Frequency Conversion With Positive Nonlinear Resistors

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A nonlinear resistor subject to an almost periodic voltage will absorb power at some frequencies, and supply power at other frequencies. Necessary and sufficient relations among these powers are found. Among the practical consequences are the results: (1) modulation efficiency cannot exceed unity, (2) subharmonics are not produced, and (3) the efficiency of generating an nth harmonic cannot exceed $1/n^2$.

1. Introduction

Positive nonlinear resistors are here defined as two-terminal devices through which the current (I) is a real finite single-valued nondecreasing function of the voltage (V) across the terminals, with the added condition that $I(0)=0$. The function $I(V)$ may have simple discontinuities.

The voltage is assumed to be uniformly almost periodic, i. e., a bounded continuous function of time representable by a uniformly convergent trigonometric series.

The resistor will absorb power, P_n (positive or negative), at each of the frequencies (ω_n) in the voltage. Relations among the P_n are of practical and theoretical interest.

The voltage can be thought of as supplied by a series-connected set of generators, some of which are real, whereas others represent the effects of voltage drop across the remaining passive elements of the network. Generators associated with positive P_n are called "sources"; those associated with negative P_n, "sinks." Continuity of $V(t)$ is provided physically by the inevitable shunt capacitance, however small, of the resistor.

$$V\equiv\sum_1^\infty V_n=\sum_1^\infty a_n \cos(\omega_n t+\theta_n). \tag{1}$$

The current through the resistor will be S^2 almost periodic,[1] as is shown in the appendix. It is given almost everywhere by

$$I(t)=\text{l. i. m.} \sum_1^\infty b_n \cos(\omega_n t+\varphi_n). \tag{2}$$

The frequencies ω_n may include some not required for the representation of V; these are included in eq (1) with vanishing coefficients. We denote the average power *absorbed* by the resistor at frequency ω_n by

$$P_n=\langle V_n I[V(t)]\rangle_t\equiv\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} V_n(t)I(t)\,dt. \tag{3}$$

2. Modulation

Partition the total voltage into two parts, a "carrier" (V_1) and the combination of signal and modulation products, such that

$$V=V_1+V_2$$
$$\langle V_2 I[V_1]\rangle=0 \tag{4}$$

defining as modulation products all frequencies (other than the signal) appearing in V but not in the current that obtains in the absence of signal.

[1] Representability by eq (2) is essentially what is meant by Stepanoff almost periodicity. See A. S. Besicovitch, Almost periodic functions, ch. II, sections 2 and 3 (Dover Publications, New York, N. Y., 1954).

THEOREM. *The sources in V_2 supply at least as much power as is absorbed by the sinks in V_2.* The total power supplied by the sources and sinks of V_2 is

$$\langle V_2 I[V_1+V_2]\rangle_t = \langle V_2\{I[V_1+V_2]-I[V_1]\}\rangle_t + \langle V_2 I[V_1]\rangle_t \geq 0. \tag{5}$$

The first term on the right is nonnegative by the nondecreasing property of I, and the second term vanishes by the definition of V_2.

The total power available in all modulation products is therefore not greater than that supplied by the signal source, or modulation efficiency cannot exceed unity.

3. Necessary Relations Among P_n

The total power absorbed by the resistor is

$$P = \langle VI\rangle_t \geq 0$$

as VI is a nonnegative function of t. Now

$$0 \leq P = \langle VI\rangle_t = \langle \Sigma V_n I\rangle_t = \Sigma \langle V_n I\rangle_t = \Sigma P_n \tag{6}$$

by virtue of uniform convergence. Further,

$$P_n = \langle a_n \cos(\omega_n t+\theta_n)\Sigma b_m \cos(\omega_m t+\phi_m)\rangle_t = \tfrac{1}{2}a_n b_n \cos(\phi_n-\theta_n)$$

so that

$$|P_n| \leq \tfrac{1}{2}|a_n b_n|$$

making

$$\Sigma |P_n| \leq \tfrac{1}{2}\Sigma |a_n b_n| \leq \sqrt{(\tfrac{1}{2}\Sigma a_n^2)(\tfrac{1}{2}\Sigma b_n^2)} = \sqrt{\langle V^2\rangle\langle I^2\rangle} < \infty \tag{7}$$

using Schwarz's inequality and the boundedness of V and I. Similarly

$$\Sigma P_n^2 \leq \tfrac{1}{4}\Sigma a_n^2 b_n^2 \leq \tfrac{1}{4}(\Sigma |a_n b_n|)^2 \leq \tfrac{1}{4}\Sigma a_n^2 \Sigma b_n^2 < \infty. \tag{8}$$

