Stress-Strain Relationships in Yarns Subjected to Rapid Impact Loading: 2. Breaking Velocities, Strain Energies, and Theory Neglecting Wave Propagation[1]

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The behavior of a yarn specimen fastened at one end to a head mass and at the other end to a small tail mass is analyzed for longitudinal impact of the specimen at the head. The analysis leads to a basic formula for "limiting breaking velocity," which is a characteristic property of the material and independent of the dimensions of the specimen. A simple procedure is described for obtaining its value. The values for cotton, nylon, and undrawn nylon yarns tested at room temperature are found to be 130, 228, and 550 meters per second, respectively. The practical application of the limiting breaking velocity to safety engineering is shown by an example on safety lines. Formulas are derived for computing energy to any strain under impact loading, and the computed values are found to agree with those obtained from the area under the impact load-extension curves.

1. Introduction

The first paper of this series [1] [2] describes equipment and procedures for studying the stress-strain behavior of a yarn specimen subjected to tensional impact. In a typical test, a 65-cm specimen was attached at one end to a head mass and at the other end to a tail mass. The head was impacted at velocities between 10 and 100 m/sec, and subsequent behavior was recorded by high-speed photography.

The behavior of the yarn specimen after longitudinal impact is treated theoretically in this paper, neglecting wave-propagation effects. The results are compared with those obtained experimentally for several materials. The practical application of this work to safety engineering is indicated.

2. Theory

Figure 1 represents a test specimen terminated by a head and a tail. The equations of motion of these masses are derived in terms of the following symbols:

$L=$ Length of specimen before impact.
$w=$ Mass of specimen.
$nw=$ Mass of tail.
$mw=$ Mass of head.
$x_{10}=$ Position of tail before impact, relative to a fixed origin, 0.
$x_{20}=$ Position of head before impact, relative to a fixed origin, O.
$t=$ Time after impact.
$x_1=$ Position of tail at time, t, relative to a fixed origin, O.
$x_2=$ Position of head at time, t, relative to a fixed origin, O.
$x_c=$ Position of center of mass of the system at time, t, relative to a fixed origin, O.
$\epsilon=$ Strain in specimen.
$F=$ Force acting on specimen at time, t.
$v_0=$ Velocity of head immediately after impact.

It is assumed here that $\frac{1}{2}w$ is concentrated at the head and the remaining $\frac{1}{2}w$ is concentrated at the tail of the specimen. Thus the problem ac-
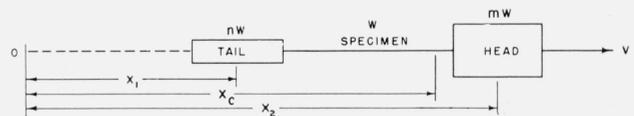

FIGURE 1. *Schematic drawing of test specimen for impact loading.*

tually treated is that of a mass-less specimen to which a head mass $(m+\frac{1}{2})w$ and a tail mass $(n+\frac{1}{2})w$ are attached.

The force equations, neglecting air friction,[3] are

$$F=(n+\tfrac{1}{2})w\frac{d^2x_1}{dt^2}, \qquad (1)$$

and

$$F=-(m+\tfrac{1}{2})w\frac{d^2x_2}{dt^2}. \qquad (2)$$

From definition of the center of mass, we obtain $(n+\frac{1}{2})wx_1+(m+\frac{1}{2})wx_2=(n+m+1)wx_c$.

Differentiating and eliminating w gives

$$(n+\tfrac{1}{2})\frac{dx_1}{dt}+(m+\tfrac{1}{2})\frac{dx_2}{dt}=(n+m+1)\frac{dx_c}{dt}. \qquad (3)$$

Because $\epsilon L=(x_2-x_1)-(x_{20}-x_{10})$, we obtain by differentiation

$$\frac{dx_1}{dt}-\frac{dx_2}{dt}=-L\frac{d\epsilon}{dt}. \qquad (4)$$

Solving eq (3) and (4) for dx_1/dt and dx_2/dt gives

$$\frac{dx_1}{dt}=\frac{dx_c}{dt}-\frac{(m+\tfrac{1}{2})L}{n+m+1}\frac{d\epsilon}{dt}, \qquad (5)$$

and

$$\frac{dx_2}{dt}=\frac{dx_c}{dt}+\frac{(n+\tfrac{1}{2})L}{n+m+1}\frac{d\epsilon}{dt}. \qquad (6)$$

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[2] Figures in brackets indicate the literature references at the end of this paper.

