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Penetration of Gamma Radiation From a Plane Monodirectional Oblique Source[1]

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The penetration of gamma radiation from a plane monodirectional oblique source is solved by the moment method of Spencer and Fano. The novel feature of the treatment is the systematic investigation and exploitation of a function-fitting technique, which consists of approximating the desired radiation-flux function, $f(x)$, by an expression of the form $\sum_n A_n\phi(B_nx)$, the parameters A_n and B_n being determined from the knowledge of the flux moments.

A discussion is given of the criteria for the proper choice of the "weight function," $\phi(x)$, and of the accuracy of the fitting procedure. The results of a sample calculation are presented for an 0.66-Mev source in water.

1. Introduction

One of the most effective techniques for calculating the diffusion of gamma rays in infinite media is the moment method [1],[2] i. e., the determination of the spatial and angular moments of the radiation flux and subsequent calculation of the flux-distribution function by moment-fitting. This technique has been successfully applied to the calculation of radiation from plane and point isotropic sources and from monodirectional plane sources emitting radiation perpendicularly to the source plane [2]. The purpose of this work is to extend the method to the determination of the propagation of radiation from monodirectional *oblique* plane sources, i. e., sources emitting photons at a specified obliquity angle with respect to the source plane. In the problems treated previously, analytical asymptotic theory was available describing the radiation field at great distances from the source [3]; this facilitated the reconstruction of the flux function from its moments. For the problem of the oblique source, reliable guidance in the form of a corresponding analytical asymptotic theory is lacking, so that increased care must be given to the moment-fitting procedures.

Although specific application of the method described in this paper will be made to the diffusion of gamma radiation, the technique used should also be useful for determining the propagation of other types of radiation, e. g., neutrons, from plane oblique sources.

We consider an infinite homogeneous scattering medium that contains, in the plane $z=0$, a radiation source emitting 1 photon per square centimeter per second. The source photons are all emitted with energy E_0, and in directions inclined at an angle α with respect to the normal to the source plane. (The distribution of the azimuthal directions with which the photons are emitted may be considered arbitrary, being irrelevant for the propagation as a function of distance from the source plane.) Our task is to determine the energy flux, $F^{(\alpha)}(E,z)dE$, at a distance z from the source plane, due to photons in the energy range $(E,E+dE)$ as a function of the obliquity α. The energy flux is defined as the flow of energy through an isotropic detector with unit cross-sectional area. The energy, E, will be expressed in million electron volts and the distance, z, in units of the mean free path of the primary radiation.

Unscattered radiation will be treated separately because its direct calculation is trivial. In order to facilitate the moment-fitting procedures, separate consideration will also be given to singly-scattered radiation. This separation is desirable because its flux distribution differs considerably from that of the multiply-scattered radiation.

[1] This work was supported by the Office of Naval Research and the Armed Forces Special Weapons Project.

[2] Figures in brackets indicate the literature references at the end of this paper.

Section 2 discusses the moment-fitting procedures appropriate to the problem. In section 3 specific application of these methods is made to radiation with a source energy of 0.66 Mev (cesium–137 radiation) in a scattering medium of water. Various energy spectra and space distributions are presented as functions of the obliquity angle. In section 4 the accuracy of the results is investigated by the comparison of various methods of moment-fitting and by comparisons with other related calculations and with an experiment. Details and subsidiary questions are given in appendixes.

In appendix A, equations are derived for the calculation of the flux of singly-scattered radiation. In appendix B it is explained how the flux moments are calculated by the Spencer-Fano method; in particular, it will be pointed out how the moments required for the treatment of the monodirectional oblique plane source radiation can be obtained from a certain set of standard moments computed on the NBS automatic computer (SEAC). Appendix C presents details of the moment-fitting procedure.

2. Moment-Fitting Methods

2.1. Introductory Remarks

We assume that the first eight spatial moments of the energy flux,

$$\overline{z^n}(E,\alpha)=\int_{-\infty}^{\infty} z^n F^{(\alpha)}(E,z)dz, \tag{1}$$

$$(n=0, 1, \ldots, 7),$$

are known. (It is assumed that $F^{(\alpha)}$ stands for the flux of singly-scattered or multiply-scattered radiation, whichever we may want to determine. For details concerning the computation of the moments, see appendix B.)

Although a well-behaved function is determined uniquely if all its moments are known the reconstruction of a function from only eight moments requires some finesse. It is desirable to incorporate prior knowledge concerning the flux that is available from related analytical calculations, physical intuition, or experience with similar problems, into a so-called weight function, which serves as the initial approximation. Successive improved approximations are then found by the use of the moments.

In order to choose this weight function properly, we must give some thought to the expected spatial dependence of the flux function. Assuming that $0 \leq \alpha \leq \pi/2$, i. e., that the photons are released by the source in the direction of positive z, we anticipate that the flux of scattered radiation will be a unimodal asymmetric function of z with a peak at a value of z somewhat larger than zero, and decreasing more rapidly on the negative than on the positive side of the peak. The asymmetry of the function will depend on the obliquity, being a maximum when $\alpha=0$, and zero when $\alpha=\pi/2$. Furthermore, it can be expected that at large distances from the source plane the flux has a dominantly exponential behavior as function of z.

To find a weight function embodying these features is difficult. The task is made easier if the flux is split into a symmetric component $f^{(\alpha)}(E,z)$ and an antisymmetric component $g^{(\alpha)}(E,z)$, so that

$$F^{(\alpha)}(E,z)=f^{(\alpha)}(E,z)+g^{(\alpha)}(E,z) \tag{2a}$$

$$F^{(\alpha)}(E,-z)=f^{(\alpha)}(E,z)-g^{(\alpha)}(E,z), \tag{2b}$$

where z denotes $|z|$. In this manner we can account for the expected peak without incorporating this feature into $f^{(\alpha)}$ and $g^{(\alpha)}$, which may be chosen to be more or less smooth, monotonic asymptotically exponential functions. There is another advantage inherent in this splitting technique. We can fit four even moments to determine $f^{(\alpha)}$ and four odd moments to determine $g^{(\alpha)}$. This is much easier than fitting eight moments simultaneously. There is a slight disadvantage also; the flux cannot be determined very accurately far behind the source because for large negative values of z the functions $f^{(\alpha)}$ and $g^{(\alpha)}$ will have nearly the same value, and their difference will be rather inaccurate. This is not a serious drawback because the small amount of radiation far behind the source usually is not of much interest.

2.2. Polynomial Expansions

We shall now discuss the method of polynomial expansion used by Spencer and Fano [1] for the solution of the plane source with $\alpha=0$ and explain why this approach cannot readily be carried over to the treatment of the source with $\alpha\neq 0$.

