}\right)=0$$

which, in virtue of the approximations adopted, becomes

$$B_2+B_3=-\frac{q\pi+4}{2q+\pi}B_1\phi^2+\frac{3\sqrt{2}\pi}{4(2q+\pi)}(B_2-B_3)\phi^2 \tag{43}$$

Our immediate purpose is to obtain two relations among the C_3'. To accomplish this, first, the following equations

$$2B_1+\frac{\pi}{4}\sqrt{2}(B_2-B_3)+\frac{\pi}{4}(B_2+B_3)\phi^2=C_0+C_2$$

$$\underline{\quad 0+\frac{\pi}{4}\sqrt{2}(B_2-B_3)-\frac{3\pi}{4}(B_2+B_3)\phi^2=C_0-C_2}$$

$$\frac{\pi}{2}B_1+\qquad 0\qquad +2(B_2+B_3)\phi^2=C_1+C_3\phi \tag{44}$$

$$\frac{\pi}{2}B_1+2\sqrt{2}(B_2-B_3)-(B_2+B_3)\phi^2=C_1-C_3\phi$$

are derived from equations (42) with a view of eliminating the terms containing ϕ^4. We remark that in equations (44) the (B_2+B_3) terms can be neglected in virtue of the equation (43). Accordingly

$$\begin{aligned} 2B_1+\frac{\pi}{4}\sqrt{2}(B_2-B_3) &= C_0+C_2 \\ \frac{\pi}{4}\sqrt{2}(B_2-B_3) &= C_0-C_2 \\ \frac{\pi}{2}B_1 \qquad\qquad &= C_1+C_3\phi \\ \frac{\pi}{2}B_1+2\sqrt{2}(B_2-B_3) &= C_1-C_3\phi \end{aligned} \tag{45}$$

from which the two desired relations

$$\begin{aligned} C_1-\frac{\pi}{2}C_2+C_3\phi &= 0 \\ C_1-C_2+\frac{\pi}{4}C_3\phi &= 0 \end{aligned} \tag{46}$$

follow. Substituting in equation (46) the expressions for the C's as given in equations (40), there result

$$\begin{aligned} 2d+\pi f\phi &= \left(\frac{\pi^2}{4}-2\right)E+\left(\frac{\pi^3}{24}-\frac{\pi}{2}\right)G\phi \\ 2d+\pi d\phi^3 &= \left(2-\frac{\pi^2}{4}\right)G+\frac{\pi^3}{24}E\phi^3 \end{aligned} \tag{47}$$

Solving for f and d, and setting the values of f and d thus secured in equations (41), there is finally obtained

$$\begin{aligned} D &= \delta_1 E\phi^2+\delta_2 G\phi^2 \\ F &= -\delta_1 G\phi^2 \end{aligned} \tag{48}$$

where

$$\delta_1=\frac{\pi^2}{8}-1,\ \delta_2=\frac{\pi^3}{12}-\frac{3\pi}{4}$$

To establish a relation between E and G there remains to be considered the end conditions given in equations (23). Accordingly, after substituting the values of F and D as expressed above, there result

$$\begin{aligned} &E\,[-\cos\mu s_1-\cosh\mu s_1+\delta_1\phi^2\,(-\cos\mu s_1+\cosh\mu s_1)] \\ &+G\,[-\sin\mu s_1-\sinh\mu s_1-\delta_1\phi^2(-\sin\mu s_1+\sinh\mu s_1) \\ &\qquad\qquad +\delta_2\phi^3\,(-\cos\mu s_1+\cosh\mu s_1)]=0 \\ &E\,[\sin\mu s_1-\sinh\mu s_1+\delta_1\phi^2\,(\sin\mu s_1+\sinh\mu s_1)]+ \\ &G\,[-\cos\mu s_1-\cosh\mu s_1-\delta_1\phi^2\,(-\cos\mu s_1+\cosh\mu s_1) \\ &\qquad\qquad +\delta_2\phi^3\,(\sin\mu s_1+\sinh\mu s_1)]=0 \end{aligned} \tag{49}$$

Hence, the condition of compatibility, which is the equation for determining μs_1, is found to be

$$\cos \mu s_1 \cosh \mu s_1 + 2 \delta_1 \mu^2 a^2 \sin \mu s_1 \sinh \mu s_1 - \delta_2 a^3 \mu^3 (\cosh \mu s_1 \sin \mu s_1 + \sinh \mu s_1 \cos \mu s_1) = -1 \tag{50}$$

Now the smallest root of the equation

$$\cos m \cosh m = -1$$

is $m = 1.8751$. Accordingly, we put

$$\mu s_1 = m + x \tag{51}$$

and substitute in the equation (50). This gives, neglecting powers of x higher than the first,

$$x = \frac{2\delta_1 m^2 \left(\frac{a}{s_1}\right)^2 \sin m \sinh m - \delta_2 m^3 \left(\frac{a}{s_1}\right)^3 (\cosh m \sin m + \cos m \sinh m)}{\sin m \cosh m - \cos m \sinh m} \tag{52}$$

Introducing the numerical values of m, δ_1, and δ_2 from above, and expressing a, the radius of the circular yoke in terms of s_0, the expression for x further becomes

$$x = 0.49\left(\frac{s_0}{s_1}\right)^2 - 0.2\left(\frac{s_0}{s_1}\right)^3 \tag{53}$$

where $\frac{s_0}{s_1}$ is the ratio of the length of the circular yoke to the total length of the bar measured on the median line.

We are now ready to give the frequency of vibration of the fundamental mode, N. This, from equation (15), is

$$N = \frac{k}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{B}{\omega\gamma}}\frac{m^2}{s_1^2}\left(1 + \frac{2x}{m}\right) \tag{54}$$

where B is the flexural rigidity, ω the cross-sectional area, γ the density, s_1, half the length of the U bar, measured on the median line, m equals 1.8751, and x which gives the effect of the yoke on the vibration, is given by equation (53).

It is now evident from equation (54), that an elongated U bar of length $2s_1$ and a clamped-free bar of length s_1 will vibrate with nearly the same frequency. In fact, when $\frac{s_0}{s_1}$ equals one-tenth the frequency of vibration of the U bar will be only 0.5 per cent higher than the frequency of vibration of a clamped-free bar.

3. THE DEFORMATION OF THE CIRCULAR YOKE

According to equations (37), the tangential and the radial displacements of points in the yoke are

$$v_1(s) = B_1 \sin n_1 s + B_2 \sin n_2 s + B_3 \sin n_3 s$$

$$u_1(s) = n_1 a B_1 \cos n_1 s + n_2 a B_2 \cos n_2 s + n_3 a B_3 \cos n_3 s$$

In elongated U bars the trigonometric functions are readily expressed as functions of ϕ^2. The method is, as before, to introduce the approximate expressions of $n_i a$, as given in equations (36). In fact, putting $s = \nu a$, we obtain,

$$\sin n_1 s = \nu\phi^2$$

$$\sin n_2 s = \sin \nu - \frac{\sqrt{2}}{2} \nu \cos \nu \phi^2$$

$$\sin n_3 s = \sin \nu + \frac{\sqrt{2}}{2} \nu \cos \nu\phi^2$$

$$\cos n_1 s = 1$$

$$\cos n_2 s = \cos \nu + \frac{\sqrt{2}}{2} \nu \sin \nu \phi^2$$

and

$$\cos n_3 s = \cos \nu - \frac{\sqrt{2}}{2} \nu \sin \nu\phi^2$$

Adopting these values of the trigonometric functions of $n_i s$, the above expressions for the tangential and radial displacements now become

$$v_1(s) = B_1 \nu\phi^2 + (B_2 + B_3)\sin\nu - \frac{\sqrt{2}}{2} (B_2 - B_3)\nu\cos\nu\phi^2$$

$$u_1(s) = B_1\phi^2 + (B_2 + B_3)\cos\nu - \sqrt{2}\,\frac{(B_2 - B_3)}{2} [\cos \nu - \nu \sin \nu]\phi^2 \qquad (55)$$

The numerical values of the constants B, etc., depend on the end deflection. We write, from equations (43) and (45),

$$B_1 = C_2$$

$$\sqrt{2}(B_2 - B_3 = - C_3\phi$$

$$B_2 + B_3 = -\frac{q\pi + 4}{2q + \pi} C_2\phi^2 - \frac{3\pi}{4(2q + \pi)} C_3\phi^3$$

The expression fcr $B_2 + B_3$ can be simplified still more.