[3] The force on the tail due to air friction was calculated to be approximately 1 percent of the yarn tension.

At maximum strain $\epsilon=\epsilon_m$ and $d\epsilon/dt=0$, therefore, $dx_1/dt=dx_2/dt=dx_c/dt$. That is, for a typical loading and unloading test under impact conditions, the velocity of the tail mass at the instant of maximum strain is equal to the velocity of the head, and both are equal to the velocity of the center of the total mass. As the total momentum imparted by the hammer to the head immediately after impact is $(m+\frac{1}{2})wv_0$, it follows from conservation of momentum that $dx_c/dt=(m+\frac{1}{2})v_0/(n+m+1)=$ constant, which is also the velocity of the head and of the tail at maximum strain. The velocity of the head, therefore, decreases from v_0 immediately after impact to $(m+\frac{1}{2})v_0/(n+m+1)$ at the time of maximum strain. It is seen that this decrease is equal to $(n+\frac{1}{2})v_0/(n+m+1)$ and thus depends upon the values of n and m.

From eq (5) and (1), or (6) and (2), it follows that

$$\frac{d^2\epsilon}{dt^2}=-\frac{n+m+1}{(n+\frac{1}{2})(m+\frac{1}{2})}\frac{F}{Lw}.$$

Integration gives

$$\left(\frac{d\epsilon}{dt}\right)^2=-\frac{2}{Lw}\frac{n+m+1}{(n+\frac{1}{2})(m+\frac{1}{2})}\int_0^\epsilon Fd\epsilon+C.$$

At $t=0$, $\epsilon=0$, and $d\epsilon/dt=v_0/L$. Therefore, $C=v_0^2/L^2$ and

$$\left(\frac{d\epsilon}{dt}\right)^2=\frac{v_0^2}{L^2}-\frac{2}{Lw}\frac{n+m+1}{(n+\frac{1}{2})(m+\frac{1}{2})}\int_0^\epsilon Fd\epsilon. \qquad (10)$$

At the maximum strain $\epsilon=\epsilon_m$ and $d\epsilon/dt=0$. It follows, therefore, that

$$\log v_0=-\tfrac{1}{2}\log\frac{(n+\frac{1}{2})(m+\frac{1}{2})}{n+m+1}+\log\sqrt{\frac{2L}{w}\int_0^{\epsilon_m}Fd\epsilon}. \qquad (11)$$

In an impact test the specimen will break if the impact velocity, v_0, is large enough that the rupture strain, ϵ_r, is attained. The minimum breaking velocity, v_n, for a given n is the velocity just sufficient to cause the specimen to break when $\epsilon_r=\epsilon_m$. Under these conditions we have

$$\log v_n=-\tfrac{1}{2}\log\frac{(n+\frac{1}{2})(m+\frac{1}{2})}{n+m+1}+\log\sqrt{\frac{2L}{w}\int_0^{\epsilon_r}Fd\epsilon}. \qquad (12)$$

Equation (12) expresses the minimum breaking velocity as a function of n. If ϵ_r and the shape of the stress-strain curve are both independent of the rate of straining, the expression under the radical sign is a constant having the dimensions of velocity squared. At high rates of straining the value of ϵ_r is less than that obtained at conventional testing rates. Also, the slope of the stress-strain curve is greater at higher rates of straining. However, for rates of straining of the same order of magnitude, these changes are small, and their effects compensate each other. Consequently, the area under the stress-strain curve should be essentially constant for the rates of straining considered here. Under these conditions a plot of $\log v_n$ versus $\log(n+\frac{1}{2})(m+\frac{1}{2})/(n+m+1)$ would be a straight line of slope $-\frac{1}{2}$.

By extrapolating from impact test data to the point at which $\log(n+\frac{1}{2})(m+\frac{1}{2})/(n+m+1)=0$, a characteristic velocity, v_b, is obtained; namely,

$$v_b=\sqrt{\frac{2L}{w}\int_0^{\epsilon_r}Fd\epsilon}. \qquad (13)$$

If we let $\rho=$density of the specimen and $\sigma=$stress, eq (13) reduces to the form

$$v_b=\sqrt{\frac{2}{\rho}\int_0^{\epsilon_r}\sigma d\epsilon}. \qquad (14)$$

It is thus seen that v_b is a quantity characteristic of the material itself, except for a possible dependence on the rate of straining.

