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Frequency Conversion With Nonlinear Reactance

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A lossless nonlinear impedance subject to an almost periodic voltage (sum of sinusoids) will absorb power at some frequencies and supply power at other frequencies. Necessary and sufficient relations among these powers are found. It is shown that simple cubic capacitors ($Q \propto V^3$) are sufficient for producing any possible conservative modulation or distortion process.

1. General Considerations

Let there be a set of linearly related positive frequencies ω_i, the relations being expressible as

$$\omega_i = \sum_j k_{ij}\omega_j, \tag{1}$$

with the k_{ij} rational numbers. Some of these frequencies (base frequencies) can be independently chosen, and the remaining ones (derived frequencies) expressed by

$$\omega_{di} = \sum_j m_{ij}\omega_{bj}, \tag{2}$$

or, in matrix notation,

$$\Omega_d = \boldsymbol{M}\Omega_b. \tag{3}$$

It is possible to have the ω_i comprise subsets such that

$$\Omega_d^{(j)} = \boldsymbol{M}^{(j)}\Omega_b^{(j)} \qquad j=1\ \ldots\ n, \tag{4}$$

but this represents a highly artificial case in practice.

Let the power absorbed by the nonlinear reactor at frequency ω_i be P_i, and for convenience let $\mathscr{P}_i \equiv P_i/\omega_i$.

Now

$$\sum_i P_i \equiv \sum_i \mathscr{P}_i\omega_i = \sum_i \mathscr{P}_{bi}\omega_{bi} + \sum_i \mathscr{P}_{di}\omega_{di} \tag{5}$$

$$= \mathfrak{P}_b\Omega_b + \mathfrak{P}_d\Omega_d$$

$$= (\mathfrak{P}_b + \mathfrak{P}_d\boldsymbol{M})\Omega_b,$$

and in the case where the ω comprise subsets:

$$\sum_i P_i = \sum_j (\mathfrak{P}_b^{(j)} + \mathfrak{P}_d^{(j)}\boldsymbol{M}^{(j)})\Omega_{bj}. \tag{5'}$$

But for a lossless device, $\sum P_i = 0$ for all choices of the independent (base) frequencies, so

$$(\mathfrak{P}_b^{(j)} + \mathfrak{P}_d^{(j)}\boldsymbol{M}^{(j)})\Omega_{bj} = 0 \qquad \text{all } j \tag{6a}$$

and

$$\mathfrak{P}_b^{(j)} + \mathfrak{P}_d^{(j)}\boldsymbol{M}^{(j)} = 0. \qquad \text{all } j \tag{6b}$$

From (6a), $\sum_i P_i^{(j)} = 0$, so power is conserved for each subset of related frequencies, as well as in toto. Henceforward we shall drop the superscript (j) for convenience, rewriting (6b) as

$$\mathfrak{P}_b + \mathfrak{P}_d\boldsymbol{M} = 0. \tag{7}$$

Hence the power at each base frequency is determined by the set of powers at the derived frequencies. Ordinarily, the base frequencies are taken as the incommensurate frequencies of the sources; the derived frequencies, those of the sinks. Hence the distribution of power load among the sources is fixed by the power dissipated in the sinks. Expanding (7) into its components:

$$\mathscr{P}_{bj}+\sum_i \mathscr{P}_{ai} m_{ij}=0 \qquad \text{all } j. \tag{8}$$

Expressing *all* ω_j in terms of the base frequencies:

$$\omega_i=\sum_j l_{ij}\,\omega_{bj}, \tag{9}$$

so that the $l_{ij}=m_{ij}$ for ω_i a derived frequency, and $l_{ij}=$

$$\delta_{ij}=\begin{cases}0 & i\neq j\\ 1 & i=j\end{cases}$$

for ω_i a base frequency.

Then $P_{bj}\equiv\sum_i \mathscr{P}_{bi} l_{ij}$ and (8) becomes

$$0=\sum_i \mathscr{P}_{bi} l_{ij}+\sum_i \mathscr{P}_{ai} l_{ij}=\sum_i \mathscr{P}_i l_{ij}, \tag{10}$$

or

$$\sum_i P_i l_{ij}/\omega_i=\sum_i\left\{P_i l_{ij}\Big/\sum_n l_{in}\omega_{bn}\right\}=0, \qquad \text{all } j. \tag{11}$$

Because the subscript j refers to base frequencies in the above, there are as many independent relations (11) among the P_i as there are base frequencies. For the particular case of two base frequencies, and the l_{ij} restricted to integers, equations (11) become equations (24) and (25) given by Manley and Rowe.[1] The result, however, is a necessary consequence of conservation of energy in any lossless system, and includes subharmonic combination frequencies.