Spencer and Fano also made use of the splitting technique and set

$$f^{(0)}(E,z)=\frac{e^{-wz}}{2}\sum_{n=0}^{\infty}c_n(E)U_n(wz) \tag{3a}$$

$$g^{(0)}(E,z)=\frac{wze^{-wz}}{2}\sum_{n=0}^{\infty}d_n(E)V_n(wz), \tag{3b}$$

where z denotes $|z|$; $e^{-wz}/2$ and $wze^{-wz}/2$ are the respective weight functions.

$$U_n(z)=\frac{(-1)^n}{2^n}\left(\frac{\partial}{\partial z}-1\right)^{2n}\sum_{j=0}^{n}\binom{n+j}{j}\frac{1}{(n-j)!}2^{-j}z^{n-j}, \tag{4a}$$

and

$$V_n(z)=\frac{2}{2(n+1)}\left(\frac{\partial}{\partial z}-1\right)^{2}U_n(z) \tag{4b}$$

$$(n=0,1,2,\cdot\cdot\cdot)$$

are orthogonal polynomial systems associated with these weight functions. The coefficients $c_n(E)$ and $d_n(E)$ are related to the moments as follows, the nth coefficient involving only moments of order not exceeding n.

$$c_n(E)=\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{(2j)!}\overline{(wz)^{2j}} \tag{5a}$$

$$d_n(E)=\sum_{j=1}^{n+1}\binom{n+1}{j}\frac{(-1)^{j-1}}{(2j-1)!}\overline{(wz)^{2j-1}} \tag{5b}$$

The validity of such a polynomial expansion depends on the correct choice of the exponent w in the weight functions. It can be shown by physical arguments and by related asymptotic penetration theory that a satisfactory choice for the plane source with obliquity zero is $w\approx(\mu_p/\mu_0)$, where μ_0 is the linear narrow-beam absorption coefficient of the primary radiation and μ_p that of the most penetrating component of the radiation. (This is usually, though not always, radiation with the primary energy.) With a correctly chosen parameter, the polynomial expansion will converge rapidly for wz up to ≈ 15, so that only a few terms requiring the use of the first few moments have to be fed into the expansion.

The parameter w is difficult to determine for the plane oblique source. Clearly, the radiation component with the highest energy (usually the component with the smallest attenuation coefficient) will not govern the deep penetration because it must travel a distance approximately equal to $z/\cos\alpha$ in order to get a distance z away from the source. On the other hand, radiation that has been turned parallel to the z-axis travels away from the source plane on the shortest possible path but has lost energy, so that its attenuation is increased. One would expect that the deep penetration is controlled by the small amount of the radiation that has been turned approximately parallel to the z-axis in a number of collisions such that it has lost very little energy. It has been found by numerical experimentation that it is difficult to obtain reliable polynomial expansions by adjusting the parameter w on the basis of convergence considerations. These difficulties can be overcome, however, by another approach [4].

2.3. Function Fitting

If eight moments are available and an even-odd split is used, we set

$$f^{(\alpha)}(E,z)=A_1(E,\alpha)\phi[zB_1(E,\alpha)]+A_2(E,\alpha)\phi[zB_2(E,\alpha)] \tag{6a}$$

$$g^{(\alpha)}(E,z)=C_1(E,\alpha)\psi[zD_1(E,\alpha)]+C_2(E,\alpha)\psi[zD_2(E,\alpha)], \tag{6b}$$

where $\phi(z)$ and $\psi(z)$ are suitable weight functions with the desired asymptotic and symmetry properties:

$$\varphi(z)=\varphi(-z) \tag{7a}$$

$$\psi(z)=-\psi(-z). \tag{7b}$$

$$\left.\begin{aligned}\varphi(z)&=\bar{\varphi}(z)e^{-z} && \text{(8a)}\\ \psi(z)&=\bar{\psi}(z)e^{-z} && \text{(8b)}\end{aligned}\right\}\text{for large } z,$$

where $\bar{\varphi}(z)$ and $\bar{\psi}(z)$ are slowly varying functions of z.

The quantities A_j and C_j are intensity (normalization) parameters, and the B_j and D_j are scale parameters. The introduction of location parameters is not necessary because the peak in the flux function will be determined by the superposition of symmetric and antisymmetric solutions. The parameters B_j and D_j, analogous to the parameter w in the method of polynomial expansion, will be effectively determined from the moments themselves, so that no "inspired guess" is necessary.

We require that

$$\overline{z^{2n}}(E,\alpha)=\sum_{j=1}^{2}A_j(E,\alpha)\int_{-\infty}^{\infty}z^{2n}\phi[zB_j(E,\alpha)]dz \tag{9a}$$

$$(n=0,1,2,3)$$

$$\overline{z^{2n+1}}(E,\alpha)=\sum_{j=1}^{2}C_j(E,\alpha)\int_{-\infty}^{\infty}z^{2n+1}\psi[zD_j(E,\alpha)]dz. \tag{9b}$$

The solution of these equations is described in appendix C. The important aspect of the solution is the fact that there are restrictions on the class of allowed weight functions $\phi(z)$ and $\psi(z)$. In order to obtain the desired asymptotic behavior, it is required that the scale parameters B_j and D_j are real-valued; otherwise, the weight functions would turn into oscillatory functions, which is inconsistent with the expected smooth behavior of the radiation density function.

As shown in appendix C, two pairs of dimensionless parameters may be introduced to characterize the relation between the function to be fitted and the proposed weight functions:

$$\left.\begin{aligned}s&=\frac{m_2^2}{m_0m_4}\,\frac{M_0M_4}{M_2^2}\\ t&=\frac{m_2^3}{m_0^2m_6}\,\frac{M_0^2M_6}{M_2^3}\end{aligned}\right\}\ (10)\qquad\qquad \left.\begin{aligned}\hat{s}&=\frac{m_3^2}{m_1m_5}\,\frac{M_1M_5}{M_3^2}\\ \hat{t}&=\frac{m_3^3}{m_1^2m_7}\,\frac{M_1^2M_7}{M_3^3}\end{aligned}\right\}\ (11)$$

where $M_n=\overline{z^n}(E,\alpha), m_{2n}=\int_{-\infty}^{\infty}z^{2n}\varphi(z)dz$, and $m_{2n+1}=\int_{-\infty}^{\infty}z^{2n+1}\psi(z)dz$.

The weight function $\varphi(z)$ belongs to the allowed class (i. e., the scale parameters B_j are real) if, and only if, the point (s,t) lies in an "allowed region" of the s-t plane, indicated by the shaded area in figure 1 and specified in appendix C. To determine whether $\psi(z)$ is allowable, we replace the point (s,t) by $(\hat{s},\hat{t})$ and use the same diagram.

It has been found in this investigation and in other related work, that considerable searching is often necessary to get an allowed weight function, but when there are several such functions they yield moment-fitting results that are in close agreement. Thus the function-fitting technique seems to operate in a sort of all-or-nothing fashion, and differs in this respect from the polynomial expansion method.

The symmetric and asymmetric weight functions have been related as follows:

$$\psi(z)=z\varphi(z). \tag{12}$$

This implies that at $z=0$, the antisymmetric component of the scattered flux vanishes and the total scattered flux is continuous.[3] The choice of z, rather than some other antisymmetric function of z as the proportionality factor in eq (12) is somewhat arbitrary but recommends itself by its simplicity and is consistent with what is known about the analytical behavior of the flux carried by photons with energies close to the source energy.

[3] The unscattered flux is obtained as space integral over the source function and the flux of the nth order of scattering as a space integral over the flux of the n—1st order of scattering. The source function is a delta function $\delta(z)$. The unscattered flux thus has a simple discontinuity at the origin, whereas the flux of higher orders of scattering is continuous.