Equation (14) can be rearranged in the form

$$\tfrac{1}{2}\rho v_b^2=\int_0^{\epsilon_r}\sigma d\epsilon, \qquad (15)$$

which states that the kinetic-energy density in the specimen when traveling at velocity v_b is just equal to the strain-energy density required to break the specimen. We thus see that if the specimen is impacted at a velocity greater then v_b, it will always be broken. For this reason, we call v_b the limiting breaking velocity.

Von Kármán [2] derived a critical velocity, v_c, at which a filament will break immediately upon impact

$$v_c=\int_0^{\epsilon_r}\sqrt{\frac{1}{\rho}\frac{d\sigma}{d\epsilon}}\,d\epsilon. \qquad (16)$$

This equation was obtained from a consideration of plastic and elastic wave propagation in a material for which the stress-strain curve is concave downward, and the stress is independent of the rate of straining.

The results calculated from eq (14) and (16) differ slightly. However, in the special case when Hooke's law is obeyed both eq (14) and (16) reduce to

$$v'=\epsilon_r\sqrt{\frac{E}{\rho}}. \qquad (17)$$

This formula may be expressed in terms of the more familiar textile quantities of tenacity and percent elongation at rupture, namely,

$$v'=29.7\sqrt{\text{tenacity}\times\text{elongation}}, \qquad (18)$$

where v' is in meters per second, tenacity is in grams per denier, and elongation is in percent.

Meredith [3] has obtained evidence that at high rates of straining the load-extension curves of some materials become more Hookean. These formulas thus assume considerable importance and by means of eq (18), estimates of the critical velocities of yarns may be computed from their tenacities and elongations at rupture under impact testing conditions.
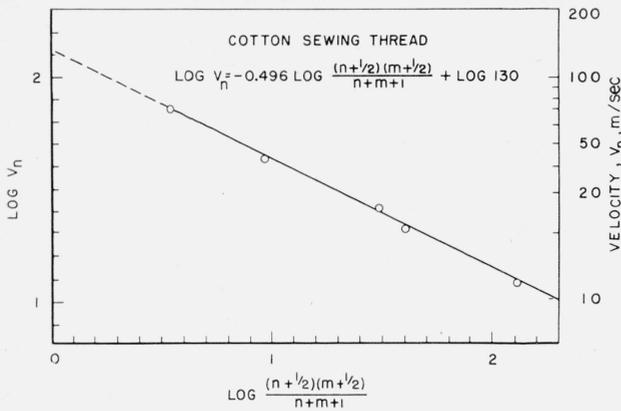
FIGURE 2. *Relationship between impact velocity to rupture, v_n, and head-tail mass parameter, $(n+\frac{1}{2})(m+\frac{1}{2})/(n+m+1)$, for cotton sewing thread.*

In a subsequent paper [4] the behavior of the specimen after longitudinal impact is treated by a theory that considers the effects of wave propagation. This treatment shows that the results given in this paper for nonwave theory are valid when the tail mass is greater than $40w$. For tail masses less than $40w$, the effects of wave propagation may become appreciable and must be considered in the theoretical treatment.

3. Experimental Results

Experimental values of v_n corresponding to different values of n were obtained on a number of different materials. These values of $\log v_n$ are plotted in figures 2, 3, and 4 against $\log(n+\frac{1}{2})(m+\frac{1}{2})/(n+m+1)$. These plots are straight lines having the empirical equations

$$\log v_n = -0.496 \log \frac{(n+\frac{1}{2})(m+\frac{1}{2})}{n+m+1} + \log 130$$

for cotton thread

$$\log v_n = -0.495 \log \frac{(n+\frac{1}{2})(m+\frac{1}{2})}{n+m+1} + \log 228$$

for nylon yarn and threads

$$\log v_n = -0.500 \log \frac{(n+\frac{1}{2})(m+\frac{1}{2})}{n+m+1} + \log 550$$

for undrawn nylon.

It should be noted that the slope of each line represented by these equations is very close to $-\frac{1}{2}$ in accordance with eq (12) for $\log v_n$.