For further results, it is convenient to re-express the linear dependence among frequencies without reference to the choice of base:

$$\boldsymbol{E}\Omega=0 \tag{12}$$

or

$$\sum_j e_{ij}\omega_j=0 \qquad \text{all } i \tag{12a}$$

with all e_{ij} integers.

If there are B base frequencies, and D derived frequencies, then $\boldsymbol{M}$ (eq(3)) is a $D\times B$ matrix of rational numbers. Let the least common denominator of the first row of $\boldsymbol{M}$ be ϵ_{11}; of the second row, ϵ_{22}; etc., up to ϵ_{DD}. Then, premultiplying (3) by the nonsingular diagonal matrix,

$$\mathfrak{E}_1=\begin{vmatrix}\epsilon_{11} & & & \mathbf{0}\\ & \cdot & & \\ & & \cdot & \\ \mathbf{0} & & & \epsilon_{DD}\end{vmatrix}$$

yields

$$\mathfrak{E}_1\Omega_d=\mathfrak{E}_1\boldsymbol{M}\Omega_b\equiv-\mathfrak{E}_2\Omega_b \tag{13}$$

$$\mathfrak{E}_1\Omega_d+\mathfrak{E}_2\Omega_b=0, \tag{13a}$$

or

$$\mathfrak{E}\Omega=0, \tag{14}$$

[1] J. M. Manley and H. E. Rowe, Some general properties of nonlinear elements. I: General energy relations, Proc. IRE **44**, 904–913 (1956).

where

$$\mathfrak{E} = \begin{array}{c|c|c|} & D & B \\ \hline D & \mathscr{E}_1 & \mathscr{E}_2 \\ \hline \end{array}$$

$$\Omega = \begin{array}{c|c|} & 1 \\ \hline D & \omega_d \\ \hline B & \omega_b \\ \hline \end{array}$$

The matrix $\mathfrak{E}$ just defined is a minimal matrix of the type $\boldsymbol{E}$ in (12); it contains no redundant relations among the ω. We make the notational distinction because later on we shall need an $\boldsymbol{E}$ containing redundancies.

Because $\mathfrak{E}_1$ is nonsingular, (13) can be solved for

$$\boldsymbol{M} = -\mathfrak{E}_1^{-1}\mathfrak{E}_2 \tag{15}$$

and (7) rewritten as

$$\mathfrak{P}_b = \mathfrak{P}_d \mathfrak{E}_1^{-1}\mathfrak{E}_2.$$

Let $\mathfrak{H} \equiv \mathfrak{P}_d \mathfrak{E}_1^{-1}$, a $1 \times D$ (row) matrix. Then

$$\mathfrak{P}_d = \mathfrak{H}\mathfrak{E}_1 \tag{16a}$$

$$\mathfrak{P}_b = \mathfrak{H}\mathfrak{E}_2 \tag{16b}$$

$$\mathfrak{P} = \mathfrak{H}\mathfrak{E}, \tag{17}$$

where

$$\mathfrak{P} = \begin{array}{c|c|c|} & D & B \\ \hline 1 & \mathscr{P}_d & \mathscr{P}_b \\ \hline \end{array}$$

The relation (17) (with $\mathfrak{E}$ defined by (14)) is not only necessary for conservation of power, but it is also sufficient for, by (14),

$$\sum_i P_i \equiv \mathfrak{P}\Omega = \mathfrak{H}\mathfrak{E}\Omega = 0.$$

Equation (17) also yields insight into the nature of the frequency mixing process, for (17) is equivalent to

$$P_i = \omega_i \sum_n h_n \epsilon_{ni}, \tag{18}$$

and defining

$$P_{in} \equiv h_n \epsilon_{ni} \omega_i,$$

we have

$$P_i = \sum_n P_{in} \tag{19a}$$

$$\sum_i P_{in} = 0, \tag{19b}$$

so that if P_{in} is considered as the power absorbed at ω_i in the nth "mixing process," the total power P_i is the net result of a set of conservative mixing processes. In any particular process, power is absorbed at those frequencies for which h_n and ϵ_{ni} are of the same sign (positive resistance presented to the ith source) and power is delivered at those frequencies for which ϵ_{ni} is of the opposite sign (negative resistance presented).