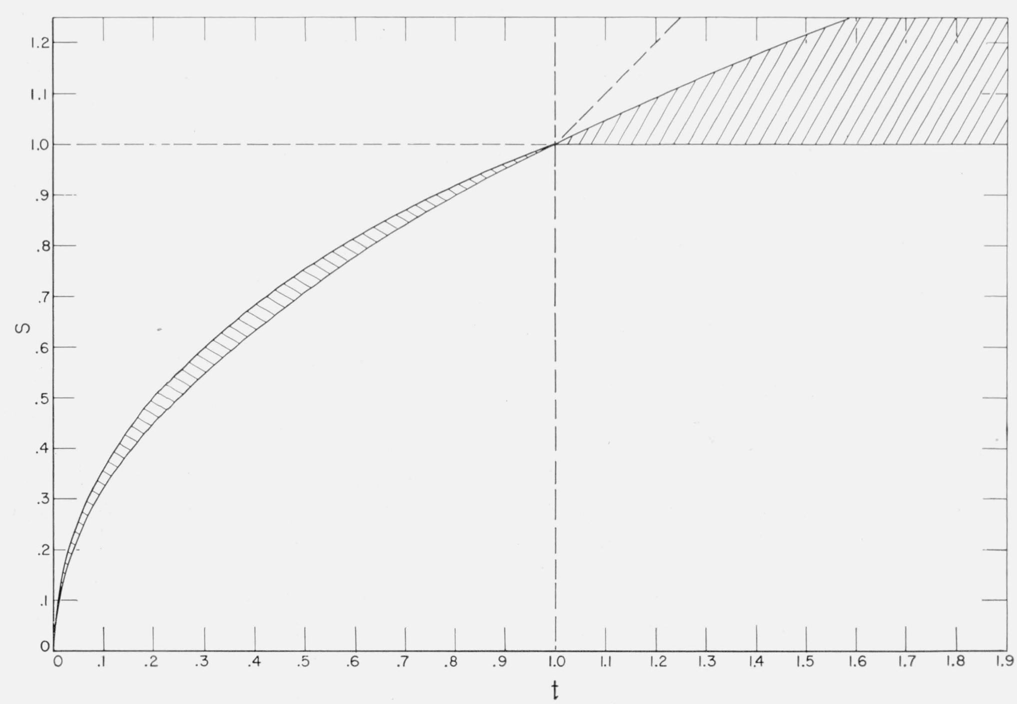

FIGURE 1. *Diagram indicating whether a weight-function is "permissible" for function-fitting.*

The diagram should extend over the region $0\leq s\leq\infty$, $0\leq t\leq\infty$, but only a portion of this region is shown. A weight-function is "permissible" if its (s, t)-point lies in the shaded region of the diagram.

3. Numerical Application: Radiation from a 0.66-Mev Plane-Oblique Source in Water

A sample calculation, based on the methods outlined in section, 2 was carried out for a source of 0.66-Mev photons (radiation from cesium-137) in water.

The flux moments needed as input data were calculated on the SEAC at energies ranging from 0.66 to 0.134 Mev. At lower energies, spectra were obtained through an extrapolation procedure based on two characteristic features of gamma-ray diffusion:

(1) At very low energies an equilibrium spectrum prevails in the sense that the flux can (at least, approximately) be factored into the product of a space-dependent function and an energy-dependent function. This has been shown to be the case for radiation from point and plane isotropic sources [2, 3], and there is no reason to doubt that it will also hold for plane oblique sources.

(2) The shape of the energy spectrum at low energies is nearly independent of the source energy. This is demonstrated by plotting in figure 2 the moment $M_0(E)$ of the flux in water as a function of energy in the range from 0.134 Mev down to 0.020 Mev for source energies of 0.255 Mev and 1.0 Mev. The two curves have been normalized in such a manner that they have the same ordinate at 0.134 Mev. It can be seen that they lie quite close together; the area under the corresponding curve for 0.66-Mev source energy, which must lie between them, can be estimated with an accuracy of 5 percent. (For the purpose of this estimate, only multiply-scattered radiation from the 0.255-Mev source should be considered because a 0.66-Mev source will produce only multiply-scattered radiation below 0.134 Mev.)

Computations were carried out for the energies and angles of incidence indicated in table 1.

Singly-scattered radiation was calculated directly according to the equations given in appendix A. Some moment calculations were also made and are discussed in section 4.

Multiply-scattered radiation was treated by the moment-method according to the function-fitting procedure outlined in section 2. It is desirable that one can fit the flux for as many

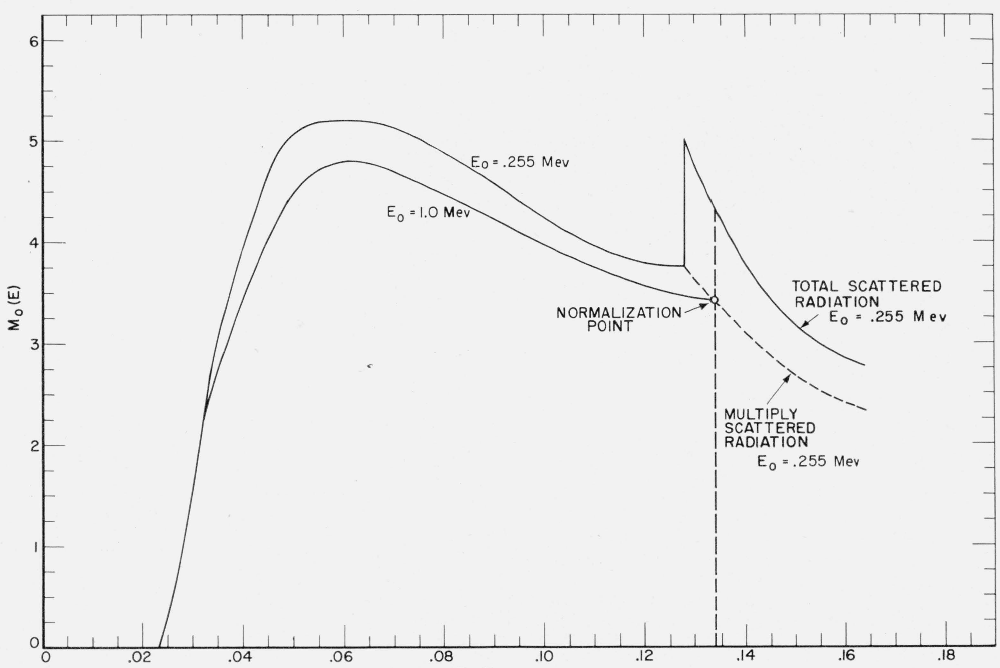

FIGURE 2. *Energy spectrum of scattered gamma radiation from 1.0- and 0.255-Mev sources in a scattering medium of water.*

These distributions were used to obtain by interpolation the low-energy part of the spectral distribution of radiation from a 0.66-Mev source.

TABLE 1. *Energies and obliquity angles for which the energy flux was calculated.*

[Asterisks indicate the values for which the multiply-scattered flux was determined by the method of function-fitting with the use of the weight function $\varphi(z)=L_{5/2}(z)$.]

E	α			
	0°	30°	60°	90°
Mev				
0.660	*	*	*	
.503	*	*		*
.325		*	*	*
.259			*	*
.215	*	*	*	*
.184	*	*	*	*
.150	*	*	*	*
.134	*	*	*	*

energies and source obliquities as possible with the use of the same weight function. This not only inspires confidence that the choice of the weight function is correct, but more importantly, it makes possible smooth interpolation. A considerable number of weight functions were tried, and the results obtained with them are described in section 4.

Preference was finally given to the function

$$\varphi(z)=z^{5/2}K_{5/2}(z)=L_{5/2}(z), \tag{13}$$

where $K_{5/2}$ is a Bessel function of the second kind with imaginary argument. This weight function was superior to other functions considered in that its region of applicability included most of the energies and obliquity angles of interest (see table 1). In the relatively rare instances of nonapplicability, interpolation with the aid of results obtained with some of the other weight functions was successfully used. In any case, it is shown in section 4.2 that the results obtained with different weight functions were generally in excellent agreement. This provides pragmatic justification for the moment-fitting technique.[4]

The results of the sample calculations for a 0.66-Mev plane monodirectional oblique source in water are illustrated in figures 3, 4, and 5. Figure 3 (a, b, c, d) shows energy spectra of the flux of scattered radiation at various distances from the source plane for obliquity angles $\alpha=0°$, 30°, 60°, and 90°.[5] Figure 4 (a, b) shows the dependence of the total scattered flux $\int_0^{0.66} F^{(\alpha)}(E, z)dE$ on the source obliquity and the distance from the source plane.