Since

$$qa = s_1 - s_0$$

$$\phi = \mu s_1 \frac{a}{s_1}$$

and

$$\mu s_1 = 1.8751 + x$$

where x is a small quantity, we can show that, also,

$$B_2 + B_3 = -(1.571 - 0.25\phi)C_2\phi^2 - 0.62\phi^4$$

Equations (40) determine C_2 and C_3 in terms of G and E. The values of G and E follow from equations (49), and, in fact, prove to be

$$G = 0.367\alpha$$

$$E = 0.500\alpha$$

when terms involving ϕ^2 are discarded. α is the amplitude of the end excursions or deflections. Thus, we write, from equations (40)

$$C_2 = (1 - 1.153\phi)\alpha$$

$$C_3 = -0.734\alpha$$

Accordingly, we have in terms of α

$$B_1 = (1 - 1.153\phi)\alpha$$

$$\sqrt{2}(B_2 - B_3) = 0.734\phi\alpha$$

$$B_2 + B_3 = -(1.571 - 2.06\phi)\phi^2\alpha \tag{56}$$

As an example, we shall evaluate the displacements of points in the yoke of a particular bar. Let the total length of the bar be ten times as great as the length of the circular yoke; that is, s_0/s_1 equals 0.1. This makes $\mu s_1 = 1.880$, and, therefore, μa or ϕ equals 0.1194. Using this value of ϕ and reckoning the end deflection α as unity, we have from (56)

$$B_1\phi^2 = 0.01232$$

$$B_2 + B_3 = -0.01895$$

$$\frac{\sqrt{2}}{2}(B_2 - B_3)\phi^2 = 0.00063$$

Substituting these in equations (55) the numerical values of the tangential and radial displacements result. These are plotted in Figure 5.

An important observation, which is of value in practice, will be made presently. It appears from Figure 5 that the radial displacement u_1 vanishes at $s = 0.05s_1$, whereas the tangential displacement v_1 at the same value of s is about (0.4 per cent) of the end deflection. We infer from this that, if the bar is held vertical, the points of the

yoke corresponding to $s=0.05s_1$ will have maximum vertical and horizontal amplitudes of equal value and of amount 0.003 mm, when the amplitude of the end deflections is 1 mm. This deduction follows from equation (2). Accordingly, to minimize the resistance of the supports to the vibratory motion of the bar, it is sufficient that the supports be introduced at the point $s=\pm 0.05s_1$.

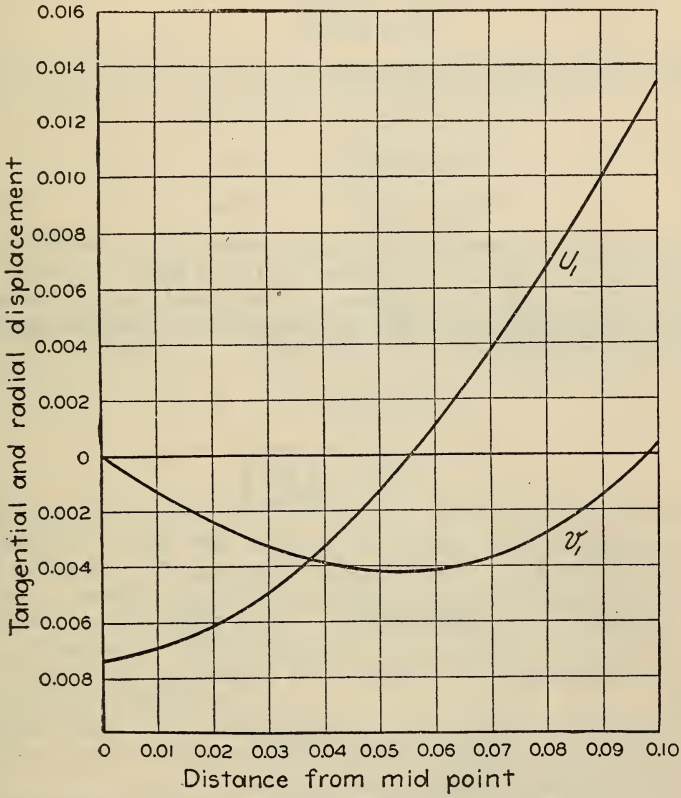

FIGURE 5.—*Tangential and radial displacement, v_1 and u_1, respectively, of the yoke of an elongated U bar of ratio $s_0/s_1=0.10$*

Displacements are expressed as fractions of the end displacement. Distance from mid-points, measured on the median line, are expressed as fractions of the half length, s_1, of the bar.

IV. RITZ'S METHOD OF APPROXIMATION

1. APPLICATION OF RITZ'S METHOD TO CURVED BARS

In the cases where the method of representing the motion of curved bars by differential equations leads to lengthy calculations in evaluating the required displacements, it is preferable to use Ritz's method of approximation. The advantage of using the differential equations is that they result in the formulation of normal functions. If, however, the interpretation of the differential equations is not clear, no real information is secured. After all, all that is necessary for practical purposes in vibration problems is an approximate evaluation

of the deformation for the fundamental mode of vibration. This information is always obtainable by Ritz's method.

Following Ritz,[3] the displacements U and V, and the change of curvature K, will be expressed as

$$U = u \cos kt$$

$$V = v \cos kt$$

$$K = \kappa \cos kt$$

where u, v, and κ are the polynomials

$$\begin{aligned} u &= c_1 u_1 + c_2 u_2 + \cdot \cdot \cdot + c_n u_n \\ v &= c_1 v_1 + c_2 v_2 + \cdot \cdot \cdot + c_n v_n \\ \kappa &= c_1 \kappa_1 + c_2 \kappa_2 + \cdot \cdot \cdot + c_n \kappa_n \end{aligned} \tag{57}$$

The related group of terms u_r, v_r, κ_r by construction satisfy all the end and continuity conditions, when the latter exist. Among themselves they also satisfy the condition of inextensibility of the median line and the definition of change of curvature. The last statement amounts mathematically to the following:

$$u_r = \rho \frac{dv_r}{ds}$$

$$\kappa_r = \frac{d^2 u_r}{ds_2} + \frac{d}{ds}\left(\frac{v_r}{\rho}\right)$$

where s is the distance along the median line and ρ the radius of curvature of the bar. Accordingly, there are two possible procedures for the construction of the terms (u_r, v_r, κ_r). Once v_r is chosen, the rest will follow by differentiation. The alternate procedure is to choose κ_r first, and arrive at v_r by integration. Whether to choose the former or the latter procedure will depend on the nature of the problem.

The constants c_r are as yet undetermined. For their evaluation two principles are available—either the principle of virtual velocities, or the Hamiltonian principle of least action.

The latter one will be used here. Representing the potential energy of deformation by W, and the kinetic energy by T, Hamilton's principle states that

$$\delta \int_{t_1}^{t_2} (W - T)\, dt = 0 \tag{58}$$

In the case at hand, the required variation will be produced through varying the constants c_r.