We also note that if we consider all frequencies involving a given base, the sum of the corresponding powers vanishes. For, from (2), the derived frequencies associated with the jth base are

$$\omega_{di,j} = m_{ij}\omega_{bj}$$

and from (8),

$$\mathscr{P}_{bj} + \sum_i \mathscr{P}_{di} m_{ij} = 0,$$

hence

$$0 = \mathscr{P}_{bj}\omega_{bj} + \sum_i \mathscr{P}_{di} m_{ij}\omega_{bj} \tag{19c}$$

$$= \mathscr{P}_{bj}\omega_{bj} + \sum_i \mathscr{P}_{di}\omega_{di,j} = P_{bj} + \sum_i P_{di,j}.$$

The equivalent conditions (7) and (17) are not only necessary in the generation of modulation and distortion products by lossless devices, they are also sufficient. That is, for any given $\boldsymbol{M}$ and $\mathscr{P}_d$, a suitable nonlinear network can be found. To prove sufficiency, we first need detailed relations for the behavior of a nonlinear reactor. A capacitor will be used in the analysis, but an inductor would be equivalent with suitable changes of notation.

2. The Mixing Process

Let the voltage

$$V = \sum_1^M a_i \cos(\omega_i t + \theta_i) \tag{20}$$

be impressed on the nonlinear capacitance

$$Q = CV^N, \qquad C > 0.$$

Let $\beta_i \equiv \omega_i t + \theta_i$ for convenience.
Then

$$Q = CV^N = C\left(\sum_1^M a_i \cos \beta_i\right)^N = C\left(\sum_1^M a_i (e^{j\beta_i} + e^{-j\beta_i})/2\right)^N. \tag{21}$$

Applying the polynomial law:

$$Q = C2^{-N}N!\sum_s \prod_{i=1}^M a_i^{s_i}(e^{j\beta_i} + e^{-j\beta_i})^{s_i}/s_i!,$$

where $\sum_s$ indicates the sum over all sets of integers s satisfying

$$s_i \geq 0, \qquad \sum s_i = N.$$

Expanding the binomial yields

$$Q = C2^{-N}N!\sum_s \prod_i a_i^{s_i} \sum_{k_i=0}^{s_i} e^{jk_i\beta_i} e^{-j(s_i-k_i)\beta_i}/k_i!(s_i-k_i)!$$

Let $l_i \equiv 2k_i - s_i$, so that

$$Q = C2^{-N}N!\sum_s \prod_i a_i^{s_i} \sum_{l_i=-s_i}^{s_i} e^{jl_i\beta_i} \Big/ \left(\frac{s_i+l_i}{2}\right)!\left(\frac{s_i-l_i}{2}\right)!, \tag{22}$$

the l_i appearing in $\sum_i$ being those integers between $-s_i$ and $+s_i$ that make $s_i + l_i$ even. For convenience, let

$$[s_i \pm l_i] \equiv \left(\frac{s_i+l_i}{2}\right)!\left(\frac{s_i-l_i}{2}\right)! \tag{23}$$

Then

$$Q=C2^{-N}N!\sum_{s}(\prod_{j} a_j^{s_j})\{\prod_{i}\sum_{l_i} e^{jl_i\beta_i}/[s_i\pm l_i]\},$$

and the term in braces can be rewritten as

$$\sum_{l_1}\sum_{l_2}\ldots\sum_{l_M}\prod_{i=1}^{M} e^{jl_i\beta_i}/[s_i\pm l_i]=\sum_{l_1}\ldots\sum_{l_M} e^{j\sum_i l_i\beta_i}\prod_i[s_i\pm l_i]^{-1},$$

making

$$Q=C2^{-N}N!\sum_{s}\sum_{l}\prod_{i} a_i^{s_i}e^{j\sum_i l_i\beta_i}/[s_i\pm l_i]. \tag{24}$$

Now in summing over l, any set of values $l_1, l_2 \ldots l_M$ may be paired with the set of equal and opposite values. Because $[s_i\pm l_i]$ is even, the odd part of the exponential ($\sin \sum_i l_i\beta_i$) will not contribute to the sum, and we can write

$$Q=C2^{-N}N!\sum_{s}\sum_{l}\prod_{i} a^{s_i}[s_i\pm l_i]^{-1}\cos\left\{\sum_i l_i(\omega_i t+\theta_i\right\}, \tag{25}$$

where $\omega_i t+\theta_i$ has been restored for β_i.