When the penetration of radiation as a function of the obliquity angle is known, one can, by superposition, determine the penetration of radiation from plane sources with arbitrary angular distribution. An example of such an application is the calculation of the ratio

$$r(\cos\alpha', z)=\frac{\int_{\cos\alpha'}^{1} d\cos\alpha \int_0^{0.66} dEF^{(\alpha)}(E,z)}{\int_{-1}^{1} d\cos\alpha \int_0^{0.66} dEF^{(\alpha)}(E,z)}. \tag{14}$$

This is the fraction of the scattered integrated energy flux from an isotropic 0.66-Mev plane source that is due to source radiation emitted within a cone of opening α' around the normal to the source plane. This ratio is exhibited in figure 5 as a function of $\cos a'$ for various values of z.

[4] One suspects that there may be a deeper-lying reason for the wide applicability of Bessel functions of the type used in eq (13). According to Spencer (private communication) such functions have also proved themselves useful for fitting flux moments for a point monodirectional source.

[5] The flux is expressed as the flow of energy, in Mev per Mev of spectral energy per unit time, through a unit area of surface perpendicular to the direction of motion of the radiation, i. e., through an isotropic detector with an efficiency independent of the direction of the incident radiation. The normalization corresponds to a source emitting one photon with an energy of 0.66 Mev per unit time and unit area. Distances from the source plane are expressed in mean free paths (mfp) of the primary radiation.

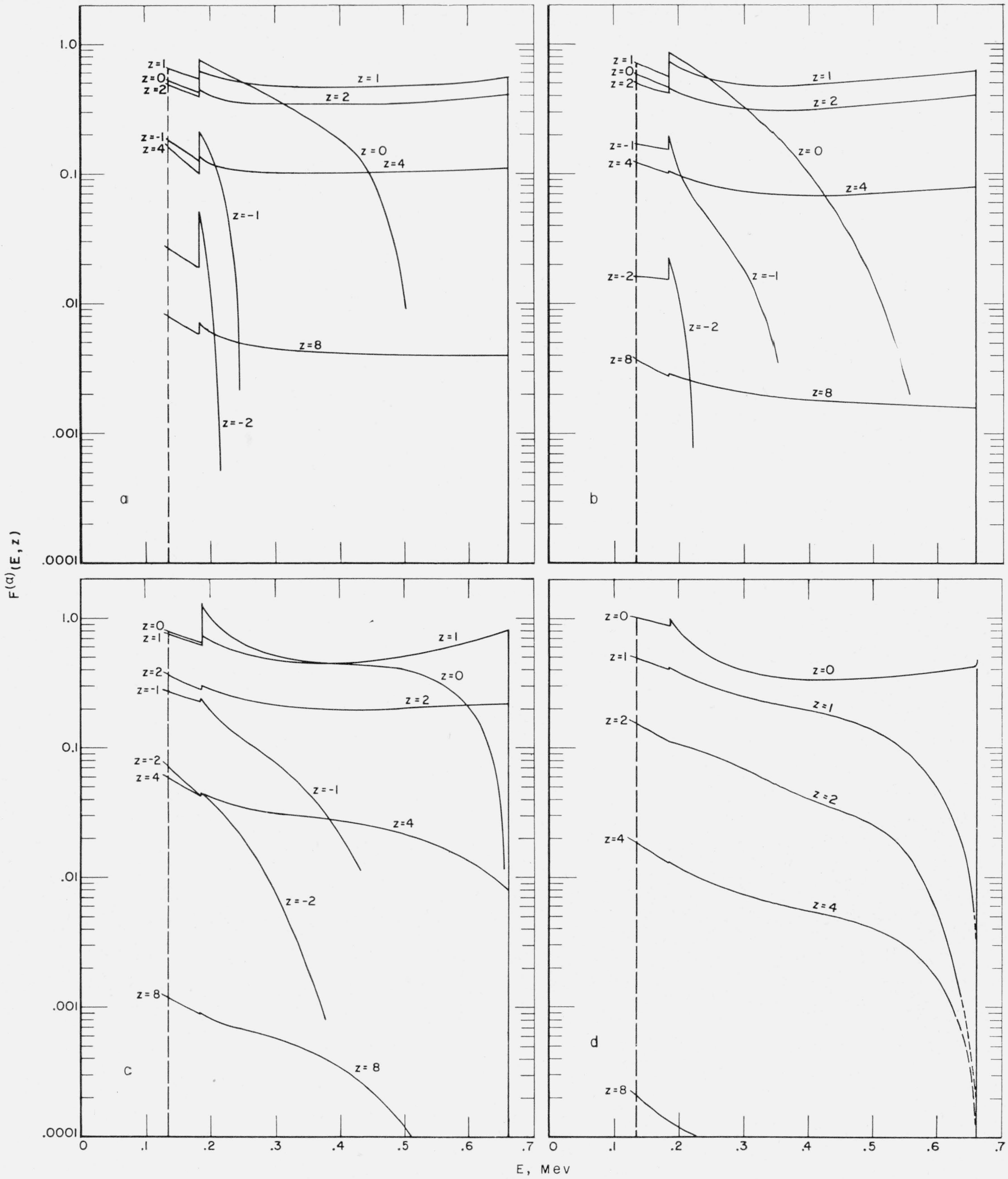

FIGURE 3. *Differential spectra $F^{(\alpha)}$ (E, z) of the scattered gamma-ray flux ((Mev/cm^2 sec)/Mev) at various distances z (mfp) from a plane monodirectional oblique source of 0.66-Mev radiation in a scattering medium of water.*

The ordinate scale corresponds to a source emitting 1 photon/cm ^{2} sec.
a, Obliquity α=0°; b, obliquity α=30°; c, obliquity α=60°; d, obliquity α=90°.

