# Formulas for Inverse Osculatory Interpolation

Herbert E. Salzer [1]

Formulas for inverse osculatory interpolation are obtained by inversion of Hermite's formula. They cover the cases for $n=2(1)7$, where $n$ is the number of points required in direct osculatory interpolation. The formulas provide an improved means for inverse interpolation in the case where the first derivative is either tabulated alongside the function or is easily obtained.

The author has previously given inverse interpolation formulas [1, 2] [2] for finding $x=x_0+ph$ from $f(x)$, in terms of $f_i\equiv f(x_i)$, where the $x_i\equiv x_0+ih$ are equally spaced at intervals of $h$. Those formulas were obtained by the inversion of Lagrange's interpolation formula. As the Hermite osculatory interpolation formula [3, 4, 5] in terms of the function and its first derivative at preassigned points $x_i$ is more accurate than the Lagrangian formula, inverse interpolation formulas obtained from the inversion of Hermite's formula would be expected to be more efficient than the corresponding formulas obtained from Lagrange's formula.

There are many tables where the first derivative is either tabulated alongside the function (e. g., Bessel functions of the first or second kind, which give $J_1(x)=-J_0'(x)$ [6, 7] or $Y_1(x)=-Y_0'(x)$ [8], respectively, or probability functions [9] ), or where the derivative is very easy to obtain (e. g., tables of the more elementary functions and their integrals, such as tables of sine, cosine, or exponential integrals [10, 11] ). In all such tables, and in many others, the user will find the inverse interpolation formulas given below to be particularly convenient, especially when the tabular interval $h$ is too large for sufficiently accurate inverse interpolation, using the formulas in terms of the functional values alone.

These present formulas give $p$ in terms of $f(x_0+ph)\equiv f$, $f(x_i)\equiv f_i$, and $f'(x_i)\equiv f_i'$ for $x_i\equiv x_0+ih$ at equally spaced intervals $h$, where the $i$ ranges from $-[(n\text{-}1)/2]$ to $[n/2]$, for $n=2(1)7$, where $n$ is the number of points required in direct osculatory interpolation. Although the direct interpolation formula for $n=6$ and $n=7$ is of the 11th- and 13th-degree accuracy, respectively, the inversion formula for $p$ that is given below does not go beyond the 10th- degree terms (and in most practical problems it is very rarely that one will go that far; in fact, the first few terms will usually suffice). Thus we do not use the coefficient of $p^{11}$ in the 6-point Hermite formula nor the coefficients of $p^{11}$, $p^{12}$, and $p^{13}$ in the 7-point Hermite formula. For every $n$, we define $r=(f-f_0)/hf_0'$, and corresponding to $n=2(1)7$, quantities $s$, $t$, $u$, $v$, $w$, $x$, $y$, $z$, and $\bar{z}$ are defined as follows:

$n=2$

$$s=(-3\{f_0-f_1\}-h\{2f_0'+f_1'\})/hf_0',$$

$$t=(2\{f_0-f_1\}+h\{f_0'+f_1'\})/hf_0',$$

$$u=v=w=x=y=z=\bar{z}=0.$$

$n=3$

$$s=\left(f_{-1}-2f_0+f_1+\frac{h}{4}\left\{f'_{-1}-f_1'\right\}\right)\Big/hf_0',$$

$$t=\left(-\frac{5}{4}\left\{f_{-1}-f_1\right\}-\frac{h}{4}\left\{f'_{-1}+8f_0'+f_1'\right\}\right)\Big/hf_0',$$

$$u=\left(-\frac{1}{2}\left\{f_{-1}-2f_0+f_1\right\}-\frac{h}{4}\left\{f'_{-1}-f_1'\right\}\right)\Big/hf_0',$$

$$v=\left(\frac{3}{4}\left\{f_{-1}-f_1\right\}+\frac{h}{4}\left\{f'_{-1}+4f_0'+f_1'\right\}\right)\Big/hf_0',$$

$$w=x=y=z=\bar{z}=0.$$

[1] Present address, Diamond Ordnance Fuze Laboratory, Department of Defense, Washington, D. C.

[2] Figures in brackets indicate the literature reference at the end of this paper.