The term of frequency ω_k arises from values of $l_1, l_2, \ldots l_M$ that make $\pm\omega_k=\sum_i l_i\omega_i$, or

$$\sum_{i\neq k} l_i\omega_i+(l_k\pm 1)\omega_k=0. \tag{26}$$

If *all* the integral linear relations among the frequencies are given by the "mixing processes,"

$$\sum_i e_{pi}\omega_i=0 \qquad (\boldsymbol{E}\Omega=0), \tag{27}$$

the ω_k terms arise from

$$l_i=e_{pi}, \qquad i\neq k$$

$$l_k=e_{pk}\pm 1,$$

and

$$Q=C2^{-N}N!\sum_{k}\sum_{s}\sum_{p}\prod_{i\neq k} a_i^{s_i}[s_i\pm e_{pi}]^{-1}\{a_k^{s_k}[s_k\pm(e_{pk}+1)]^{-1}\cos(\omega_k t+\theta_k+\sum_j e_{pj}\theta_j)$$
$$+a_k^{s_k}[s_k\pm(e_{pk}-1)]^{-1}\cos(\omega_k t+\theta_k-\sum_j e_{pj}\theta_j)\}. \tag{28}$$

Expanding the cosines, the expression in the braces becomes

$$a_k^{s_k}\cos(\omega_k t+\theta_k)\cos\left(\sum_j e_{pj}\theta_j\right)\{[s_k\pm(e_{pk}+1)]^{-1}+[s_k\pm(e_{pk}-1)]^{-1}\}$$
$$+a_k^{s_k}\sin(\omega_k t+\theta_k)\sin\left(\sum_j e_{pj}\theta_j\right)\{-[s_k\pm(e_{pk}+1)]^{-1}+[s_k\pm(e_{pk}+1)]^{-1}\}.$$

Recalling the definition (23) of $[s\pm l]$, the coefficients in braces can be simplified, making the final form of Q become

$$Q=C2^{-N}N!\sum_{k}\sum_{s}\sum_{p}\{\prod_{i\neq k} a_i^{s_i}/[s_i\pm e_{pi}]\}a_k^{s_k}[(s_k+1)\pm e_{pk}]^{-1}$$
$$\{(s_k+1)\cos(\omega_k t+\theta_k)\cos(\sum_j e_{pj}\theta_j)+e_{pk}\sin(\omega_k t+\theta_k)\sin\sum_j e_{pj}\theta_j)\}. \tag{29}$$

Differentiating for $I=\dot{Q}$, and writing $a_k^{s_k}=a_k^{s_k+1}/a_k$ for symmetry:

$$I=C2^{-N}N!\sum_{k}\sum_{s}\sum_{p}\omega_k a_k^{-1}\{\prod_{i\neq k} a_i^{s_i}/[s_i\pm e_{pi}]\}a_k^{s_k+1}[(s_k+1)\pm e_{pk}]^{-1}$$
$$\{e_{pk}\cos(\omega_k t+\theta_k)\sin\varphi_p-(s_k+1)\sin(\omega_k t+\theta_k)\cos\varphi_p\}, \tag{30}$$

with $\varphi_p\equiv\sum_j e_{pj}\theta_j$.

If we let

$$s_i' = s_i \qquad i \neq k$$

$$s_k' = s_k + 1,$$

the summation $\sum_s$ with $\sum_i s_i = N$ is the same as $\sum_{s'}$ with $\sum_i s_i' = N+1$ and the restriction $s_k' > 0$. But if we let $s_k' = 0$, the coefficient of sin $(\omega_k t + \theta_k)$ vanishes, and $[s_k' \pm e_{pk}]^{-1}$ vanishes unless $e_{pk} = 0$; but $e_{pk} = 0$ eliminates the cos $(\omega_k t + \theta_k)$ term. Hence $s_k' = 0$ contributes nothing, and

$$I = C2^{-N}N! \sum_k \sum_p \sum_{s'} \omega_k a_k^{-1} \{ \Pi_i a_i^{s_i'} / [s_i' \pm e_{pi}] \} \{ e_{pk} \cos(\omega_k t + \theta_k) \sin \varphi_p - s_k' \sin(\omega_k t + \theta_k) \sin \varphi_p \}, \tag{31}$$

with the integers $s_i' \geq 0$, $\sum_i s_i' = N+1$.