The final condition follows from the nondecreasing behavior of I:

$$\theta(x) \equiv \langle\{V(t)-V(t-x)\}\{I[V(t)]-I[V(t-x)]\}\rangle_t \geq 0. \tag{9}$$

Now

$$V(t)-V(t-x) = \Sigma a_n\{\cos(\omega_n t+\theta_n)-\cos(\omega_n t-\omega_n x+\theta_n)\}$$
$$= \Sigma a_n\{[1-\cos\omega_n x]\cos(\omega_n t+\theta_n)-\sin\omega_n x \sin(\omega_n t+\theta_n)\}$$

with a similar expression for $I(t)-I(t-x)$, making

$$\theta(x) = \Sigma \tfrac{1}{2}a_n b_n \cos(\phi_n-\theta_n)\{(1-\cos\omega_n x)^2-\sin^2\omega_n x\}$$
$$= 2\Sigma P_n(1-\cos\omega_n x). \tag{10}$$

These results can be collected as the four conditions:

(a) $\Sigma P_n \geq 0$
(b) $\Sigma |P_n| < \infty$
(c) $\Sigma P_n^2 < \infty$
(d) $\Sigma P_n(1-\cos\omega_n x) \geq 0$ all x.

4. Sufficiency

The above necessary conditions are also sufficient; given a set of ω_n and P_n satisfying $(a)-(d)$, there exists at least one combination of allowed voltage and resistor producing a set of P_n arbitrarily close to the specified values.

For consider the voltage

$$V = A^2 \Sigma P_n \cos \omega_n t. \tag{11}$$

Condition (c) guarantees the Parseval equation and (b) makes V uniformly almost periodic. Condition (d) shows that V never exceeds its value at $t=0 (V_0 = A^2 \Sigma P_n)$ and condition (a) makes V_0 positive. Let V be applied across the resistor described by

$$I = \begin{cases} 0, & V \leq V_0 - \delta \\ 1, & V > V_0 - \delta \end{cases} \tag{12}$$

where δ is a small positive constant. There is a theorem [2] that V possesses translation numbers such that

$$|\omega_n \tau| < \epsilon \ (\text{mod } 2\pi) \qquad \text{for } n < N; \tag{13}$$

hence a relatively dense set of time intervals for which $V_0 - \delta < V < V_0$, yielding unit current. By setting $1/A^2$ equal to the fraction of the time that the current is unity, the power at ω_n is given by

$$P_n' = P_n \int_{I=1} \cos \omega_n t \, dt \Big/ \int_{I=1} dt, \tag{14}$$

that is, P_n'/P_n is equal to the average value of $\cos \omega_n t$ on the time intervals of unit current. This ratio can be made as close to unity as desired for any number (N) of frequencies, by reducing δ, for the continuity of V in eq (11) provides a continuous reduction of the intervals of integration. For $n > N$, the arithmetic error $P_n' - P_n$ can be made arbitrarily small by a sufficiently large choice of N, because by eq (14), $|P_n'/P_n| \leq 1$, and $\Sigma |P_n|$ converges.

5. Practical Theorems

Three practical theorems can be deduced directly from the necessary and sufficient relations among the P_n. For the modulation theorem, we consider a sinusoidal carrier and signal, so that all ω_n are of the form $n\omega_c + m\omega_s$, $(-\infty < n, m < \infty)$. Condition (d) yields

$$\Sigma \Sigma P_{nm} [1 - \cos (n\omega_c + m\omega_s) x] \geq 0. \tag{15}$$

For $x = 2\pi k / \omega_c$, k an integer, this becomes

$$\Sigma \Sigma P_{nm} [1 - \cos 2\pi m k \omega_s / \omega_c] \geq 0 \tag{16}$$

and we can choose k to make the cosine less than ϵ for $m < M$ excepting the values $m = l\omega_c / \omega_s$, for which the cosine becomes unity. Therefore $\Sigma \Sigma P_{nm} \geq 0$, summing over all frequencies that are not harmonics of the carrier.

The second theorem states that subharmonics cannot be generated. Let τ be the least common period of the sources (or in general the least translation number of the source voltage). Then $P_n (1 - \cos \omega_n \tau)$ vanishes for all positive P_n, and condition (d) requires this expression to vanish for each of the negative P_n. Hence all sinks have the period τ; all generated frequencies are harmonics of difference frequencies.