$n=4$

$$s=\left(\frac{1}{108}\left\{56f_{-1}-297f_0+216f_1+25f_2\right\}+\frac{h}{18}\left\{2f'_{-1}-18f'_0-18f'_1-f'_2\right\}\right)\Big/hf'_0,$$

$$t=\left(-\frac{1}{108}\left\{124f_{-1}-27f_0-108f_1+11f_2\right\}-\frac{h}{36}\left\{8f'_{-1}+63f'_0-f'_2\right\}\right)\Big/hf'_0,$$

$$u=\left(\frac{1}{54}\left\{25(f_{-1}-f_2)+135(f_0-f_1)\right\}+\frac{h}{36}\left\{f'_{-1}+72f'_0+63f'_1+4f'_2\right\}\right)\Big/hf'_0,$$

$$v=\left(\frac{1}{108}\left\{59f_{-1}-54f_0-27f_1+22f_2\right\}+\frac{h}{36}\left\{7f'_{-1}+18f'_0-9f'_1-2f'_2\right\}\right)\Big/hf'_0,$$

$$w=\left(-\frac{1}{108}\left\{52f_{-1}+81f_0-108f_1-25f_2\right\}-\frac{h}{36}\left\{5f'_{-1}+36f'_0+27f'_1+2f'_2\right\}\right)\Big/hf'_0,$$

$$x=\left(\frac{1}{108}\left\{11(f_{-1}-f_2)+27(f_0-f_1)\right\}+\frac{h}{36}\left\{f'_{-1}+9f'_0+9f'_1+f'_2\right\}\right)\Big/hf'_0,$$

$$y=z=\bar{z}=0.$$

$n=5$

$$s=\left(\frac{1}{108}\left\{7(f_{-2}+f_2)+128(f_{-1}+f_1)-270f_0\right\}+\frac{h}{72}\left\{(f'_{-2}-f'_2)+32(f'_{-1}-f'_1)\right\}\right)\Big/hf'_0,$$

$$t=\left(-\frac{1}{864}\left\{31(f_{-2}-f_2)+1408(f_{-1}-f_1)\right\}-\frac{h}{144}\left\{(f'_{-2}+f'_2)+64(f'_{-1}+f'_1)+360f'_0\right\}\right)\Big/hf'_0,$$

$$u=\left(-\frac{1}{288}\left\{41(f_{-2}+f_2)+256(f_{-1}+f_1)-594f_0\right\}-\frac{h}{96}\left\{3(f'_{-2}-f'_2)+64(f'_{-1}-f'_1)\right\}\right)\Big/hf'_0,$$

$$v=\left(\frac{1}{1152}\left\{91(f_{-2}-f_2)+1792(f_{-1}-f_1)\right\}+\frac{h}{192}\left\{3(f'_{-2}+f'_2)+128(f'_{-1}+f'_1)+396f'_0\right\}\right)\Big/hf'_0,$$

$$w=\left(\frac{1}{144}\left\{13(f_{-2}+f_2)+32(f_{-1}+f_1)-90f_0\right\}+\frac{h}{48}\left\{(f'_{-2}-f'_2)+12(f'_{-1}-f'_1)\right\}\right)\Big/hf'_0,$$

$$x=\left(-\frac{1}{576}\left\{29(f_{-2}-f_2)+272(f_{-1}-f_1)\right\}-\frac{h}{96}\left\{(f'_{-2}+f'_2)+24(f'_{-1}+f'_1)+60f'_0\right\}\right)\Big/hf'_0,$$

$$y=\left(-\frac{1}{864}\left\{11(f_{-2}+f_2)+16(f_{-1}+f_1)-54f_0\right\}-\frac{h}{288}\left\{(f'_{-2}-f'_2)+8(f'_{-1}-f'_1)\right\}\right)\Big/hf'_0,$$

$$z=\left(\frac{1}{3456}\left\{25(f_{-2}-f_2)+160(f_{-1}-f_1)\right\}+\frac{h}{576}\left\{(f'_{-2}+f'_2)+16(f'_{-1}+f'_1)+36f'_0\right\}\right)\Big/hf'_0,$$

$$\bar{z}=0.$$

$n=6$

$$\begin{aligned}s=\Bigg(\frac{1}{3000}&\left\{76f_{-2}+2375f_{-1}-8500f_0+5000f_1+1000f_2+49f_3\right\}\\&+\frac{h}{600}\left\{3f'_{-2}+150f'_{-1}-400f'_0-600f'_1-75f'_2-2f'_3\right\}\Bigg)\Big/hf'_0,\end{aligned}$$

$$\begin{aligned}t=\Bigg(-\frac{1}{108000}&\left\{3327f_{-2}+169500f_{-1}-8000f_0-168000f_1+2625f_2+548f_3\right\}\\&-\frac{h}{3600}\left\{21f'_{-2}+1500f'_{-1}+8600f'_0+1200f'_1-75f'_2-4f'_3\right\}\Bigg)\Big/hf'_0,\end{aligned}$$

$$u=\left(-\frac{1}{21600}\left\{943f_{-2}-2050f_{-1}-62550f_0+45200f_1+17575f_2+882f_3\right\}\right.$$
$$\left.-\frac{h}{1440}\left\{13f'_{-2}+260f'_{-1}-2400f'_0-2960f'_1-445f'_2-12f'_3\right\}\right)\Big/hf'_0,$$