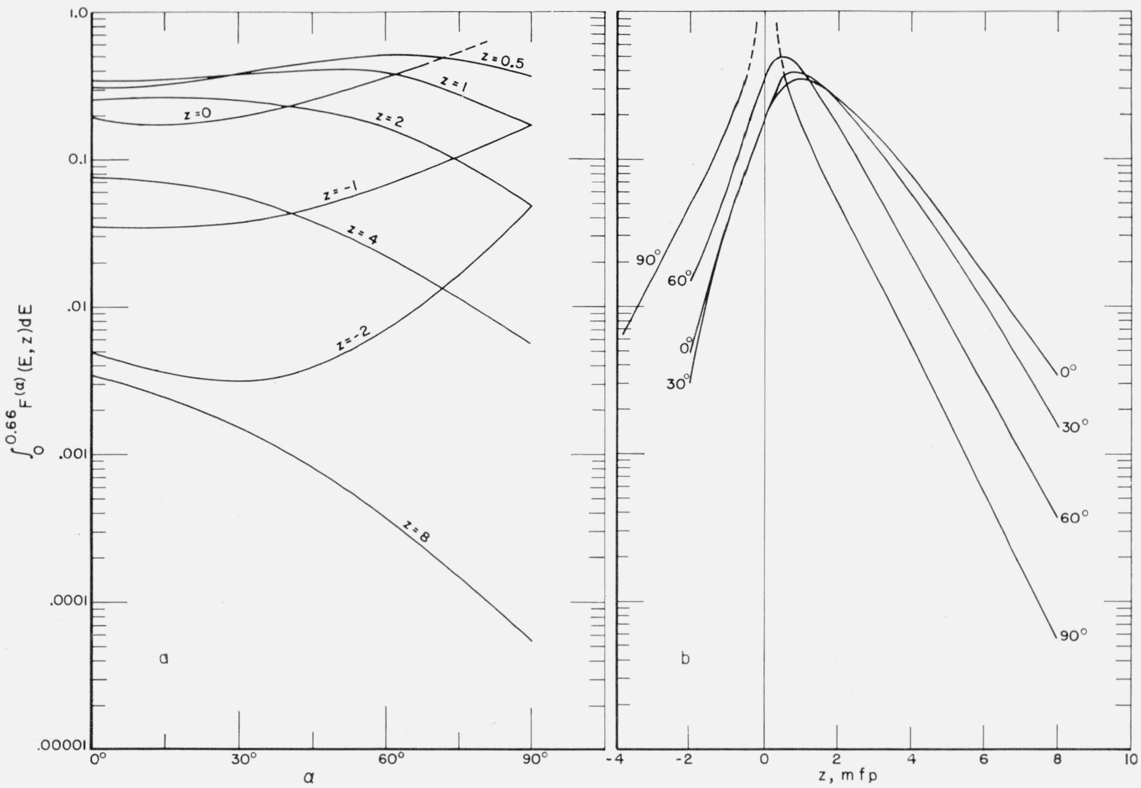

FIGURE 4. *Total scattered gamma-ray flux from a 0.66-Mev plane monodirectional oblique source in a scattering medium of water.*

The ordinate scale (in Mev/cm² sec) corresponds to a source emitting 1 photon/cm² sec.
a, Flux versus obliquity; b, flux versus distance from source plane (mfp).

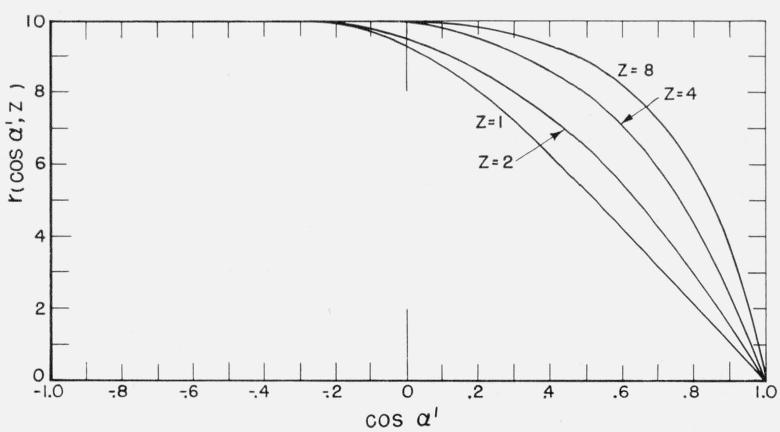

FIGURE 5. *Fraction of the scattered energy flux from a 0.66-Mev plane isotropic source in water that is due to source radiation emitted within a cone of opening α′ around the normal to the source plane.*

4. Checks and Comparisons

4.1. An Experimental Comparison

Experimental evidence with which the theory could be compared is unfortunately quite scarce, in fact is nonexistent for the exact conditions of the present problem. The only relevant published work seems to be that of Kirn, Kennedy, and Wyckoff [5], who measured the penetration of 0.66-Mev gamma radiation from oblique plane sources through slabs of concrete.

A comparison will be made by considering experimental and theoretical buildup factors. The energy buildup factor—a characteristic parameter widely used in shielding work—is defined as the ratio

$$\frac{\text{Total energy flux (scattered plus unscattered)}}{\text{Unscattered energy flux}}.$$

The fractional energy-transmission through a slab is the product of the buildup factor times the quantity exp $(-z/\cos\alpha)$.

The experiment of Kirn et al. and the calculation differ in a number of respects: (a) The calculation for an infinite medium allows for the possibility that radiation can penetrate beyond the detector and then be backscattered into it. In the transmission experiment, however, with the detector behind a concrete slab and with no further backing behind the detector, such backscattering is impossible. The buildup factor for a slab is therefore smaller than that for an infinite medium. In a Monte Carlo calculation [6] it has been found that the ratio of the two buildup factors becomes nearly constant for penetrations greater than 4 mfp. (b) It is only approximately true that concrete may be considered a water-equivalent medium (by expressing all distances in mfp of the primary radiation). Actually, there is enough difference at low energies in the respective absorption coefficients (per electron) so that the buildup factors for concrete are appreciably lower than those for water. By referring to the extensive tabulations of Goldstein and Wilkins [2] it can be found that this reduction factor is a very slowly varying function of the penetration distance (except in the vicinity of the source).

The reason for the approximate constancy of the buildup reductions due to effects (a) and (b) lies in the fact that both are reductions of the low-energy end of the gamma-ray spectrum, but this part of the spectrum—as pointed out in section 3—is an approximate equilibrium spectrum, i. e., rather independent of the depth of penetration.

(c) The calculation assumes a monoenergetic source spectrum. In the experiment the source was physically large, and the source radiation was unfiltered, so that it presumably contained an appreciable amount of low-energy scattered radiation. Additional scattered radiation could have been introduced by the collimating arrangement.

(d) The calculation is for the energy flux, whereas the experimental detector response is proportional to the energy flux for energies down to 200 kev only, but rises somewhat more steeply at lower energies.

Figure 6 shows semilogarithmic plots of the experimental and theoretical buildup factors versus oblique penetration distance $z/\cos\alpha$ (in mfp), for obliquity angles $\alpha=0°$ and $\alpha=60°$. It can be seen that in both cases the experimental curves lie below the theoretical curves, and that for $(z/\cos\alpha)\gtrapprox 3$ the corresponding curves are almost parallel to each other, in conformity with the expectation of a constant buildup-factor reduction. For medium and deep penetration the theory therefore correctly predicts the relative attenuation of the energy flux at different depths.[6]

[6] For a more detailed quantitative comparison for obliquity $\alpha=0°$, taking into account the boundary effect according to the Monte Carlo method, see [6].