The third theorem limits the efficiency of harmonic generation to $1/n^2$. For

$$P_1 (1 - \cos \omega x) - |P_n| (1 - \cos n\omega x) \geq 0$$

making

$$\frac{|P_n|}{P_1} \leq \frac{1 - \cos \omega x}{1 - \cos n\omega x} = \frac{\sin^2 \omega x/2}{\sin^2 n\omega x/2}. \tag{17}$$

[2] A. S. Besicovitch, Almost periodic functions, p. 53 (Dover Publications, New York, N. Y., 1954).

As $x \to 0$, the ratio of sines squared goes to its minimum value of $1/n^2$.

The positive definite form (d) can be generalized:

THEOREM. *If* $g(\omega)=\omega\int_0^\infty \varphi(x)\sin\omega x\,dx$, *where* $\varphi(x)$ *is a nonincreasing positive function, and the integral is summed in the Cesàro sense, then* $J[g]\equiv\Sigma g(\omega_n)P_n$ *is positive definite.*

For

$$g(\omega)=\lim_{\lambda\to\infty}\int_0^\lambda(1-x/\lambda)\varphi(x)d(1-\cos\omega x)$$

$$=\lim_{\lambda\to\infty}\left\{\frac{1}{\lambda}\int_0^\lambda\varphi(x)(1-\cos\omega x)dx+\int_0^\lambda(1-x/\lambda)(1-\cos\omega x)(-d\varphi)\right\} \tag{18}$$

upon integrating by parts. Then

$$J[g]=\lim\frac{1}{\lambda}\int_0^\lambda\varphi(x)\Sigma P_n(1-\cos\omega_n x)dx+\lim\int_0^\lambda(1-x/\lambda)\Sigma P_n(1-\cos\omega_n x)(-d\varphi) \tag{19}$$

using the uniform convergence of (d). By (d) and the restrictions on φ, there are no negative terms on the right, so

$$J[g]\geq 0. \tag{20}$$

COROLLARY:

$$J(\alpha)\equiv\Sigma\omega_n^\alpha P_n\geq 0 \qquad \text{for } 0\leq\alpha<2.$$

In this range of α, J vanishes only for the trivial case of all P_n zero. The closure $J(2)$ is positive semidefinite; it vanishes for $V=\sin^3\omega t$, $I(V)=0$ for $V\leq 0$, $I(V)=1$ for $V>0$.

6. Appendix

Let $I(V)$ have only a finite number of simple discontinuities in any finite range of V. Let these be at the points V_i. Let $0<\delta<\text{g.l.b.}|V_i-V_j|$ and let l.u.b. $|I(V+\delta)-I(V)|=B$ so that for any $0<\epsilon<\delta$, $|I(V+\epsilon)-I(V)|\leq B$. For $|V-V_i|>\delta$, I is continuous in the interval $[V, V\pm\epsilon]$, and ϵ can be chosen to make $|I(V\pm\epsilon)-I(V)|<\zeta$, where ζ is any assigned positive number. Therefore,

$$|I(V\pm\epsilon)-I(V)|\leq\begin{cases}\zeta, & |V-V_i|>\delta\\ B, & |V-V_i|\leq\delta.\end{cases}$$

Now if $V(t)$ is uniformly almost periodic, but not periodic, we can choose δ so that on the interval $t_0\leq t\leq t_0+l$, the measure of t for which $|V-V_i|\leq\delta$ is less than $l\zeta^2$.

Then

$$\frac{1}{l}\int_{t_0}^{t_0+l}|I[V(t)\pm\epsilon]-I[V(t)]|^2dt\leq B^2\zeta^2+\zeta^2$$

and

$$\underset{-\infty<t_0<\infty}{\text{l.u.b.}}\left\{\frac{1}{l}\int_{t_0}^{t_0+l}|I[V(t)\pm\epsilon]-I[V(t)]|^2dt\right\}^{1/2}<\zeta\sqrt{B^2+1}$$

Now the translation numbers τ of $V(t)$ make $|V(t+\tau)-V(t)|<\epsilon$ so that

$$\text{l.u.b.}\left\{\frac{1}{l}\int_{t_0}^{t_0+l}|I[V(t+\tau)]-I[V(t)]|^2dt\right\}^{1/2}<\zeta\sqrt{B^2+1}$$

and $I(t)$ is Stepanoff almost periodic, with a Fourier Series which converges to $I(t)$ in the mean square.

If $V(t)$ is periodic, then $I(t)$ is also periodic, hence Stepanoff a. p. a fortiori.

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