A Matrix with Real Characteristic Roots

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It is proved that a certain matrix, which is the coefficient matrix of a differential equation found in the theory of dielectric relaxation, has only real characteristic roots. This is done by finding a real symmetric matrix with the same principal minors and thus the same characteristic roots.

In papers [1, 2, 3] [1] by J. D. Hoffman and B. M. Axilrod a certain differential equation has a real, constant coefficient matrix $A=(a_{ij})$ with the properties

$$a_{ii}\geq 0 \qquad \text{for all } i \tag{1}$$

$$a_{ij}\leq 0 \qquad \text{for all } i\neq j \tag{2}$$

$$\sum_i a_{ij}=0 \qquad \text{for all } j \tag{3}$$

$$a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1}=a_{i_2i_1}a_{i_3i_2}\ldots a_{i_1i_k} \qquad \text{for all } i_1,i_2,\ldots,i_k. \tag{4}$$

In this note we prove the conjecture of J. D. Hoffman and B. M. Axilrod that a matrix with these properties has only real characteristic roots and that these characteristic roots lie between zero and twice the maximum diagonal element.

The second statement is a direct consequence of the first and properties (1), (2), and (3) by the theorem of S. Gershgorin [4] and A. Brauer [5]. This theorem states that all the characteristic roots of a matrix $A=(a_{ij})$ lie in the area bounded by the circles $|z-a_{jj}|\leq\sum_i|a_{ij}|-|a_{jj}|$.

Therefore, we may concentrate on proving that all the characteristic roots are real. We shall do this by exhibiting a symmetric matrix with the same characteristic roots as one with properties (2) and (4).

Let $B=(b_{ij})$ be a matrix of the same order as the matrix $A=(a_{ij})$, which has properties (2) and (4), such that

$$b_{ij}=-(a_{ij}a_{ji})^{1/2}$$

with the square root assumed to be positive. Then

$$\begin{aligned} b_{i_1i_2}b_{i_2i_3}\ldots b_{i_ki_1} &=(-1)^k(a_{i_1i_2}a_{i_2i_1}a_{i_2i_3}a_{i_3i_2}\ldots a_{i_ki_1}a_{i_1i_k})^{1/2}\\ &=(-1)^k|a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1}|\\ &=a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1}\end{aligned}$$

for all $i_1,i_2,\ldots,i_k$.

Now consider any principal minor

$$|A(s_1,s_2,\ldots,s_m)|=|a_{ij}|_{i,j=s_1,s_2,\ldots,s_m} \text{ of } A.$$

From the definition of a determinant we have

$$|A(s_1,s_2,\ldots,s_m)|=\sum\pm a_{s_1t_1}a_{s_2t_2}\ldots a_{s_mt_m}$$

where the sum is taken over all permutations

$$\begin{pmatrix} s_1s_2 & \ldots & s_m\\ t_1t_2 & \ldots & t_m\end{pmatrix}$$

of $s_1,s_2,\ldots,s_m$, and the sign is positive if the permutation is even, and negative if it is odd.

Each permutation is the product of cycles. Thus

$$\begin{pmatrix} s_1s_2 & \ldots & s_m\\ t_1t_2 & \ldots & t_m\end{pmatrix}=\begin{pmatrix} i_1i_2 & \ldots & i_k\\ i_2i_3 & \ldots & i_1\end{pmatrix}\ldots,$$

so that

$$\begin{aligned} a_{s_1t_1}a_{s_2t_2}\ldots a_{s_mt_m} &=a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1}\ldots\\ &=b_{i_1i_2}b_{i_2i_3}\ldots b_{i_ki_1}\ldots\\ &=b_{s_1t_1}b_{s_2t_2}\ldots b_{s_mt_m}.\end{aligned}$$

It follows that

$$|A(s_1,s_2\ldots,s_m)|=|B(s_1,s_2,\ldots,s_m)|,$$

that is, that the corresponding principal minors of A and B are equal.

This implies that A and B have the same characteristic equation and thus have the same characteristic roots.

[1] J. D. Hoffman and B. M. Axilrod, Dielectric relaxation for spherical molecules in a crystalline field: Theory for two simple models, J. Research NBS **54,** 375 (1955) RP2598.

[2] J. D. Hoffman, Theory of dielectric relaxation for a single-axis rotator in a crystalline field. II, J. Chem. Phys. **23,** 1331 (1955).

[3] B. M. Axilrod, Dielectric relaxation for a three-dimensional rotator in a crystalline field: Theory for a general six-site model, J. Research NBS **56,** 81 (1956) RP2651.

[4] S. Gershgorin, Über die Abgrenzung Engenwerte einer Matrix, Izvest. Akad. Nauk, S.S.S.R. **7,** 749 (1931).

[5] A. Brauer, Limits for the characteristic roots of a matrix, Duke Math. J. **13,** 387 (1946).

WASHINGTON, October 20, 1955.

[1] Figures in brackets indicate the literature references at the end of this paper.