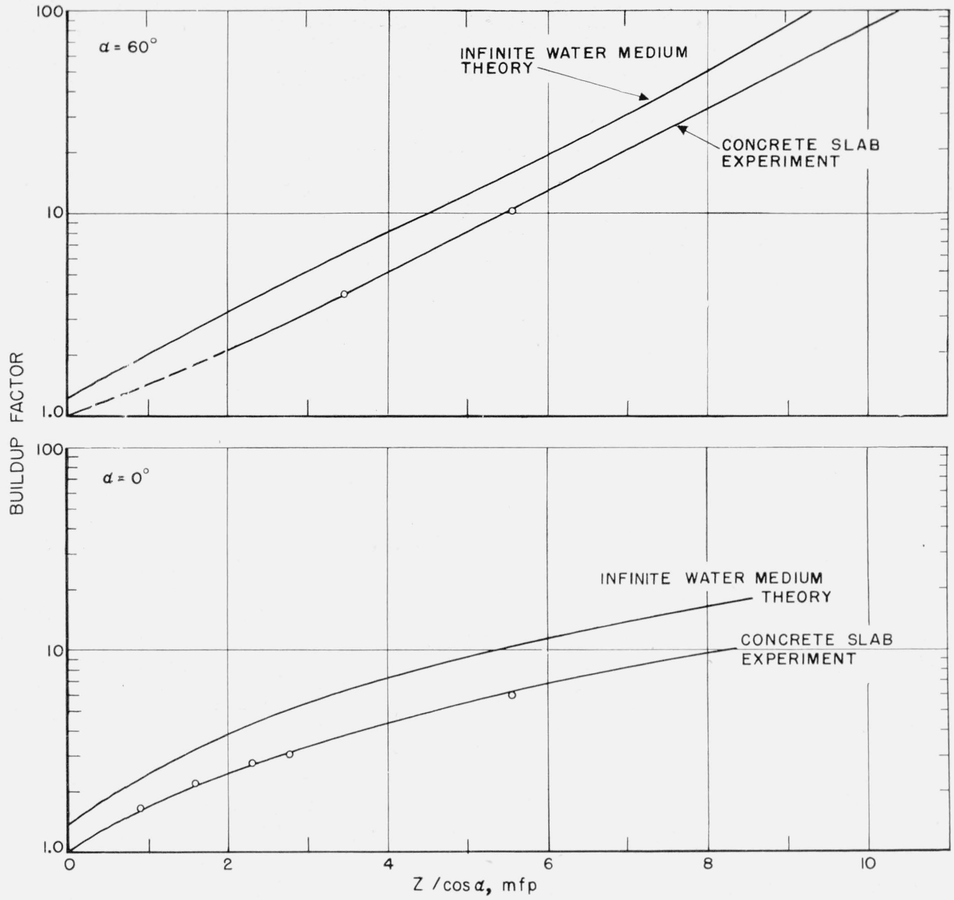

FIGURE 6. *Theoretical and experimental buildup factors for radiation from a 0.66-Mev plane monodirectional oblique source, plotted versus oblique penetration distance.*

The theoretical curves pertain to an infinite scattering medium of water, the experimental curves to a plane-parallel concrete barrier.

4.2. Comparison of Different Weight Functions and Moment-Fitting Procedures

In the absence of much experimental evidence that would provide a check on the theory, one is forced to rely heavily on tests of internal consistency. Hence a good deal of numerical experimentation has been accomplished, employing various weight functions to be used in the method of function-fitting, including also polynomial expansions whenever possible.

Listed below are the relevant properties of a few functions that were found to be useful as weight functions in fitting the flux distribution from monodirectional plane oblique sources. These functions were selected because they have the proper asymptotic behavior and are either available in tabulated form or easily evaluated directly. In the formulas below, z will be understood to mean $|z|$.

Bessel Functions:

$$\varphi(z)=L_p(z)=z^pK_p(z), \tag{15}$$

where K_p is a Bessel function of the second kind with imaginary argument.

$$m_{2n}=\int_{-\infty}^{\infty}dz z^{2n}\varphi(z)=2^{2n+p}\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}+n+p\right). \tag{16}$$

$$\lim_{z\to\infty} L_p(z)=\left(\frac{\pi}{2}\right)^{1/2} z^{p-1/2}e^{-z}. \tag{17}$$

If $p=q+\frac{1}{2}$, $(q=0,1,2, \ldots)$,

$$L_{q+1/2}(z)=\left(\frac{\pi}{2}\right)^{1/2} e^{-z}\sum_{i=0}^{q}\frac{(q+i)!}{(q-i)!i!}(1/2)^i z^{q-i}. \tag{18}$$

Generalized Exponential Integral:

$$\varphi(z)=E_p(z)=\int_1^\infty e^{-zt}\frac{dt}{t^p}. \tag{19}$$

(For $p=1$, this is the ordinary exponential integral).

$$\lim_{z\to\infty} E_p(z)=\frac{e^{-z}}{z} \tag{20}$$

$$m_{2n}=\frac{(4n)!}{2n+p}. \tag{21}$$

Error Function:

$$\varphi(z)=1-\operatorname{erf}(\sqrt{z})=1-\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt, \tag{22}$$

$$\lim_{z\to\infty}\left[1-\operatorname{erf}(\sqrt{z})\right]=\frac{e^{-z}}{\sqrt{\pi z}}, \tag{23}$$

$$M_{2n}=\frac{(4n+1)!}{2^{4n}(2n+1)!}. \tag{24}$$

Gaussian Weight Function:

$$\varphi(z)=G(z)=\frac{e^{-z^2/2}}{\sqrt{2\pi}} \tag{25}$$

$$m_{2n}=\frac{(2n)!}{2^n n!} \tag{26}$$

Table 2 gives a comparison of moment-fitting results obtained at an energy of 0.184 Mev for obliquity angles $\alpha=0°$ and 60°. Results for the symmetric and antisymmetric components are shown separately. For comparative purposes the normalization has been chosen so that the results obtained with the weight function $L_{5/2}$ are the standard of comparison and have the value unity. The over-all conclusion can be reached from an inspection of table 2 that the various methods are in good agreement; nonnegligible discrepancies occur only very close to or very far from the source. It is to be noted that at $\alpha=0°$ there is a good agreement with the well-established method of polynomial expansion into U- and V-polynomials. Even a Gaussian weight function yields essentially the same results as the others at distances not too far from the source. The results of table 2 are typical and have been confirmed by other similar calculations.

It should be mentioned that function-fitting could also be carried out with the use of a larger number of moments than four by representing the flux by a functional series of more than two terms. If the use of the additional moments makes little difference, the fitting procedure is presumably accurate. Such a check would be equivalent to establishing the validity of a polynomial expansion by examining its convergence. In the sample calculation the number of available moments unfortunately was too small to do this.

TABLE 2. *Comparison of the results of moment fitting with different weight functions*

z	Weight function (normal incidence: $\alpha=0°$) $L_{5/2}$	$L_{7/2}$	$L_{9/2}$	G	e^{-z} (U-polynomial expansion)
0	1.00	0.981	0.970	0.918	1.06
0.5	1.00	.992	.986	.946	1.00
1	1.00	1.02	1.00	1.00	0.978
2	1.00	0.998	1.01	1.07	.978
4	1.00	.957	0.949	0.908	.974
8	1.00	.995	1.01	1.12	.977
16	1.00	.915	0.849	0.212	1.03

$f^{(0)}$ (0.184, z)

z	Weight function (oblique incidence: $\alpha=60°$) $L_{1/2}$	$L_{3/2}$	$L_{5/2}$	$L_{7/2}$	$L_{9/2}$	E_4	E_5	$1-erf(\sqrt{z})$
0	1.15	1.04	1.00	0.976	0.961	1.30	1.12	1.09
0.5	0.987	1.00	1.00	.999	.994	1.03	0.992	0.993
1	.936	0.962	1.00	1.01	1.02	0.920	.940	.939
2	.998	.997	1.00	1.00	1.02	.911	.996	.992
4	1.03	1.02	1.00	0.984	0.974	.903	1.02	.963
8	0.904	0.926	1.00	.960	.972	.780	1.15	.901
16	1.53	1.19	1.00	.680	.532	1.20	1.58	1.50

$f^{(60)}$ (0.184, z)

z	$zL_{5/2}$	$zL_{7/2}$	$zL_{9/2}$	ze^{-z} (V-polynomial expansion)
0.5	1.00	1.09	1.04	1.51
1/2	1.00	1.12	1.09	1.25
2	1.00	1.09	1.10	1.04
4	1.00	1.01	0.933	1.11
8	1.00	0.825	1.11	0.811
16	1.00	.731	0.695	.846

$g^{(0)}$ (0.184, z)

z	$zL_{1/2}$	$zL_{3/2}$	$zL_{5/2}$	$zL_{7/2}$	$zL_{9/2}$	zE_4	zE_5
0.5	1.30	1.11	1.00	0.920	0.833	1.46	1.25
1	1.02	1.02	1.00	.978	.951	0.992	1.06
2	1.12	0.963	1.00	1.03	1.02	.910	0.877
4	1.05	1.02	1.00	0.978	0.950	1.06	1.19
8	0.947	0.975	1.00	1.06	.107	0.881	1.32
16	1.47	1.20	1.00	0.823	0.698	1.63	1.78

$g^{(60)}$ (0.184, z)

4.3. Other Comparisons

The penetration of radiation from a plane isotropic source can be determined through the superposition of results for plane monodirectional oblique sources. Good agreement has been found between the results of such a superposition and a direct calculation based on a polynomial expansion with the use of the appropriate plane isotropic moments. The ratio of the respective results for the total scattered energy flux at various distances from the source is shown in table 3.