Noting that the coefficient of the cosine term can be summed over s' independently of k, we define

$$A_p \equiv \sum_{s'} \Pi_i a_i^{s_i'} \Big/ [s_i' \pm e_{pi}], \tag{32}$$

the strength of the pth process, and

$$B_{pk} \equiv \sum_{s'} s_k' \Pi_i a_i^{s_i'} \Big/ [s_i' \pm e_{pi}] = a_k \frac{\partial A_p}{\partial a_k}, \tag{33}$$

making

$$I = C2^{-N}N! \sum_k \sum_p \omega_k a_k^{-1} \{ A_p e_{pk} \cos(\omega_k t + \theta_k) \sin \varphi_p - B_{pk} \sin(\omega_k t + \theta_k) \cos \varphi_p \}. \tag{34}$$

The power absorbed at frequency ω_k is

$$P_k = \tfrac{1}{2} C2^{-N}N! \omega_k \sum_p A_p e_{pk} \sin \varphi_p, \tag{35}$$

and if we write $H_p \equiv \frac{CN!}{2^{N+1}} A_p \sin \varphi_p$,

$$P_k / \omega_k = \sum_p H_p e_{pk} \tag{36a}$$

$$\mathfrak{P} = \boldsymbol{HE}. \tag{36b}$$

3. Order of Processes

In eq (32) for A_p, the nonvanishing terms are those for $|e_{pi}| \leq s_i'$, hence $\sum_i |e_{pi}| \leq \sum_i s_i' = N+1$ and we define the order of a process to be $\sum_i |e_{pi}| - 1$. A $Q = CV^N$ device gives rise to processes of order N, $N-2$, $N-4$, etc, because e_{pi} must be of the same parity as s_i'.

The result of a process of given order can be produced by the superposition of processes of other orders, introducing new frequencies for which $P_i = 0$. That is, high-order processes can be indirectly yielded by low-order devices by cascaded modulation, if suitable reactance loads are added to the circuit and d-c bias to obtain even-order processes from odd. By introducing dummy frequencies involving only reactive power, the result of any desired set of processes can be achieved using only simple nonlinear devices of the form $Q = CV^3$. A constructive demonstration of this will constitute our proof of sufficiency.

4. Redundancies

The relation (36b) represents the net effect of all mixing processes, both the minimal set and the redundant ones. Partitioning into

$$\mathfrak{P}_a = \boldsymbol{HE}_1 \tag{37a}$$

$$\mathfrak{P}_b = \boldsymbol{HE}_2 \tag{37b}$$

leaves redundancies in $\boldsymbol{E}_1$, making it singular, so that $\boldsymbol{H}$ cannot be found from the given $\mathfrak{P}_d$. In certain special cases, the frequencies are such that there are no redundant relations, and $\boldsymbol{E}\equiv\mathfrak{E}$, so that $\boldsymbol{H}=\mathfrak{H}=\mathfrak{P}_d\mathfrak{E}_1^{-1}$, if the processes of $\mathfrak{E}$ are all of order N. Then the amplitudes a_i can be chosen arbitrarily and the A_p computed (one process for each derived frequency). The sin φ_p are then known, and a sufficiently large choice of C will make the φ_p real.

Because

$$\Phi=\boldsymbol{E}\Theta=\boldsymbol{E}_1\Theta_d+\boldsymbol{E}_2\Theta_b, \tag{38}$$

the phases at base frequencies can be assigned, and the remainder are determined by

$$\Theta_d=\boldsymbol{E}_1^{-1}(\Phi-\boldsymbol{E}_2\Theta_b)=\boldsymbol{E}_1^{-1}\Phi+\boldsymbol{M}\Theta_b. \tag{39}$$

Equation (34) yields both current components, so the sink impedances are readily computed.

In the general case of mixed-order $\mathfrak{E}$, the introduction of additional derived frequencies with corresponding null components of $\mathfrak{P}_d$ can convert (17) into

$$\mathfrak{P}'=\boldsymbol{H}'\boldsymbol{E}', \tag{40}$$

with $\boldsymbol{E}'$ of (say) third order, but the set of frequencies will usually allow redundant processes. Our plan of attack is to express the given $\mathfrak{E}$ in terms of third-order processes alone, and to subdivide the resulting set of frequencies into subsets not possessing third-order redundancies. Then for these subsets, each complete $\boldsymbol{E}$ is also a minimal $\mathfrak{E}$, and the subsets of mixing processes can be carried out by separate cubic capacitors, using common base-frequency sources.