$$v=\left(\frac{1}{86400}\left\{5729f_{-2}+121400f_{-1}-16000f_0-118400f_1+6175f_2+1096f_3\right\}\right.$$
$$\left.+\frac{h}{2880}\left\{37f'_{-2}+1720f'_{-1}+5140f'_0+1120f'_1-155f'_2-8f'_3\right\}\right)\Big/hf'_0,$$

$$w=\left(\frac{1}{144000}\left\{1274f_{-2}-84875f_{-1}-189000f_0+175000f_1+92750f_2+4851f_3\right\}\right.$$
$$\left.+\frac{h}{14400}\left\{36f'_{-2}-2175f'_{-1}-19800f'_0-20100f'_1-3600f'_2-99f'_3\right\}\right)\Big/hf'_0,$$

$$x=\left(-\frac{1}{144000}\left\{5743f_{-2}+50125f_{-1}-22000f_0-46000f_1+10625f_2+1507f_3\right\}\right.$$
$$\left.-\frac{h}{4800}\left\{39f'_{-2}+925f'_{-1}+1900f'_0+100f'_1-225f'_2-11f'_3\right\}\right)\Big/hf'_0,$$

$$y=\left(\frac{1}{14400}\left\{172f_{-2}+3275f_{-1}+3900f_0-4600f_1-2600f_2-147f_3\right\}\right.$$
$$\left.+\frac{h}{480}\left\{f'_{-2}+45f'_{-1}+200f'_0+180f'_1+35f'_2+f'_3\right\}\right)\Big/hf'_0,$$

$$z=\left(\frac{1}{86400}\left\{351f_{-2}+750f_{-1}-4000f_0+2625f_2+274f_3\right\}\right.$$
$$\left.+\frac{h}{2880}\left\{3f'_{-2}+30f'_{-1}-20f'_0-120f'_1-45f'_2-2f'_3\right\}\right)\Big/hf'_0,$$

$$\bar{z}=\left(-\frac{1}{432000}\left\{1066f_{-2}+10625f_{-1}+9000f_0-13000f_1-7250f_2-441f_3\right\}\right.$$
$$\left.-\frac{h}{14400}\left\{8f'_{-2}+175f'_{-1}+600f'_0+500f'_1+100f'_2+3f'_3\right\}\right)\Big/hf'_0.$$

**$n=7$**

$$s=\left(\frac{1}{36000}\left\{157(f_{-3}+f_3)+4968(f_{-2}+f_2)+43875(f_{-1}+f_1)-98000f_0\right\}\right.$$
$$\left.+\frac{h}{1200}\left\{(f'_{-3}-f'_3)+54(f'_{-2}-f'_2)+675(f'_{-1}-f'_1)\right\}\right)\Big/hf'_0,$$

$$t=\left(-\frac{1}{108000}\left\{167(f_{-3}-f_3)+8667(f_{-2}-f_2)+192375(f_{-1}-f_1)\right\}\right.$$
$$\left.-\frac{h}{3600}\left\{(f'_{-3}+f'_3)+81(f'_{-2}+f'_2)+2025(f'_{-1}+f'_1)+9800f'_0\right\}\right)\Big/hf'_0,$$

$$u=\left(-\frac{1}{648000}\left\{7339(f_{-3}+f_3)+213786(f_{-2}+f_2)+631125(f_{-1}+f_1)-1704500f_0\right\}\right.$$
$$\left.-\frac{h}{21600}\left\{47(f'_{-3}-f'_3)+2403(f'_{-2}-f'_2)+20925(f'_{-1}-f'_1)\right\}\right)\Big/hf'_0,$$

$$v=\left(\frac{1}{1296000}\left\{5206(f_{-3}-f_3)+249831(f_{-2}-f_2)+2517750(f_{-1}-f_1)\right\}\right.$$
$$\left.+\frac{h}{129600}\left\{94(f'_{-3}+f'_3)+7209(f'_{-2}+f'_2)+125550(f'_{-1}+f'_1)+340900f'_0\right\}\right)\Big/hf'_0,$$

$$w=\left(\frac{1}{5184000}\left\{52109(f_{-3}+f_3)+1310016(f_{-2}+f_2)+1525875(f_{-1}+f_1)-5776000f_0\right\}\right.$$
$$\left.+\frac{h}{172800}\left\{337(f'_{-3}-f'_3)+15648(f'_{-2}-f'_2)+88275(f'_{-1}-f'_1)\right\}\right)\Big/hf'_0,$$

$$x=\left(-\frac{1}{5184000}\left\{18493(f_{-3}-f_3)+772368(f_{-2}-f_2)+4174125(f_{-1}-f_1)\right\}\right.$$
$$\left.-\frac{h}{518400}\left\{337(f'_{-3}+f'_3)+23472(f'_{-2}+f'_2)+264825(f'_{-1}+f'_1)+577600f'_0\right\}\right)\Big/hf'_0,$$