As a further check, moment calculations were also carried out for singly-scattered radiation, which could be compared with the results of direct evaluations of the formulas of appendix A. Table 3 also contains sample results of such calculations; again, the agreement is good. It should be mentioned that the direct evaluation of the flux of singly-scattered radiation was rather laborious because of the necessary numerical integration. It would have been more expeditious, and would have involved only a minor loss of accuracy, to use moment-fitting making use of weight functions suggested by the exact analytical expressions or physical considerations.

TABLE 3. *Sample calculations to test the accuracy of the moment-fitting procedure*

z	A	B	C
0	0.922	----------	----------
0.5	.960	0.935	----------
1	1.09	.979	1.04
2	0.971	1.04	1.01
4	1.02	1.12	1.03
8	0.908	1.15	1.07

Ratio of results for singly-scattered flux from a monodirectional plane oblique source:
A: Moment calculation/direct calculation ($\alpha=0°$, $E=0.325$ Mev; weight function $L_{5/2}(z)$).
B: Moment calculation/direct calculation ($\alpha=90°$; $E=0.215$ Mev; weight function $E_1(z)$).

Ratio of results for scattered flux from plane isotropic source, $\int_0^\pi \sin\alpha d\alpha \int_0^{0.66} dE F^{(\alpha)}(E,z)$:
C: Superposition of plane oblique source results/direct calculation based on plane isotropic moments.

5. Appendix A. Singly-Scattered Radiation

Consider a photon emitted by a plane source at $z=0$, with energy E_0 and obliquity α. Assume that the photon travels to a depth s without interaction, undergoes a Compton scattering between s and $s+ds$ that changes its energy from E_0 to a value in the range $(E, E+dE)$, and crosses the z-plane with obliquity ω without further interaction. The probability for this contingency is

$$PdEds=\frac{K(E_0,E)}{\cos\alpha}\exp\left\{-\frac{\mu(E_0)s}{\cos\alpha}-\frac{\mu(E)(z-s)}{\cos\omega}\right\}U[(z-s)\cos\omega]dEds, \tag{A1}$$

where $\mu(E)$ is the narrow-beam linear total-absorption coefficient, $K(E_0,E)$ is the Klein-Nishina differential coefficient for Compton scattering with an energy change from E_0 to E, and $U(z)$ is the unit step function ($U=1$, if $z>0$; $U=0$, if $z<0$). According to the law of Compton scattering

$$\cos\omega=\cos\alpha\cos\left(1-\frac{mc^2}{E}+\frac{mc^2}{E_0}\right)+\sin\alpha\sin\left(1-\frac{mc^2}{E}+\frac{mc^2}{E_0}\right)\cos\varphi, \tag{A2}$$

where mc^2 is the electron rest energy, and φ is an azimuthal angle distributed uniformly between 0 and 2π.

In order to obtain the energy flux $F_1^{(\alpha)}(E,z)$ of singly-scattered radiation, one must multiply eq (A1) by E, integrate over all values of s between 0 and ∞ (assuming that $0\leq\alpha\leq\pi/2$), multiply the result by sec ω [7], and average over all possible azimuthal angles φ. With distances expressed in mean free paths of the primary radiation, the final result for the singly-scattered flux can be written

$$F_1^{(\alpha)}(E,z)=\frac{EK(E_0,E)}{\mu(E_0)\cos\alpha}e^{-(z/\cos\alpha)}\frac{1}{2\pi}\int_0^{2\pi}d\varphi H(z,E,\alpha), \tag{A3}$$

where

$$\begin{aligned} H(z,E,\alpha)&=\frac{1}{t\cos\omega}\{(e^{zt}-1)U(\cos\omega)-U(-\cos\omega)\} && (z>0)\\ &=-\frac{1}{t\cos\omega}e^{zt}U(-\cos\omega) && (z<0). \qquad \text{(A4)}\\ t&=\frac{1}{\cos\alpha}-\frac{\mu(E)}{\mu(E_0)}\frac{1}{\cos\omega}. \end{aligned}$$

The integral in eq (A3) has been evaluated numerically, except when $\alpha=0°$; in this case

$$\cos\omega=\cos\left(1-\frac{mc^2}{E}+\frac{mc^2}{E_0}\right), \tag{A5}$$

and the integration over φ is trivial.

6. Appendix B. Calculation of the Spatial Flux Moments

The following brief discussion of the moment equations is presented to make this paper self-contained. It will also bring out some minor points in which the procedure followed here differs in details from that given in the basic reference [1]. These are the use of an energy flux instead of a number flux, the separation of unscattered and singly-scattered flux, and the extension to plane sources with arbitrary angular distributions with cylindrical symmetry.

[7] Without the factor sec ϖ one would obtain an energy current across the z-plane instead of the energy flux through surface perpendicular to the direction of propagation of the radiation.

Let $F_k(z, \cos\theta, E)$ be the energy flux of photons that have been scattered k times. In terms of the F_k's, the transport equation has the form

$$\left.\begin{aligned}
&\frac{\partial F_k}{\partial z}\cos\theta+\mu F_0=S(z,E,\cos\theta)\\
&\frac{\partial F_k}{\partial z}\cos\theta+\mu F_k=\int_E^{E_0}dE'\,\frac{E}{E'}\,K(E',E).\\
&\qquad\int_0^{2\pi}d\varphi'\int_{-1}^{1}d\cos\theta'\,\frac{1}{2\pi}\,\delta\left(1-\frac{mc^2}{E}+\frac{mc^2}{E'}-\cos\theta\cos\theta'-\right.\\
&\qquad\left.\sin\theta\sin\theta'\cos\varphi'\right)F_{k-1}\qquad(k>0)
\end{aligned}\right\}\tag{B1}$$

where δ is the Dirac delta function, and S is the source function.

We shall consider a set of elementary source functions

$$S_L=E_0\delta(E-E_0)P_L(\cos\theta)\delta(z)\,\frac{2L+1}{4\pi}\qquad L=0,1,2,\ldots,\tag{B2}$$

where P_L is a Legendre polynomial. These elementary sources have no direct physical meaning except when $L=0$ but are convenient mathematical fictions.

We shall indicate the source type from which a given flux is derived by an index L.