Since at the instants t_1 and t_2 the system must be in its natural configuration, we take

$$\begin{aligned} t_1 &= (2m+1)\frac{\pi}{2k} \\ t_2 &= (2m+5)\frac{\pi}{2k} \end{aligned} \tag{59}$$

[3] Walter Ritz, Uber eine neue Methode zur Lösung gewisser Variations probleme der mathematischen physik; J. für die reine und angewandte Mathematik; Band 135; 1909.

where m is a positive integer. The potential energy of deformation and the kinetic energy in a vibrating bar are given by the expressions

$$W=\frac{1}{2}\int_{-s_1}^{s_1} B\,K^2\,ds$$

$$T=\frac{1}{2}\int_{-s_1}^{+s_1} \omega\gamma\,(U^2+V^2)\,ds$$

where B is the flexural rigidity, γ the density and ω the cross-sectional area, and $2s_1$ the total length of the bar. Defining

$$W_c=\frac{1}{2}\int_{-s_1}^{+s_1} \kappa^2\,ds$$

$$T_c=\frac{1}{2}\int_{-s_1}^{+s_1} (u^2+v^2)\,ds \tag{60}$$

and, assuming that the quantities B, γ, ω are uniform throughout the bar, the expressions for the potential and the kinetic energies now become

$$W=BW_c\,\mathrm{cas}^2\,kt$$

$$T=\omega\gamma k^2 T_c\,\sin^2\,kt \tag{61}$$

In view of equations (59) and (61), the Hamiltonian equation (58) gives

$$\delta\int_{(2m+1)\frac{\pi}{2k}}^{(2m+5)\frac{\pi}{2k}} (BW_c\cos^2 kt-k^2\,\omega\gamma\,T_c\sin^2 kt)\,dt=0$$

which after integration is

$$\delta\,(BW_c-k^2\,\omega\gamma\,T_c)=0$$

or

$$\delta\,(W_c-\mu^4\,T_c)=0 \tag{62}$$

since,

$$\mu^4=\frac{k^2\omega\gamma}{B}$$

according to the equation (15). Expanding equation (62), since W_c and T_c are functions of c_s,

$$\delta\,(W_c-\mu^4 T_c)=\Sigma\frac{d}{dc_s}(W_c-\mu^4 T_c)\,\delta c_s=0 \tag{63}$$

As all the values of c_s are independent of each other, it suffices, in order to produce the variation required in equation (63), to put $c_s=0$ for all values of s except where $s=r$. Thus,

$$\frac{d}{dc_r}(W_c-\mu^4 T_c)=0,\quad (r=1,\,2\,\cdots\,n) \tag{64}$$

Now, W_c and T_c are each quadratic in the c_s. In fact, when the definitions

$$T_{sr}=\int_0^{s^1}(u_r\,u_s+v_r\,v_s)\,ds$$

$$W_{sr}=\int_0^{s_1}\kappa_r\,\kappa_s\,ds$$

are written, equations (60) yield

$$W_c=\sum_{s=1}^{n}\alpha_{rs}\,c_r\,c_s\,W_{rs}+W_1$$

$$T_c=\sum_{s=1}^{n}\alpha_{rs}\,c_r\,c_s\,T_{rs}+T_1$$

$$\begin{pmatrix}\alpha_{rs}=2, \text{ when } r\pm s\\ \alpha_{rs}=1, \text{ when } r=s\end{pmatrix}$$

where W_1 and T_1 both are free from the particular constant C_r. Accordingly, the differentiation in (64) gives the following n homogeneous equations.

$$\sum_{r=1}^{n}c_s\,(W_{rs}-\mu^4\,T_{rs})=0\ (s=1\cdots n) \tag{65a}$$

Introducing the dimensional quantities

$$w_{rs}=s_1\,W_{sr}$$

$$t_{rs}=s_1^{-3}\,T_{sr}$$

the n homogeneous equations now become

$$\sum_{r=1}^{r=n}c_s\,[w_{rs}-(\mu s_1)^4t_{rs}]=0\ (s=1,\,2\,\cdots,\,n) \tag{65b}$$

The condition of compatibility among these n equations leads to the determinant:

$$\begin{vmatrix} w_{11}-(\mu s_1)^4t_{11} & w_{12}-(\mu s_1)^4t_{12} & \cdots\cdots & w_{1n}-(\mu s_1)^4t_{1n}\\ \cdot & \cdot & \cdots\cdots & \cdot\\ w_{n1}-(\mu s_1)^4t_{n1} & w_{n2}-(\mu s_1)^4t_{n2} & \cdots\cdots & w_{nn}-(\mu s_1)^4t_{nn}\end{vmatrix}=0 \tag{66}$$

which is an algebraic equation of the nth degree in $(\mu s_1)^4$ the frequency characteristics. Theoretically, the number n ought to be increased indefinitely. Physically, the number of degrees of freedom of the vibrating bar ought to be made correspondingly large. In practice, a determinant of rank 2 or an algebraic equation of the second degree in $(\mu s_1)^4$ suffices amply to determine the particular value of μs_1 which relates to the fundamental mode of vibration.

2. AN ILLUSTRATION FROM A CLASSICAL PROBLEM

The modes of vibration of a straight bar, both ends free, are well known. A new solution will be worked out as an illustration of the preceding theory. This particular example has been chosen since a straight bar may be thought of as a limiting case of a U bar. Here, only the symmetrical (that is, fundamental) vibration will be considered.

Let the straight bar be of length $2s_1$. Let U be the transversal displacement.

$$U = u \cos k\, t \tag{67}$$

As before, let B, ω, γ be the flexural rigidity, the cross-sectional area, and the density of the bar, respectively. Putting $s = s_1 x$, we choose for the change of curvature the polynomial series

$$\kappa = \frac{d^2u}{ds^2} = \frac{d^2u}{s_1^2 dx^2} = \frac{1}{s_1^2}\left[c_1(1+\cos \pi x) + c_2\,(1-\cos 2\pi x) + \cdots\right] \tag{68}$$

since the bending moment G and the shear N vanish at $x = \pm 1$. To point out the analogy with the third of equations (57), equation (68) may also be written

$$\kappa = \frac{1}{s^2}\,(c_1\kappa_1 + c_2\kappa_2 + \cdots + c_n\kappa_n) \tag{68a}$$

Integrating equation (68) with respect to x,

$$\frac{du}{dx} = c_1\left(x + \frac{\sin \pi x}{\pi}\right) + c_2\left(x - \frac{\sin 2\pi x}{2\pi}\right) + \cdots \tag{69}$$

The constant of integration vanishes by virtue of the fact that the bar experiences no change of inclination at the mid-point $s = 0$, or $x = 0$. A further integration with respect to x gives

$$u = c_1\left(a_1 + \frac{x^2}{2} - \frac{\cos \pi x}{\pi^2}\right) + c_2\left(a_2 + \frac{x^2}{2} + \frac{\cos 2\pi x}{4\pi^2}\right) + \cdots \tag{70}$$

To determine the constants of integration, a_s, the condition for the conservation of the center of gravity is employed, which in the present example will be expressed by the relation

$$\int_0^1 u dx = 0$$

As this is to be satisfied whatever the value of the constants a_s, individually

$$\int_0^1 \left(a_1 + \frac{x^2}{2} - \frac{\cos \pi x}{\pi^2}\right) dx = 0$$

and

$$\int_0^1 \left(a_2 + \frac{x^2}{2} + \frac{\cos 2\pi x}{4\pi^2}\right) dx = 0$$

from which

$$a_1 = a_2 = -\frac{1}{6}$$

So, finally, the expression for the displacement assumes the form

$$u = c_1\left(-\frac{1}{6}+\frac{x^2}{2}-\frac{\cos \pi x}{\pi^2}\right)+c_2\left(-\frac{1}{6}+\frac{x^2}{2}+\frac{\cos 2\pi x}{4\pi^2}\right)+\ldots \tag{71}$$

which may also be written, in analogy with the first of equations (57),

$$u = c_1u_1 + c_2u_2 + \ldots . \tag{71a}$$

Rewriting the definitions

$$t_{rs} = \int_0^1 u_r u_s dx$$

$$w_{rs} = \int_0^1 \kappa_r \kappa_s dx$$

the quantities t_{11}, w_{11}, etc., on the basis of equations (68a) and (71a) emerge with the following numerical values:

$$\begin{aligned} t_{11} &= \left(\frac{\pi^4}{45}+\frac{5}{2}\right)\frac{1}{\pi^4} = 4.66462\,\frac{1}{\pi^4},\ w_{11} = \frac{3}{2} \\ t_{12} &= \left(\frac{\pi^4}{45}+\frac{17}{16}\right)\frac{1}{\pi^4} = 3.22712\,\frac{1}{\pi^4},\ w_{12} = 1 \\ t_{22} &= \left(\frac{\pi^4}{45}+\frac{5}{32}\right)\frac{1}{\pi^4} = 2.32087\,\frac{1}{\pi^4},\ w_{22} = \frac{3}{2} \end{aligned} \tag{71}$$