4. Design of Safety Lines

The empirical equation derived for $\log v_n$ has many practical applications. Consider, for example, a safety line of length, L, that connects a construction worker to a rigid point of the structure. Under these conditions, m is infinite. The maximum free
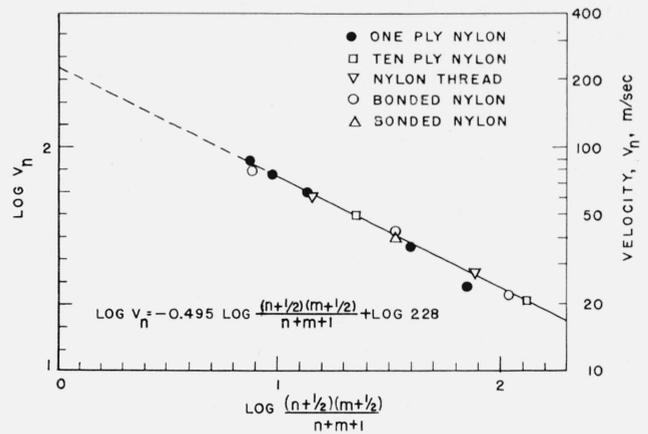

FIGURE 3. *Relationship between impact velocity to rupture, v_n, and head-tail mass parameter, $(n+\frac{1}{2})(m+\frac{1}{2})/(n+m+1)$, for nylon yarns and threads.*
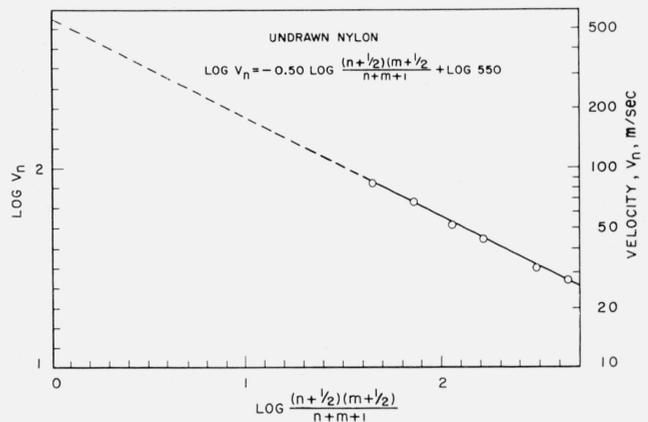

FIGURE 4. *Relationship between impact velocity to rupture, v_n, and head-tail mass parameter, $(n+\frac{1}{2})(m+\frac{1}{2})/(n+m+1)$, for undrawn nylon.*

fall of the worker would be $2L$, and the maximum free-fall velocity $2\sqrt{gL}$. As m becomes infinite, eq (12) reduces to

$$\log v_n = -\frac{1}{2}\log(n+\frac{1}{2}) + \log v_b. \qquad (19)$$

Making the assumption (which must be checked by experiment) that the safety line behaves under impact like the single yarn considered in the derivation of eq (19), substituting $v_n = 2\sqrt{gL}$ and solving for n, one obtains

$$n = \frac{v_b^2 - 2gL}{4gL}.$$

If v_b is 200 m/sec for the material of the safety line, $L=10$m, and $g=9.8$ m/sec^2, the computed value of n is 101. If the weight of the worker, nw, is 202 lb, the minimum weight of 10 m of safety line, w, to just support the worker at impact is 2 lb, according to this computation.

This weight is approximately that of a rope $\frac{1}{2}$ in. in diameter [5]. A nylon rope $\frac{7}{16}$ in. in diameter has a static breaking strength of 4,000 lb [6], or 20 times

the weight of the worker. However, under the dynamic conditions just assumed, a nylon rope with $v_b=200$ m/sec would have no factor of safety whatever according to this computation.

If v_b for the material of the safety line is 100, $n=25$, and for the same weight of worker, w would have to be 8 lb. This weight is approximately that of a rope about 1 in. in diameter [5]. A manila rope 1 in. in diameter has a static breaking strength of 11,000 lb, or 50 times the estimated weight of the worker. Again, under dynamic conditions, this manila rope would not provide a reasonable factor of safety according to this computation. Results obtained by Newman [6] showed that a 9/16-in.-diam sisal rope 10 ft long broke when a weight of 142 lb was dropped a height of only 20 ft.