The only first-order process that is possible is the identity $\omega_i=\omega_i$, because the ω were defined to be positive and distinct. Because a third-order device can produce only first- and third-order processes, the problem of enumerating producible processes among a set of frequencies is relatively straightforward. Although a second-order device has the same simplicity, using $Q=CV^3$ in the sufficiency proof automatically extends the sufficiency to (a) symmetrical devices and (b) devices with positive slope throughout.

5. Minimal Third-Order Processes

Instead of choosing the source frequencies as bases, we take suitable subharmonics of the sources, so that all derived frequencies will be integral harmonics and combinations. We start by expressing harmonics of ω in terms of minimal third-order processes, i. e., third-order processes defining a set of frequencies not possessing any redundant processes of third order.

For generating an odd harmonic, $O_1\omega$, we take a sequence of odd harmonics $O_n\omega$ generated by

$$O_{n+1}=O_n/3 \tag{41}$$

if integral, and otherwise

$$O_{n+1}=(O_n\pm 1)/2,$$

using the sign that makes O_{n+1} odd. This procedure yields a sequence that can be generated from ω by third-order processes in one and only one way.

For an even harmonic, we again divide by 3 if possible, and repeat until we have an even term not containing 3 as a factor. We then start an odd series with

$$O_1=\frac{E\pm 1}{3}, \qquad O_2=E-O_1$$

and continue down from O_2 to ω as before, but add zero (d-c) to the sequence. In reverse, we find that O_1 and E are generated by

$$2O_1 = O_2 \pm 1$$

$$E = O_1 + O_2 + (\text{zero}).$$

For combination tones, $A\omega_a + B\omega_b + \ldots$, we partition the sum into a sum of three combination tones, then repartition these in the same way, etc. A partial sum of two terms can be partitioned by using the artifice $A\omega_a + B\omega_b = A\omega_a + B\omega_b + 0$, as was done for even harmonics. The original combination tone can thus be reduced to nonredundant third-order processes that use harmonics of the base frequencies for starting points. These harmonics can then be reduced as before.

6. Choice of Base Frequencies

Our procedure for expressing various frequencies by minimal processes was based on those frequencies being integral harmonics. In terms of the original arbitrary base frequencies, we had

$$\omega_{di} = \sum_j m_{ij}\omega_{bj}, \tag{2}$$

where m_{ij} are rational fractions. For each j, let the L.C.D. of the m_{ij} be D_{jj}; then

$$\omega_{di} = \sum_j r_{ij}\omega_{bj}/D_{jj}, \tag{43}$$

where r_{ij} are integers. Let the new bases be $\omega_{Bj} \equiv \omega_{bj}/D_{jj}$ so that

$$\Omega_d = \boldsymbol{R}\Omega_B,\ \Omega_b = \boldsymbol{D}\Omega_B. \tag{44}$$

Some of the frequencies of ω_B are new frequencies (associated with zero power) and some may be members of the old sets ω_d and ω_b. All remaining members of ω_d and ω_b comprise the components of a new ω_D:

$$\Omega_D = \mathfrak{M}\Omega_B. \tag{45}$$

By the minimal process procedure of the previous section, the frequencies in ω_D may be associated in sets

$$\Omega_{Dl} = \mathfrak{M}_l\Omega_B, \tag{46}$$

where m_{lij} are integers, so that the corresponding $\mathfrak{E}_l$ as determined by the procedure leading to (13) and (14) is

$$\mathfrak{E}_l = \boxed{\begin{array}{c|c} I & -\mathscr{M}_l \end{array}} \tag{47}$$

A given frequency may appear in several vectors ω_{Dl}.

Let the voltages at frequencies appearing in ω_{Dl} be impressed on the lth cubic capacitor, $Q = C_l V^3$. For each of these capacitors, the relation (16a) becomes

$$\mathfrak{H}_l = \mathfrak{P}_{Dl}$$

and all components are known. Assigning arbitrary nonzero amplitudes to all voltages, the A_{li} are readily computed. The phases of the base frequencies can be assigned; for convenience we let each $\theta_{Bli} = 0$. Then (39) and (35) yield

$$\Theta_{Dl} = \Phi_l$$

$$\theta_{Dli} = \sin^{-1} 2^4\mathscr{P}_{Dli}/3!C_lA_{li}.$$

Using zero-impedance generators at all frequencies involved, their interconnections with the l capacitors is a trivial problem.