Thus, neglecting the constants c_r, r larger than 2, the system of homogeneous equations corresponding to equation (65b) becomes in this example

$$\begin{aligned} c_1[w_{11}-(\mu s_1)^4 t_{11}] + c_2[w_{12}-(\mu s_1)^4 t_{12}] &= 0 \\ c_1[w_{12}-(\mu s_1)^4 t_{12}] + c_2[w_{22}-(\mu s^1)^4 t_{22}] &= 0 \end{aligned} \tag{72}$$

The condition of compatibility leads to the characteristic relation, expressed as a determinant

$$\begin{vmatrix} w_{11}-(\mu s_1)^4 t_{11} & w_{12}-(\mu s_1)^4 t_{12} \\ w_{12}-(\mu s_1)^4 t_{12} & w_{22}-(\mu s_1)^4 t_{22} \end{vmatrix} = 0$$

which when expanded yields, after introducing the numerical values for t_{11}, etc., indicated in equation (71), the quadratic equation

$$1.646704\left(\frac{\mu s_1}{\pi}\right)^8 - 16.09600\left(\frac{\mu s_1}{\pi}\right)^4 + 5.00000 = 0 \tag{73}$$

with the roots

$$(\mu s_1)_1 = 2.36506$$
$$(\mu s_1)_2 = 5.50895 \tag{74}$$

The smaller root gives the frequency characteristic of the fundamental mode of vibration. The larger, that of the next overtone. To determine the ratio of c_2 to c_1, we substitute $\mu s_1 = 2.36506$, in the first or the second of the equations (72). The substitution in the first gives for c_2/c_1 the value 0.07; in the second, the value 0.05. In view of the smallness of this ratio, it is sufficient to take

$$u = c_1\left(-\frac{1}{6} + \frac{x^2}{2} - \frac{\cos \pi x}{\pi^2}\right) \tag{75}$$

as the expression for the transverse displacement of the bar when vibrating in its fundamental mode.

In order to effect a comparison with the classical theory of the bar with free ends, the consideration of the latter will be restricted to the solution for the case of symmetrical vibration. As $u(s) = -u(-s)$, it follows from equation (22) that

$$u = D(\cos\mu s + \cosh \mu s) + E(\cos\mu s - \cosh \mu s)$$

As the ends of the bar are free,

$$D(-\cos\mu s_1 + \cosh \mu s_1) + E - \cos\mu s_1 - \cosh \mu s_1) = 0$$

$$D(-\sin\mu s_1 + \sinh \mu s_1) + E(\sin\mu s_1 - \sinh \mu s_1) = 0$$

with the condition of compatibility

$$\cot\mu s_1 \tanh \mu s_1 = -1$$

The two smallest roots of this equation are

$$(\mu s_1)_1 = 2.3650204 \tag{76}$$
$$(\mu s_1)_2 = 5.4978039$$

which are in good agreement with the values already found by the approximate method of Ritz and given in equations (74). The values in equations (76) are taken from Rayleigh[4] and are the results of dividing his values of m_1 and m_3 by 2.

To compare the displacements obtained by the two methods, Ritz's and the classical, Table 1 has been prepared, in which the results in the second column have been calculated on the basis of the classical solution and those in the third have been calculated on the basis of equation (75). The end deflections were chosen to be the same for both methods. The agreement appears to be good, the deviations of the results in the two cases being less than 1 per cent of the end deflection.

[4] Rayleigh The Theory of Sound, 5, 2d ed., p. 278.

TABLE 1.—*The lateral displacements in a free-free bar computed by the classical and by Ritz's method*

Distance from mid-point	Displacements	
	Classical method	Ritz's method
0.0	−0.6078	−0.6167
.1	− .5857	− .5937
.2	− .5202	− .5260
.3	− .4142	− .4169
.4	− .2720	− .2714
.5	− .0992	− .0959
.6	+ .0977	+ .1027
.7	+ .3120	+ .3172
.8	+ .5372	+ .5413
.9	+ .76775	+ .7469
1.0	+1.0000	+1.0000

V. RAYLEIGH'S METHOD FOR DETERMINING THE FUNDAMENTAL MODE OF VIBRATION

If in equation (57) the displacements u_r, v_r are those due to the rth overtone in the rth mode of vibration, the quantities T_{rs} and W_{rs} vanish and the determinant (66) reduces to

$$\prod_{r=1}^{r=\infty} (W_{rr} - \mu^4 T_{rr}) = 0$$

where

$$T_{rr} = \int_0^{s_1} (U_r^2 + V_r^2) ds$$

$$W_{rr} = \int_0^{s_1} \kappa_r^2 ds$$

κ_r being the change of curvature. The roots μ_r^4 now are

$$\mu_r^4 = \frac{W_{rr}}{T_{rr}}; \quad (\mu_1^4 < \mu_2^4 < \ldots) \tag{77}$$

Suppose that for the fundamental mode of vibration a radial displacement u, is assumed which is a slight modification of u_1, the actual value, and with the same end conditions as in u_1. The quantities T_{11} and W_{11} are evaluated on the basis of u, and the ratio

$$\bar{\mu}^4 = \frac{\overline{W}_{11}}{\overline{T}_{11}}$$

is formed. Is the approximate value $\bar{\mu}_1$ thus formed, larger or smaller than the true value μ_1? To answer this question we again resort to Ritz's method of approximation. Since u_1 is only slightly in error, it is sufficient to put

$$u_1 = c_1 u_1 + c_2 u_2$$

where u_2 gives the displacement for the mode of vibration next higher than the fundamental. The quantities v_1 and κ_1, the tangential displacement and the change of curvature, respectively, follow from u_1.

The quantities W_{12}, T_{12}, etc., are formed which give rise, as in equation (66), to the determinant

$$\begin{vmatrix} \overline{W}_{11}-\mu^4\overline{T}_{11} & W_{12}-\mu^4 T_{12} \\ W_{12}-\mu^4 T_{12} & W_{22}-\mu^4 T_{22} \end{vmatrix}=0$$

which expands into

$$\overline{W}_{11}-\mu^4\overline{T}_{11}=\frac{(W_{12}-\mu^4 T_{12})^2}{W_{22}-\mu^4 T_{22}} \tag{78}$$

To determine the smallest root μ_1^4, we write

$$\mu_1^4=\overline{\mu}_1^4+\Delta\mu=\frac{\overline{W}_{11}}{\overline{T}_{11}}+\Delta\mu$$

and set it in equation (78), with the result that

$$-\Delta\mu\overline{T}_{11}=\frac{(W_{12}-\mu_1^4 T_{12})^2}{(W_{22}-\mu_1^4 T_{22})} \tag{79}$$

It is easily seen, by virtue of equation (77), that the denominator in the right member of equation (79) is positive. The numerator, being a squared quantity, is also positive. Accordingly $\Delta\mu$ is a negative quantity, since $\overline{T}_{11}$ is positive. Hence, any assumed configuration for the fundamental mode of vibration which is a departure from the actual configuration will give a larger value for μ_1^4 calculated on the basis of

$$\overline{\mu}_1^4=\frac{\overline{W}_{11}}{\overline{T}_{11}} \tag{80}$$

where

$$\overline{W}_{11}=\int_0^{s_1}(u^2+v^2)ds)$$

$$\overline{T}_{11}=\int_0^{s_1}\kappa^2 ds$$

The above result, which we obtain by means of Ritz's method, was stated, originally, by Rayleigh as a corollary to a general theorem on vibrations.[5] In Rayleigh's method, therefore, μ_1, will be made to depend on a parameter β. Then, to obtain the configuration which is the nearest to the actual configuration present during the fundamental mode of vibration, we choose that value of β which makes the right-hand member in (80) a minimum.