$$y=\left(-\frac{1}{1728000}\left\{6217(f_{-3}+f_3)+117408(f_{-2}+f_2)+72375(f_{-1}+f_1)-392000f_0\right\}\right.$$
$$\left.-\frac{h}{57600}\left\{41(f'_{-3}-f'_3)+1584(f'_{-2}-f'_2)+6675(f'_{-1}-f'_1)\right\}\right)\Big/hf'_0,$$

$$z=\left(\frac{1}{1728000}\left\{2209(f_{-3}-f_3)+70584(f_{-2}-f_2)+272625(f_{-1}-f_1)\right\}\right.$$
$$\left.+\frac{h}{172800}\left\{41(f'_{-3}+f'_3)+2376(f'_{-2}+f'_2)+20025(f'_{-1}+f'_1)+39200f'_0\right\}\right)\Big/hf'_0,$$

$$\overline{z}=\left(\frac{1}{5184000}\left\{2783(f_{-3}+f_3)+38592(f_{-2}+f_2)+14625(f_{-1}+f_1)-112000f_0\right\}\right.$$
$$\left.+\frac{h}{172800}\left\{19(f'_{-3}-f'_3)+576(f'_{-2}-f'_2)+2025(f'_{-1}-f'_1)\right\}\right)\Big/hf'_0.$$

For every $n$, the formula for $p$ is the following:

$$p=r-r^2s+r^3(2s^2-t)+r^4(-5s^3+5st-u)+r^5(14s^4-21s^2t+3t^2+6su-v)+r^6(-42s^5+84s^3t$$
$$-28st^2-28s^2u+7tu+7sv-w)+r^7(132s^6-330s^4t+180s^2t^2+120s^3u-12t^3-72stu-36s^2v+4u^2$$
$$+8tv+8sw-x)+r^8(-429s^7+1287s^5t-990s^3t^2-495s^4u+495s^2tu+165st^3+165s^3v-45t^2u$$
$$-45su^2-90stv-45s^2w+9uv+9tw+9sx-y)+r^9(1430s^8-5005s^6t+5005s^4t^2+2002s^5u$$
$$-1430s^2t^3-2860s^3tu-715s^4v+55t^4+660st^2u+330s^2u^2+660s^2tv+220s^3w-55tu^2-55t^2v$$
$$-110suv-110stw-55s^2x+5v^2+10uw+10tx+10sy-z)+r^{10}(-4862s^9+19448s^7t-24024s^5t^2$$
$$-8008s^6u+10010s^3t^3+15015s^4tu+3003s^5v-1001st^4-6006s^2t^2u-2002s^3u^2-4004s^3tv$$
$$-1001s^4w+286t^3u+858stu^2+858s^2tw+858st^2v+858s^2uv+286s^3x-22u^3-132tuv-66t^2w$$
$$-66sv^2-132suw-132stx-66s^2y+11vw+11ux+11ty+11sz-\overline{z})+\cdot\cdot\cdot.$$


[1] H. E. Salzer, A new formula for inverse interpolation, Bul. Am. Math. Soc. **50,** 513 (1944).

[2] H. E. Salzer, Inverse interpolation for eight-, nine-, ten-, and eleven-point direct interpolation, J. Math. Phys. **24,** 106 (1945).

[3] T. Fort, Finite differences, p. 85 (Clarendon Press, Oxford, England, 1948).

[4] J. F. Steffensen, Interpolation, p. 33 (Williams & Wilkins, Baltimore, Md., 1927).

[5] H. E. Salzer, New formulas for facilitating osculatory interpolation, J. Research NBS **52,** 211 (1954) RP2491.

[6] National Bureau of Standards, Table of the Bessel functions $J_0(z)$ and $J_1(z)$ for complex arguments (Columbia University Press, New York, N. Y., 1943; 2d ed. 1947).

[7] Harvard Computation Laboratory, The annals of the computation laboratory of Harvard University **III,** Tables of the Bessel functions of the first kind of orders zero and one (Harvard University Press, Cambridge, Mass., 1947).

[8] National Bureau of Standards, Table of the Bessel functions $Y_0(z)$ and $Y_1(z)$ for complex arguments (Columbia University Press, New York, N. Y., 1950).

[9] Tables of normal probability functions, NBS AMS23 (1953); Tables of the error function and its derivative, NBS AMS41 (1954).

[10] Tables of sine, cosine, and exponential integrals, NBS MT5; NBS MT6 (1940).

[11] Table of sine and cosine integrals for arguments from 10 to 100, NBS AMS32 (1954).


WASHINGTON, September 27, 1955.



U. S. GOVERNMENT PRINTING OFFICE: 1956