It has been demonstrated that any modulation-distortion operation that satisfies the necessary conditions for lossless devices can be produced with nonlinear capacitors, and in particular, with cubic capacitors. The demonstration of sufficiency, however, utilized perfect generators. It would be more satisfying to demonstrate the operation using generators at only those frequencies at which the net power input is positive, and using passive linear impedances at all sink and null-power frequencies. This can be done for any *given* set of base frequencies; variation of frequencies may lead to variation of the strengths of the several mixing processes, depending on the reactance-versus-frequency behavior of the elements and the particular circuit configuration used.

The discussion is simplest when there is only one base frequency; all sinks are rational harmonics of a given source. The procedure we have used for generating minimal processes was such that if any base harmonic was common to two minimal processes, all lower frequencies involved were also common. (Except for the d-c bias needed in some processes, which can be handled by connecting individual bias batteries immediately adjacent to the capacitors.) Thus an ordered tree of generators can be assembled, with various capacitors connected at appropriate points on the "trunk" or upright "limbs." Equation (34) gives both components of current at each frequency (for arbitrary assigned voltages), so impedances can be calculated for replacing all generators except the power source. The impedances for the null-power dummy frequencies will be purely reactive; those for sink frequencies will have positive resistance. By associating each sink with one capacitor only, all common impedances are purely reactive. The necessary reactances are computed from the base of the trunk upward, leaving the "terminal twigs" for sink loads and independent reactance adjustments. Such adjustments can be made for any *given* base frequency, but cannot be guaranteed for frequency variations.

Because the general combination tones are sums and differences of harmonics of the bases, the harmonic-generating trees are to be interconnected with combination-tone sinks and their associated capacitors. In the case of harmonics that are both sink frequencies and steps in the development of combination tones, extra branches can be added to the harmonic-generating trees, so that the harmonic sink loads do not carry the harmonic current used in generating combination tones. Thus *all* mutual impedances can be made purely reactive, and *all* loads isolated. Thus the necessary relations among frequencies and powers are also sufficient for realization with cubic capacitors, at any set of incommensurate spot frequencies.

7. Regeneration and Stability

It is apparent from eq (19b) that in a given process power is absorbed by the capacitor at some frequencies and delivered at other frequencies, or that in each mixing process the capacitor presents positive and negative resistance components at various frequencies. At a given frequency, some processes may absorb energy, and some may deliver energy. At a frequency associated with passive impedance (no generator), any power dissipated in the impedance is the net power delivered at this frequency by the capacitor. The total effective resistance of the circuit at this frequency is zero. The impressed voltage at this frequency is zero and the current is uniquely determined by the processes; the circuit is stable.

On the other hand, if for some processes the capacitor delivers power at the frequency of one of the generators, we call it a regenerative process; it presents negative resistance to the generator. If the total resistance so presented is zero or negative, the circuit is sometimes said to be unstable.[2] If a circuit is unstable in this sense, i. e., if it can supply power to an

[2] Manley & Rowe, op. cit., p. 908.

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external generator, then there is current at this frequency even in the absence of the generator. Hence the unstable frequency is merely any one of the frequencies generated by the network. If this frequency is a rational harmonic or combination tone of the other generators, the effect is entirely expected. But the unstable frequency may be incommensurate with all the source frequencies. In this case the network is said to oscillate. This condition occurs when the set of frequencies involved in mixing processes is such that the number of base frequencies exceeds the number of independent sources. Then by (19c) at least one frequency involving the excess base is associated with negative power (unless *all* such frequencies are associated with zero power, in which case the extra base is not needed).

This type of oscillation can occur in systems that are ordinarily considered stable, i. e., one that exhibits bounded responses to all bounded driving forces, and possesses no free oscillations. The presence of such oscillations is, however, dependent on threshold values of generator voltages, because for sufficiently small generator voltages, only integral harmonics and combination tones can appear.[3]

WASHINGTON, October 3, 1956.

[3] Unpublished work, to appear in J. Wash. Acad. Sci.