[5] Rayleigh, Some General Theorems Relating to Vibrations. London Math. Soc. Proc. IV, pp. 357–368; 1873 (sci. papers, 1, p. 21)

VI. VIBRATION OF A SHORT U BAR

The bar to be considered now is of length $2s_1$. The yoke is a semicircle of radius a and of length $2s_0$. The ratio $\frac{s_1}{s_0}=2$. (See fig. 6.) Hence,

$$\rho=a \qquad \text{when } 0\leqq s\leqq s_0$$

$$\rho=\infty \qquad \text{when } s_0\leqq s\leqq s_1$$

$$s_0=\frac{s_1}{2} \qquad \frac{s_1}{a}=\pi$$

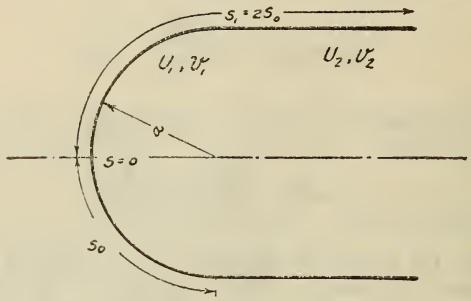

FIGURE 6.—*Median line of the short* U *bar*

The frequency characteristic of vibration of the fundamental mode and the displacements present will be determined by Ritz's method.

The shear is assumed to be given by a sine series, as in the case of the straight bar previously considered. Accordingly,

$$\frac{d\kappa}{ds}=\frac{d^2}{ds^2}\left(\frac{du}{ds}+\frac{v}{\rho}\right)=\sum_{q=1}^{n}\frac{A_2}{s_1^2}\sin q\pi\frac{s}{s_1} \tag{81}$$

Putting $s=s_1x$, a first integration with respect to x gives

$$\kappa=\kappa s_1=\frac{d}{dx}\left(\frac{du}{ds}+\frac{v}{\rho}\right)=\frac{\sum_{q=1}^{n}}{}\pi A_q(C_q+\cos_q\pi x) \tag{82}$$

As the bending moment vanishes at $s=\pm s_1$, or $x=\pm 1$, C_q has the value $+1$ or -1 depending on q. Thus, introducing new constants a_q, equation (82) becomes

$$\begin{aligned}\kappa=\kappa s_1=\frac{d}{dx}\left(\frac{du}{ds}+\frac{v}{\rho}\right)=&a_1(1+\cos\pi x)+\\ &\sum_n a_n(1+\cos n\pi x)+\\ &\sum_m a_m(1-\cos m\pi x)\\ &n=3,\ 5,\ 7\ \ldots\\ &m=2,\ 4,\ 6\ \ldots\end{aligned} \tag{83}$$

Integrating equation (83) with respect to x,

$$\frac{du}{ds}+\frac{v}{\rho}=a_1\left[x+\frac{\sin \pi x}{\pi}\right]+$$
$$\sum_n a_n\left[x+\frac{\sin n\pi x}{n\pi}\right]+$$
$$\sum_m a_m\left[x-\frac{\sin m\pi x}{m\pi}\right]+ \tag{84}$$
$$n=3,\ 5,\ 7\ \ldots$$
$$m=2,\ 4,\ 6\ \ldots$$

The constant of integration vanishes, since the inclination of the bar at the mid-point of the yoke, or at $x=0$, experiences no change. Next, putting

$$\bar{u}=\frac{u}{s_1},\quad \bar{v}=\frac{v}{s}$$

the condition for the inextensibility of the median line gives, for the curved portion

$$\frac{du}{ds}=\frac{1}{\pi}\frac{d^2\bar{v}}{dx^2}$$

Also, in the curved portion, from equation (84)

$$\frac{d^2\bar{v}}{dx^2}+\pi^2\bar{v}=a_1\pi(x+\sin \pi x)+$$
$$\sum_n a_n\pi\left(x+\frac{\sin n\pi x}{n\pi}\right)+$$
$$\sum_m a_m\pi\left(x-\frac{\sin m\pi x}{m\pi}\right)+$$
$$n=3,\ 5,\ 7\ \ldots$$
$$m=2,\ 4,\ 6\ \ldots$$

Solving for $\bar{v}$ in the usual manner, we obtain for the curved portion

$$\bar{v}=\frac{a_1}{\pi^2}\left(\pi x-\frac{\pi x\cos \pi x}{2}+b_1\sin \pi x\right)+$$
$$\sum_n \frac{a_n}{\pi^2}\left(\pi x-\frac{\sin n\pi x}{n(n^2-1)}+b_n\sin \pi x\right)+\ldots \tag{85}$$
$$\sum_m \frac{a_m}{\pi^2}\left(\pi x+\frac{\sin m\pi x}{m(m^2-1)}+b_m\sin \pi x\right)+\ldots$$
$$n=3,\ 5,\ 7\ \ldots$$
$$m=2,\ 4,\ 6\ \ldots$$

since the tangential displacement v vanishes at $x=0$. The constants b_1, b_n, and b_m are evaluated on the basis of the condition that the

center of gravity of the bar remains at rest independently of a_1, a_n, and a_m. Accordingly, we have, from equation (26),

$$b_1 = -\frac{2}{\pi} - \frac{\pi}{4} + \frac{1}{8}$$

$$b_n = -\frac{2}{\pi} - \frac{\pi}{4} + \frac{\sin \frac{n\pi}{2}}{2n(n^2-1)}$$

$$b_m = -\frac{2}{\pi} - \frac{\pi}{4} + \frac{2}{\pi}\frac{\cos \frac{m\pi}{2}}{(m^2-1)^2}$$

$$n = 3, 5, 7 \ldots$$

$$m = 2, 4, 6 \ldots$$

The radial displacement u divided by s_1 in the curved portion follows from the condition of inextensibly, and, therefore, is

$$\bar{u} = \frac{a_1}{\pi^2}\left(1 - \frac{\cos \pi x}{2} + \frac{\pi x \sin \pi x}{2} + b_1 \cos \pi x\right) +$$
$$\sum_n \frac{a_n}{\pi^2}\left(1 - \frac{\cos n\pi x}{n^2-1} + b_n \cos \pi x\right) + \ldots$$
$$\sum_m \frac{a_m}{\pi^2}\left(1 + \frac{\cos m\pi x}{m^2-1} + b_m \cos \pi x\right) + \ldots \tag{86}$$
$$n = 3, 5, 7 \ldots$$
$$m = 2, 4, 6 \ldots$$

The expression which gives the change of inclination in the straight portion of the bar simplifies to

$$\frac{d\bar{u}}{dx} = a_1[x + \sin \pi x] +$$
$$\Sigma a_n\left[x + \frac{\sin n\pi x}{n\pi}\right] + \ldots$$
$$\Sigma a_m\left[x - \frac{\sin m\pi x}{m\pi}\right] + \ldots$$
$$n = 3, 5, 7 \ldots \tag{87}$$
$$m = 2, 4, 6 \ldots$$

since the radius of curvature ρ is infinite. Accordingly, the lateral displacement in the straight portion is given by

$$\bar{u} = \frac{a_1}{\pi^2}\left(\frac{\pi^2 x^2}{2} - \cos \pi x + c_1\right) +$$
$$\Sigma \frac{a_n}{\pi^2}\left(\frac{\pi^2 x^2}{2} - \frac{\cos n\pi x}{n^2} + c_n\right) + \ldots$$
$$\Sigma \frac{a_m}{\pi^2}\left(\frac{\pi^2 x^2}{2} + \frac{\cos m\pi x}{m^2} + c_m\right) + \ldots$$
$$n = 3, 5, 7 \ldots$$
$$m = 2, 4, 6 \ldots$$

The normal displacement is continuous everywhere. Consequently, the different expressions (86) and (87) for u give the same numerical value when x is put equal to 1/2. This determines the constants c_1, c_n, and c_m, with the following values:

$$c_1 = 1 + \frac{\pi}{4} - \frac{\pi^2}{8}$$

$$c_n = 1 - \frac{\pi^2}{8}$$

$$c_m = 1 - \frac{\pi^2}{8} + \frac{\cos\frac{m\pi}{2}}{m^2(m^2-1)}$$

$$m = 2, 4, 6 \ldots$$

The tangential displacement in the straight portion of the bar is constant. Since the tangential displacement is continuous everywhere, it follows from equation (85) that

$$\bar{v} = \sum_{r=1}^{n+1} \frac{a_r}{\pi^2}\left(\frac{\pi}{2} + b_r\right) \tag{88}$$

We next write, in analogy with equation (57)

$$u = (a_1\bar{u}_1 + a_2\bar{u}_2 + a_3\bar{u}_3 + \ldots)\frac{s_1}{\pi^2}$$

$$v = (a_1\bar{v}_1 + a_2\bar{v}_2 + a_3\bar{v}_3 + \ldots)\frac{s_1}{\pi^2}$$

$$\kappa = (a_1\bar{\kappa}_1 + a_2\bar{\kappa}_2 + a_3\bar{\kappa}_3 = \ldots)\frac{1}{s_1}$$

where the expressions for the typical terms u_r, v_r, κ_r are obtained by comparison with equations (85), (86), (87), (88), and (83).

