Journal of Research of the National Bureau of Standards Vol. 58, No. 3, March 1957 Research Paper 2747

Numerical Computation of the Transfinite Diameter of Two Collinear Line Segments

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Orthonormal polynomials are employed to compute the transfinite diameter of a set consisting of two collinear line segments. With polynomials of degree 10, two-place accuracy has been achieved.

The concept of the transfinite diameter $\tau(E)$ of a closed bounded point set E was introduced by M. Fekete [5],[1] and the relation between this domain functional and conformal mapping and potential theory has been stressed by both Fekete and Szegö [7]. According to Fekete's original definition,

$$\tau(E)=\lim_{n\to\infty}\sqrt[\binom{n}{2}]{V_n}, \tag{1}$$

where

$$V_n=\max_{z_i\epsilon E}|V(z_1,z_2,\ldots,z_n)|, \tag{2}$$

and where V is the Vandermonde determinant.

The transfinite diameter is known explicitly for a number of elementary geometrical configurations, but in general, its numerical evaluation is attended by considerable difficulty. In a previous paper [4], Davis and Rabinowitz showed that complex orthogonal polynomials may be employed profitably for its numerical determination in the case of certain simply connected domains, a square and nonconvex "bean-shaped" domain having been tested. Recently, Professor Fekete called the attention of the author to the possibility of using an analogous relationship in the case where E consists of a finite number of rectifiable Jordan arcs. Making use of very general theorems recently established by Fekete and Walsh [6, p. 61], we may assert:

Let $p_n(z)=a_nz^n+\ldots$, $a_n>0$ $(n=0,\ldots)$ be complex polynomials that are orthonormal over E in the sense that

$$\int_E p_m(z)\overline{p_n(z)}ds=\delta_{mn}. \tag{3}$$

Then we have

$$\lim_{n\to\infty}(1/a_n)^{1/n}=\tau(E). \tag{4}$$

For two collinear line segments of equal length placed symmetrically with respect to 0, say, E: $-1\leq x\leq -a, a\leq x\leq 1$, $0<a<1$, the value of $\tau(E)$ is known theoretically and is simply

$$\tau(E)=\tfrac{1}{2}\sqrt{1-a^2}. \tag{5}$$

For two collinear line segments of unequal length, its value has been obtained by N. Achiezer (1) and can be expressed as the ratio of certain elliptic functions. For more than two line segments a closed-form value of $\tau(E)$ is not known to the author.

The relationship (4) was tested numerically on SEAC to see what could be achieved by way of accuracy, using single precision codes. In these computations, the value $a=1/2$ was selected, leading in (5) to

$$\tau=\tfrac{1}{4}\sqrt{3}\approx 0.4330127. \tag{6}$$

The computation was carried out by making use of a single precision (8 significant decimals) floating point orthonormalizing routine developed by J. Bram. This multiple-purpose code has already been described in [3]. The inner products (3) were computed by means of a 10-point Gaussian quadrature rule on each of the two segments $(-1,-\frac{1}{2})$ and $(\frac{1}{2},1)$. The machine was programmed to print out the coefficients of the orthonormal polynomials, as well as the values of these polynomials at the Gaussian abscissas employed. In this way, it was possible to monitor the obvious global properties of the orthonormal polynomials, as well as to see where the accumulated round-off began to vitiate the computations. One way in which this was done was as follows: The polynomials $p_n(x)$, orthonormal over the set E:$(-1,-\frac{1}{2})$, $(1\frac{1}{2},1)$, are alternately even and odd. As n increases, the theoretical zero values for the alternate coefficients of p_n become contaminated by round-off, and these "zeros" begin to assume the proportions of the nonzero coefficients and of the values of the orthonormal polynomials themselves (see table 1). It was found that all significance was lost when attempt was made to go beyond $n=10$. The last value of $(1/a_n)^{1/n}$ gave the value of $\tau(E)$ correct to within 0.009. Additional accuracy is obtainable from (4) only by employing double-precision coding and going beyond $n=10$.

Although closed-form expressions for the orthonormal polynomials over E:$(-1,-\frac{1}{2}),(\frac{1}{2},1)$ are not available, such expressions are available for the Tschebyscheff polynomials for E [2, p. 287]. Here the even and odd polynomials have a totally different structure. Using the latter polynomials as a guide (in the theory of domain polynomials these two sets frequently behave alike), we can confirm the slightly higher values for $(1/a_n)^{1/n}$ registered in table 2 for

[1] Figures in brackets indicate the literature references at the end of this paper.

TABLE 1. *Orthonormal polynomials on $(-1, -\frac{1}{2}), (\frac{1}{2}, 1)$ exhibiting effect of roundoff*

Coefficient of—	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
p_0	**1.0000000**										
p_1	0.0000000001	**1.3093073**									
p_2	**−2.6843775**	−0.00000000003	**4.6017900**								
p_3	−0.0000000002	**−4.1140778**	0.0000000002	**6.1932355**							
p_4	**7.6721535**	−0.0000000004	**−28.177018**	0.00000000004	**23.822212**						
p_5	0.00000007	**11.694401**	−0.0000003	**−40.918493**	0.0000003	**31.868611**					
p_6	**−22.558405**	−0.0000005	**131.20021**	0.0000017	**−230.99467**	−0.0000014	**125.35317**				
p_7	−0.0000083	**−34.093102**	0.000048	**189.08749**	−0.000085	**−319.27433**	0.000046	**167.38838**			
p_8	**66.959357**	0.000078	**−524.56943**	−0.00043	**1435.1255**	0.00071	**−1637.5651**	−0.00037	**663.46546**		
p_9	0.00065	**100.83287**	−0.005	**−763.6782**	0.013	**2022.6110**	−0.015	**−2241.6237**	0.0059	**885.36970**	
p_{10}	**−199.48833**	−0.0021	**1963.8362**	0.015	**−7312.7657**	−0.039	**12922.714**	0.041	**−10891.666**	−0.0157	**3521.1597**

TABLE 2. *Computation of transfinite diameter from leading coefficients of orthonormal polynomials*

n	a_n	$1/\sqrt[n]{a_n}$	$\sqrt{a_n/a_{n+2}}$
1	1.3093073	0.76376	0.45979
2	4.6017900	.46616	.43951
3	6.1932355	.54454	.44084
4	23.822212	.45264	.43594
5	31.868611	.50041	.43633
6	125.35317	.44700	.43467
7	167.38858	.48120	.43481
8	663.46546	.44389	.43408
9	885.36970	.47048	----------
10	3521.1597	.44191	----------
		Theoretical value: 0.4330127	

odd n. We can also conclude from the form of the Tschebyscheff polynomials that the ratio $(a_n/a_{n+2})^{1/2}$ would be a good estimator for $\tau(E)$. Table 2 also presents these values, and it will be seen that the last entry, $(a_8/a_{10})^{1/2}$, yields $\tau(E)$ correctly to within 0.001.

The author thanks Morris Newman and Patricia Ferrant for their assistance with the numerical portions of this paper.

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WASHINGTON, October 18, 1956.