We expand F_{kL} into a Legendre series in the angular variable and take spatial moments of the Legendre expansion coefficients; i. e., we introduce the transform

$$G_{kL}(n,l,E)=\int_{-\infty}^{\infty}dz\int_{-1}^{1}d\cos\theta\, z^nP_l(\cos\theta)\,F_{kL}(z,\cos\theta,E)\tag{B3}$$

and a similar transform for the source function

$$T_L(n,l,E)=\int_{-\infty}^{\infty}dz\int_{-1}^{2}d\cos\theta\, z^nP_l(\cos\theta)S_L(z,\cos\theta,E)=\frac{1}{2\pi}E_0\delta(E-E_0)\,\delta_{n0}\,\delta_{lL}.\tag{B4}$$

It follows from the transport eq (B1) and sundry properties of Legendre polynomials, that the transforms G_{kL} obey the equation

$$\begin{aligned}
&(\Delta+\mu)G_{0L}=T_L && (k=0)\\
&(\Delta+\mu)G_{kL}=\int_E^{E_0}dE\,\frac{E}{E'}\,K(E',E)P_l\left(1-\frac{mc^2}{E}+\frac{mc^2}{E'}\right)G_{k-1,L} && (k>0),
\end{aligned}\tag{B5}$$

where Δ is an operator defined by the relation

$$\Delta G_{kL}(n,l,E)=n\,\frac{l+1}{2l+1}\,G_{kL}(n-1,l+1,E)+n\,\frac{l}{2l+1}\,G_{kL}(n-1,l-1,E).\tag{B6}$$

The transform of the multiply scattered flux is found by summation

$$H_1(n,l,E)=\sum_{k=2}^{\infty}G_{kL}(n,l,E).\tag{B7}$$

It follows from eq (B5) and (B7) that

$$(\Delta+\mu)H_L(n,l,E)=\int_E^{E_0}dE'\,\frac{E}{E'}\,K(E',E)P_l\left(1-\frac{mc^2}{E}+\frac{mc^2}{E'}\right)H_L(n,l,E')+T_L^*(n,l,E),\tag{B8}$$

where the source function

$$T_L^*(n,l,E)=\int_E^{E_0} dE' \frac{E}{E'} k(E',E) P_l\left(1-\frac{mc^2}{E}+\frac{mc^2}{E'}\right) G_{1L}(n,l,E'). \tag{B9}$$

The calculation thus proceeds as follows: One computes G_{0L} and G_{1L} with the use of eq (B5), then the source function according to eq (B9), and finally H_L according to eq (B7). The equation for G_{0L} is algebraic; all others are linear integral equations of the Volterra type, and are solved numerically on the NBS automatic computer (SEAC). The equations are interlinked, but the nature of the coupling operator Δ is such that they need not be solved simultaneously. Indicated below are the chains of equations that must be solved to calculate $H_L(n,l,E)$ or $G_{kL}(n,l,E)$; for the sake of brevity only the (n,l) values are shown.

			00
		11	
	22		20
33		31	
	42		40
		51	
			60

$L=0$

		02		
	13		11	
24		22		20
	33		31	
		42		40
			51	
				60

$L=2$

	04				
15		13			
	24		22		
		33		31	
			42		40
				51	
					60

$L=4$

06						
	15					
		24				
			33			
				42		
					51	
						60

$L=6$

			01	
		12		10
	23		21	
34		32		30
	43		41	
		52		50
			61	
				70

$L=1$

		03			
	14		12		
25		23		21	
	34		32		30
		43		41	
			52		50
				61	
					70

$L=3$

	05					
16		14				
	25		23			
		34		32		
			43		41	
				52		50
					61	
						70

$L=5$

07							
	16						
		25					
			34				
				43			
					52		
						61	
							70

$L=7$

For example, in order to compute $H_0(4, 0, E)$, one must first calculate in turn the quantities $H_0(0, 0, E)$, $H_0(1, 1, E)$, $\mathrm{H}_0(2, 2, E)$, and $H_0(3, 1, E)$.

It can be shown that

$$H_L(n,l,E)=G_{kL}(n,l,E)=0 \tag{B10}$$

unless $n-|L-l|$ is a nonnegative even integer.

Now consider a monoenergetic plane source with arbitrary distribution, expanded into a Legendre series:

$$S(z, \cos\theta, E)=E_0\delta(E-E_0)\delta(z)\sum_{L=0}^{\infty}\frac{2L+1}{4\pi}A_LP_L(\cos\theta). \tag{B11}$$

It follows from the linearity of the transport equation that the moments of the flux for such a source can be obtained through the addition of moments for "elementary" sources. Taking into account (B10), one finds that

$$\left.\begin{aligned}\overline{z^{2n}}(E)&=\sum_{L=0}^{n}A_{2L}H_{2L}(2n,0,E)\\ \overline{z^{2n+1}}(E)&=\sum_{L=0}^{n}A_{2L+1}H_{2L+1}(2n+1,0,E).\end{aligned}\right\}\tag{B12}$$

It is remarkable that in order to determine the first n even (odd) moments, one need specify the source only by its first n even (odd) expansion coefficients A_L.

Finally, we note that for a monodirectional source with angular part δ (cos θ—cos α),

$$A_L=P_L\ (\cos\alpha).\tag{B13}$$

7. Appendix C. Details of the Function-Fitting Procedure

For the solution of the moment eqs (9a, b) it is convenient to adopt the following notation:

$$\left.\begin{aligned}A_j&=\alpha_jB_j\frac{M_0}{m_0}\\ B_j&=\left[\frac{m_0}{m_2}\frac{M_2}{M_0}\beta_j\right]^{-1/2}\end{aligned}\right\}\ \text{(C1)}\qquad \left.\begin{aligned}C_j&=\gamma_jD_j^3\frac{M_1}{m_1}\\ D_j&=\left[\frac{m_1}{m_3}\frac{M_3}{M_1}\delta_j\right]^{-1/2}\end{aligned}\right\}\ \text{(C2)}$$

$$(j=1,2)\qquad\qquad (j=1,2)$$

Equations (9a, b) can then be written in the form

$$\left.\begin{aligned}\alpha_1+\alpha_2&=1\\ \alpha_1\beta_1+\alpha_2\beta_2&=1\\ \alpha_1\beta_1^2+\alpha_2\beta_2^2&=s\\ \alpha_1\beta_1^3+\alpha_2\beta_2^3&=t\end{aligned}\right\}\ \text{(C3)}\qquad \left.\begin{aligned}\gamma_1+\gamma_2&=1\\ \gamma_1\delta_1+\gamma_2\delta_2&=1\\ \gamma_1\delta_1^2+\gamma_2\delta_2^2&=\hat{s}\\ \gamma_1\delta_1^3+\gamma_2\delta_2^3&=\hat{t}\end{aligned}\right\}\ \text{(C4)}$$

Equations (C3) and (C4) being formally the same, further discussion is confined to (C3). A solution can be obtained by a method given by Chandrasekhar [7]. Let

$$\omega_0=\frac{t-s^2}{s-1}\ \text{(C5)}\qquad \omega_1=\frac{s-t}{s-1}\ \text{(C6)}$$

It follows from (C3) that β_1 and β_2 are the roots of the quadratic equation

$$\beta^2+\omega_1\beta+\omega_0=0,\tag{C7}$$

and that

$$\alpha_2=\frac{\beta_1-1}{\beta_1-\beta_2}\ \text{(C8)}\qquad \alpha_1=1-\alpha_2\ \text{(C9)}$$

B_1 and B_2 are real-valued if and only if β_1 and β_2 are real and positive. This requires that

$$\left.\begin{aligned}\omega_1&\leq 0\\ \omega_0&\geq 0\\ \omega_1^2-4\omega_0&\geq 0\end{aligned}\right\}.\tag{C10}$$

Equations (C5), (C6), and (C10) imply the following conditions on s and t:

If $0 \leq s \leq 1$, $0 \leq t \leq 1$, the allowed region lies between the parabola $s^2=t$, and the curve whose parametric representation is

$$\left.\begin{aligned} s&=\frac{1}{8}\{3+6u-u^2-[(3+6u-u^2)^2-64u]^{1/2}\} \\ t&=us \end{aligned}\right\} \qquad (0 \leq u \leq 1) \tag{C11}$$

If $s \geq 1$ and $t \geq 1$, the allowed region lies between the line $s=1$ and the parabola $s^2=t$.

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8. References

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