Introducing the definitions

$$T_{rs} = s_1^2 \int_0^{s_1} (\bar{v}_r\bar{v}_s + \bar{u}_r\bar{u}_s)\, ds = s_1^3 \int_0^{\pi} (\bar{v}_r\bar{v}_s + \bar{u}_r\bar{u}_s) dx$$

$$W_{rs} = \frac{1}{s_1^2}\int_0^{s_1} \bar{\kappa}_r\bar{\kappa}_s d_s = \frac{1}{s_1}\int_0^{\pi} \bar{\kappa}_r\bar{\kappa}_s dx$$

the quantities T_{11}, W_{11}, etc., prove to have the numerical values

$$T_{11} = 9.64721\frac{s_1^3}{\pi^4}, \quad W_{11} = \frac{3}{2}s^{-1}$$

$$T_{12} = 6.17078\frac{s_1^3}{\pi^4}, \quad W_{12} = s_1^{-1} \tag{89}$$

$$T_{22} = 4.09341\frac{s_1^3}{\pi^4}, \quad W_{22} = \frac{3}{2}s^{-1}$$

Neglecting in the expansion for the displacement the terms with the coefficients a_g, where g is larger than 2, the system of homogeneous equations in a_g corresponding to equations (65) reduce to

$$a_1(W_{11}-\mu^4T_{11})+a_2(W_{12}-\mu^4T_{12})=0$$
$$a_1(W_{12}-\mu^4T_{12})+a_2(W_{22}-\mu^4T_{22})=0$$

with the condition of compatibility

$$\begin{vmatrix} W_{11}-\mu^4T_{11} & W_{12}-\mu^4T_{12} \\ W_{12}-\mu^4T_{12} & W_{22}-\mu^4T_{22} \end{vmatrix}=0$$

which, with the numerical values of W_{11}, etc., given above, reduces to

$$1.41146\left(\frac{\mu s_1}{\pi}\right)^8-8.25937\left(\frac{\mu s_1}{\pi}\right)^4+1.25=0. \tag{90}$$

To find the root of the smallest value; that is, the frequency characteristic of the fundamental mode, we take first

$$W_{11}-\mu^4T_{11}=0$$

or, with numerical values substituted,

$$\left(\frac{\mu s_1}{\pi}\right)^4=0.15548.$$

Next, we take

$$\left(\frac{\mu s_1}{\pi}\right)^4=0.15548+z$$

and substitute in the equation (90). The substitution gives,

$$z=0.000006.$$

Accordingly, the first approximations give a value for the frequency characteristic μs_1 of 1.97275, which is sufficiently accurate for practical purposes. We also infer that there is no need to keep the terms containing a_2 in the expressions for the displacements for the fundamental mode of vibration.

Hence, the radial and tangential displacements, u and v, in the fundamental mode of vibration of the short U bar, may be written

$$u=a_1\left[1-\frac{\cos\pi x}{2}+\frac{\pi x\sin\pi x}{2}+\left(\frac{1}{8}-\frac{2}{\pi}-\frac{\pi}{4}\right)\cos\pi x\right],\text{ when } 0\leqq x\leqq\tfrac{1}{2}$$

$$v=a_1\left[\pi x-\frac{\pi x\cos\pi x}{2}+\left(\frac{1}{8}-\frac{2}{\pi}-\frac{\pi}{4}\right)\sin\pi x\right]\text{ when } 0\leqq x\leqq\tfrac{1}{2}$$

$$u=a_1\left[\frac{\pi^2x^2}{2}-\cos\pi x+1+\frac{\pi}{4}-\frac{\pi^2}{8}\right]\text{ when } \tfrac{1}{2}\leqq x\leqq 1 \tag{91}$$

$$v=a_1\left[\frac{1}{8}-\frac{2}{\pi}+\frac{\pi}{2}\right]\text{ when } \tfrac{1}{2}\leqq x\leqq 1$$

where $x=\frac{s}{s_1}$, s_1 is the half length of the bar, s is the distance of a point from the mid-point of the yoke, measured on the median line. As for the frequence of vibration, N,

$$N=\frac{3.8917}{2\pi}\sqrt{\frac{B}{\omega\gamma}}\cdot\frac{1}{s_1^2}$$

where B is the flexural rigidity, ω the cross-sectional area, and γ the density of the bar.

The numerical values of the displacements, evaluated from the formula (91) are given in Table 2. The radial or the lateral displacement of the ends is assumed to be unity. It now appears from this table that the radial displacement vanishes at the point $s=0.244s_1$. Also, that the tangential displacement is 0.0594. We infer from this that, when the short U bar is held vertical, the point $s=0.24s_1$ executes maximum vertical and horizontal displacements of equal value and of amount 0.042 mm when the amplitude of the end deflections is 1 mm. This follows from equation (2).

TABLE 2.—*The tangential and radial displacements of a vibrating short* U *bar*

Fractional distance from mid-point	Radial displacement	Tangential displacement
0.0	−0.1229	0.0000
.1	−.1018	−.0365
.2	−.0415	−.0599
.3	+.0501	−.0591
.4	+.1761	−.0264
.5	+.2752	+.0422
.6	+.4065	+.0422
.7	+.5487	+.0422
.8	+.6966	+.0422
.9	+.8479	+.0422
1.0	+1.0000	+.0422

VII. EFFECT OF THE YOKE ON VIBRATION

It was shown by Lamb [6] that when a straight bar is curved slightly the frequency of vibration is lowered and the nodal points are brought together slightly. The same effects are illustrated here. As one passes from the example of the straight bar, to the short U bar, defined above, and then to the elongated U bar with a length ten times as great as that of the yoke, the frequency characteristic μs_1 takes the values 2.365, 1.973, 1.880, successively. With regard to the variation of displacements in these examples, as effected by the presence of a yoke, we refer to Figure 7, where the radial displacements are plotted against the fractional distances from the mid-point. This figure illustrates two important effects, as resulting from increased length of the yoke, while the total length of the bar is kept the same; that is, the points, where the radial displacements vanish, are brought nearer to the mid-point of the yoke, and the ratio of the radial displacement at the mid-point of the yoke to the end deflection is made smaller.

[6] Horace Lamb, On the Flexure of a Curved Bar Proc. London Math. Soc., **14**, p. 367; 1888.

VIII. REPRODUCTION BY A SINGLE LOAD OF DEFORMATIONS DUE TO VIBRATION

Theoretically, to reproduce in static tests the strained form of the vibrating U bar, a continuous distribution of externally applied forces is required. In practice, however, it is sufficient to have the form obtained under the action of a single load. If the point of application of the single load is properly chosen, the form thus secured will not differ very much from the dynamic form during vibration. This is true only for the fundamental mode of vibration, of course. For the determination of the proper location of the single load Rayleigh's method is of great value.