5. Energy of Straining

The energy per unit length, Q, to stress the specimen to a strain ϵ is given from eq (10) by the equation

$$Q=\int_0^\epsilon Fd\epsilon=\frac{Lw}{2}\frac{(n+\frac{1}{2})(m+\frac{1}{2})}{n+m+1}\left[\left(\frac{v_0}{L}\right)^2-\left(\frac{d\epsilon}{dt}\right)^2\right], \quad (20)$$

where $d\epsilon/dt$ is the slope of the strain-time curve at the strain ϵ. If the specimen is not ruptured but is loaded to a maximum strain, ϵ_m, and then unloaded, the energy, Q_m, to the maximum strain, ϵ_m, is given by the equation

$$Q_m=\int_0^{\epsilon_m} Fd\epsilon=\frac{Lw}{2}\frac{(n+\frac{1}{2})(m+\frac{1}{2})}{n+m+1}\left(\frac{v_0}{L}\right)^2, \quad (21)$$

since $(d\epsilon/dt)_{\epsilon=\epsilon_m}$ is zero. If the specimen is ruptured, the energy, Q_r, to the rupture strain, ϵ_r, is given by the equation

$$Q_r=\int_0^{\epsilon_r} Fd\epsilon=\frac{Lw}{2}\frac{(n+\frac{1}{2})(m+\frac{1}{2})}{n+m+1}\left[\left(\frac{v_0}{L}\right)^2-\left(\frac{d\epsilon}{dt}\right)^2_{\epsilon=\epsilon_r}\right] \quad (22)$$

The value of $(d\epsilon/dt)_{\epsilon=\epsilon_r}$ will be zero, if the impact velocity of the head is just sufficient to produce rupture. Above this velocity $(d\epsilon/dt)_{\epsilon=\epsilon_r}$ is a positive quantity.

By means of eq (20), (21), and (22), it is possible to compute Q, Q_m, and Q_r from the conditions of a test. For Q_m it is only necessary to know the velocity, v_0, of the head at impact. For Q and Q_r, the value of $d\epsilon/dt$ and $(d\epsilon/dt)_{\epsilon=\epsilon_r}$ must be known in addition to the value of v_0. The values of v_0, $d\epsilon/dt$, and $(d\epsilon/dt)_{\epsilon=\epsilon_r}$ are readily obtained from the positions of the head and of the tail on the high-speed motion pictures taken during the test. The energy values computed by means of these equations for different materials and impact testing conditions are plotted in figure 5 against those derived from the area of the impact-load–extension curves. The plotted points fall near a straight line having a slope equal to 1.
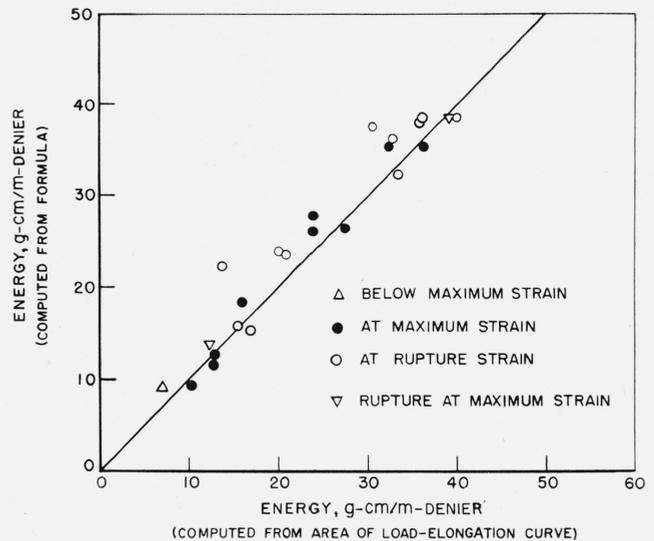

FIGURE 5. *Energy values computed from formulas plotted against those based upon the area under the impact load-extension curves.*

6. Summary and Conclusions

The theoretical treatment given in this paper for longitudinal impact of a yarn leads to a formula for the limiting breaking velocity that is a characteristic property of the material and is independent of the dimensions of the specimen. A specimen impacted at a velocity greater than its limiting breaking velocity will always be broken. A simple procedure is described for obtaining this limiting breaking velocity from impact-test data, and results are given for cotton, nylon, and undrawn nylon yarns. The potential importance of the limiting breaking velocity to safety engineering is indicated by its application to estimates of margin of safety of rupture of safety lines for construction workers. The theoretical treatment also leads to formulas for obtaining energy to any given strain, including rupture strain, under different impact conditions. These values agree reasonably well with those obtained from the area under the impact-load-extension curves.

7. References

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