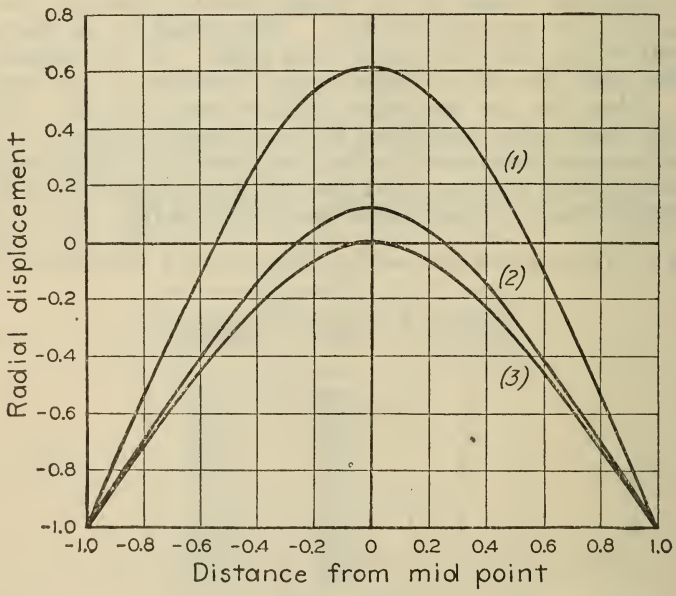

FIGURE 7.—*Effect of the presence of the yoke on the radial displacement u*

Curve (1) refers to a straight bar; curved (2) to the short U bar; curve (3) to an elongated U bar of ratio $s_0/s_1=0.10$. The apex of the curve (3) is slightly above the zero line of the radial displacement. The distance from the mid-point is given as the fractional value of the half length of the bar, measured on the median line and the radial displacement as the fractional value of the end displacement.

Let s_2 be the distance measured on the median line of the point of application of the single load X from the mid-point of the yoke. (See fig. 8.) Also let $2s_1$ be the total length of the bar and $2s_0$, the length of the semicircular yoke of radius a. Denote the radial and tangential displacements in the yoke by u_1 and v_1; in the portion from $s=s_0$ to $s=s_2$ by u_2 and v_2; and in the portion $s=s_2$ to $s=s_1$ by u_3 and v_3.

It is clear that the bending moment G at s, when s is in the yoke, is given by

$$G=X\left(s_2-s_0+a\cos\frac{s}{a}\right)$$

and the change of curvature κ_1, by

$$\kappa_1 = \frac{d}{ds}\left(\frac{du_1}{ds} + \frac{v_1}{a}\right)$$

Hence

$$\frac{d}{ds}\left(\frac{du_1}{ds} + \frac{v_1}{a}\right) = \frac{X}{B}\left(s_2 - s_0 + a \operatorname{cas}\frac{s}{a}\right)$$

where B represents the flexural rigidity of the bar, uniform throughout. Making the substitutions

$$s = a\theta$$

$$\frac{s_2 - s_0}{a} = n$$

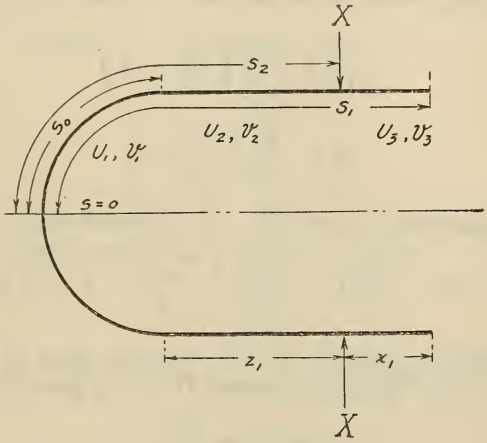

FIGURE 8.—*Position of the point of application of a single load acting on a U bar*

the equation (91) becomes

$$\frac{d}{d\theta}\left(\frac{du_1}{d\theta} + v_1\right) = \frac{X}{B}a^3(n + \cos\theta)$$

the integration of which, with respect to θ, gives, since there is no change of inclination at the mid-point of the yoke ($\theta = 0$)

$$\frac{du_1}{d\theta} + v_1 = \frac{X}{B}a^3(n\theta + \sin\theta)$$

Or, introducing the condition of inextensibility,

$$\frac{d^2v_1}{d\theta} + v_1 = \frac{X}{B}a^3(n\theta + \sin\theta)$$

Solving for v_1, in the usual manner

$$v_1(\theta) = \frac{X\,a^3}{B}\left(n\theta - \frac{\theta\cos\theta}{2} + A\sin\theta\right) \tag{92}$$

since v_1 vanishes for $\theta=0$. The term $\frac{AXa^3}{B}\sin\theta$ represents a rigid displacement. In fact, it will be the displacement caused by moving the bar along its axis by an amount $\frac{AXa^3}{B}$. As the present analysis is for the purpose of finding the static deformation as nearly as possible like the deformation during vibration, it is evident that the determination of the constants of integration should be made by using the appropriate conditions implied by the dynamic deformation. As will be remembered, the condition of constancy of the center of gravity

$$v_1\left(\frac{\pi}{2}\right)=-\frac{2a}{s_1-s_0}\int_0^{\frac{\pi}{2}} v_1 \sin\theta\, d\theta$$

is one of these, and serves to determine A. Accordingly

$$v_1\left(\frac{\pi}{2}\right)=\frac{Xa^3}{B}\left(\frac{\pi n}{2}+A\right)$$

and, hence

$$A=-\frac{n}{2}\left(\frac{\pi}{2}+\frac{4}{\pi}\right)+\frac{1}{8} \tag{93}$$

The radial displacement u_1 results, by differentiating v_1, and is

$$u_1(\theta)=\frac{Xa^3}{B}\left(n+\frac{\theta\sin\theta}{2}-\frac{\cos\theta}{2}+A\cos\theta\right) \tag{94}$$

Next consider the portion in the straight part of the bar, from $s=s_0$ to $s=s_2$. The bending moment G at s is given by

$$G=X(s_2-s)$$

and the change of curvature κ_2 by

$$\kappa_2=\frac{d^2u_2}{ds^2}$$

Hence,

$$\frac{du_2}{ds}=\frac{X}{B}(s_2-s)$$

which, after making the substitution

$$z=s_2-s$$

becomes

$$\frac{du_2}{dz}=\frac{X}{B}z$$

Two successive integrations with respect to z give

$$u_2=\frac{X}{B}\left(\frac{2^3}{6}+\alpha z_1^2 z+\beta z_1^3\right) \tag{95}$$

where

$$z_1=s_2-s_0$$

The constants of integration α and β are to be determined from the conditions of continuity of the radial displacement and the change of inclination at $s=s_0$:

$$(u_1)_{\theta=\frac{\pi}{2}}=(u_2)_{z=z1}$$

$$\left(\frac{du_1}{ds}+\frac{v_1}{a}\right)_{\theta=\frac{\pi}{2}}=-\left(\frac{du_2}{dz}\right)_{Z=z1}$$

Accordingly

$$a^3\left(n+\frac{\pi}{4}\right)=z_1^3\left(\frac{1}{6}+\alpha+\beta\right)$$

$$a^2\left(\frac{n\pi}{2}+1\right)=z_1^2\left(\frac{1}{2}+\alpha\right)$$

As $na=z_1$, the constants α and β prove to be

$$-\alpha=\frac{1}{2}+\frac{1}{n^2}+\frac{\pi}{2n}$$

$$\beta=\frac{1}{3}+\frac{2}{n^2}+\frac{\pi}{4n^3}+\frac{\pi}{2n}$$

The tangential displacement v_2 is constant, and is

$$v_2=v_1\left(\frac{\pi}{2}\right) \tag{96}$$

Finally, for the portion from s_2 to s_1, the radial displacement u_3 is given by

$$u_3=\frac{X}{B}(-\alpha z_1^3x+\beta z_1^3) \tag{97}$$

where

$$x=s-s_2$$

The displacement in this portion is a rigid displacement and fulfills the conditions

$$(u_3)_{x=0}=(u_2)_{z=0}$$

$$\left(\frac{du_3}{dx}\right)_{x=0}=-\left(\frac{du_2}{dz}\right)_{z=0}$$

The tangential displacement is constant, and is

$$v_3=v_1\left(\frac{\pi}{2}\right) \tag{97}$$

The quantities n, x_1 and z_1, enter variously in the separate expressions of the displacements. The quantities x_1 and z_1 depend on n

$$na=z_1$$

$$x_1/z_1=\frac{\pi}{2n}\left(\frac{s_1}{s_0}-1\right)$$

Accordingly when n is given, x_1 and z_1 follow. To determine n, we remark that if the same bar is put in vibration the deformations will be given by

$$u=u_1 \cos kt \qquad v=v_1 \cos kt$$

$$u=u_2 \cos kt \qquad v=v_2 \cos kt$$

$$u=u_3 \cos kt \qquad v=v_3 \cos kt$$

when

$$0 \leqq s \leqq s_0$$

$$s_0 \leqq s \leqq s_2$$

$$s_2 \leqq s \leqq s_1$$

and proceed according to Rayleigh's method.

Introducing the definitions (given in connection with equation (80)).

$$\overline{T}_{11}=\int_0^{s_1}(u^2+v^2)ds$$

$$\overline{W}_{11}=\int_0^{s}\kappa^2 ds$$

and substituting the appropriate value of the displacements and of the change of curvature from above, it is found that

$$\overline{T}_{11}=\frac{X^2}{B^2}(a^7e_1+a^7e_2+z_1{}^7e_3+z_1{}^7e^4)$$

$$\overline{W}_{11}=\frac{X^2}{B^2}(a^3w_1+z_1{}^3w_2)$$

where

$$e_1=n^2\left(\frac{\pi}{2}+\frac{\pi^3}{24}\right)+n\left(2-\frac{\pi^2}{4}+4A\right)+A_2\frac{\pi}{2}-A\frac{\pi}{4}+\frac{\pi^3}{96}$$

$$e_2=\frac{\pi}{2}\left(\frac{n\pi}{2}+A\right)^2$$

$$e_3=\frac{1}{252}+\frac{\alpha^2}{3}+\beta^2+\frac{\alpha}{15}+\frac{\beta}{12}+\alpha\beta$$

$$e^4=\frac{\alpha^2}{3}\left(\frac{x_1}{z_1}\right)^3-\alpha\beta\left(\frac{x_1}{z_1}\right)^2+\beta^2\frac{x_1}{z_1}$$

$$w_1=\frac{n^2\pi}{2}+2n+\frac{\pi}{4}$$

$$w_2=\frac{1}{3}$$

We require the ratio of W_{11} to T_{11}. This is from the above

$$\frac{\overline{W}_{11}}{\overline{T}_{11}}=\frac{a^3w_{11}+z_1{}^3w_2}{a^7(e_1+e_2)+z_1{}^7(e_3+e_4)}$$

Also, since $z_1 = na$ and $s_0 = \frac{\pi}{2} a$

$$\frac{\overline{W}_{11}}{\overline{T}_{11}} = \left(\frac{\pi}{s_1}\right)^4 \left(\frac{s_1}{2s_0}\right)^4 \frac{N_1(n)}{N_2(n)} \tag{99}$$

where

$$N_1(n) = w_1 + n^3 w_2$$

$$N_2(n) = e_1 + e_2 + n^7(e_3 + e_4)$$

Thus, for the frequency characteristic, we obtain from equation (99) the expression

$$\left(\frac{\mu s_1}{\pi}\right)^4 = \left(\frac{s_1}{2s_0}\right)^4 \frac{N_1(n_1)}{N_2(n_1)} \tag{100}$$

where n_1 is the root of the equation

$$\frac{d}{dn} \frac{N_1(n)}{N_2(n)} = 0$$

as is required by Rayleigh's method. This particular value of $n = n_1$, then, determines the position s_2

$$s_2 = \left(n_1 + \frac{\pi}{2}\right) a$$

of the point of application of the single load to obtain a statical deformation reproducing the deformation present during vibration.

We shall now determine the value of $n = n_1$, for the example of the short U bar $\left(\frac{s_1}{s_0} = 2\right)$, already treated by Ritz's method. The value of n_1 is best obtained graphically. Selecting various values for n, the quantities $N_1(n)$ and $N_2(n)$ are calculated and also the ratios of $N_1(n)$ to $N_2(n)$. These calculated values are given in Table 3. Plotting in Figure 9 the ratio $N_1(n)$ to $N_2(n)$ against n, we find that the minimum value of the ratio occurs nearly at $n = 1$. Accordingly when the single load is applied at the point

$$s_2 = 0.815 s_1$$

the resulting static deformation will for the short U bar most closely approximate the deformation realized during vibration.

This particular value gives for the frequency characteristic the value

$$\left(\frac{\mu s_1}{\pi}\right)^4 = 0.15600$$

which is only slightly higher than the value obtained by Ritz's method (=0.15548).

Similarly as for the short U bar (fig. 8), let s_2 be the point of application on an elongated U bar of a single load which gives the same form of deformation as that when the bar is in vibration.

TABLE 3.—*Evaluation of the ratio $N_1(n)$ to $N_2(n)$*

n	$N_1(n)$	$N_2(n)$	$\frac{N_1(n)}{N_2(n)}$
1.5708	9.0948	56.938	0.15973
1.3090	6.8383	43.519	.15714
1.0472	4.9852	31.950	.15604
1.0000	4.6895	30.061	.15600
.9500	4.3888	28.131	.15601
.7854	3.4866	22.265	.15659
.5236	2.3111	14.497	.15942

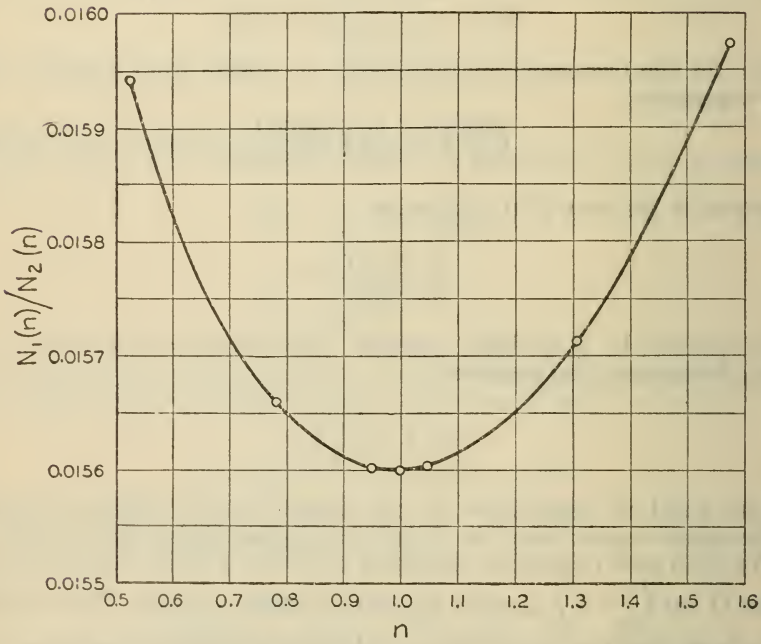

FIGURE 9.—*Graph to determine the minimum value of the ratio $N_1(n)$ to $N_2(n)$*

n is given by the relation $s_2-s_0=na$, where s_2 is the distance of the point of application of the single load X from the mid-point of the yoke, measured on the median line; $2s_0$ is the length of the yoke; a is the radius of the yoke.

Rayleigh,[7] as an illustration of his method, has considered the example of clamped-free bar of length s_1 and found that when the single load is applied at a point whose distance from the clamped end is 3/4 s_1, the deformation thus obtained approximates the form of the vibrating bar. Now, it was shown in the treatment of the elongated U bar of length $2s_1$, that the frequency of vibration is practically that of a clamped-free bar of length s_1. Accordingly, to obtain in statical tests on elongated U bars a deformation which approximates most closely that when the same bar is vibrating, it suffices to apply the load at a point $s_2=0.75S_1$.

IX. ACKNOWLEDGMENT

It is a pleasure to acknowledge the encouragement and assistance of Prof. F. D. Murnaghan in the preparation of this paper.

WASHINGTON, December 11, 1930.

[7] Rayleigh, The Theory of Sound. 1, 2d ed., p. 284.