Osculatory Interpolation in the Complex Plane

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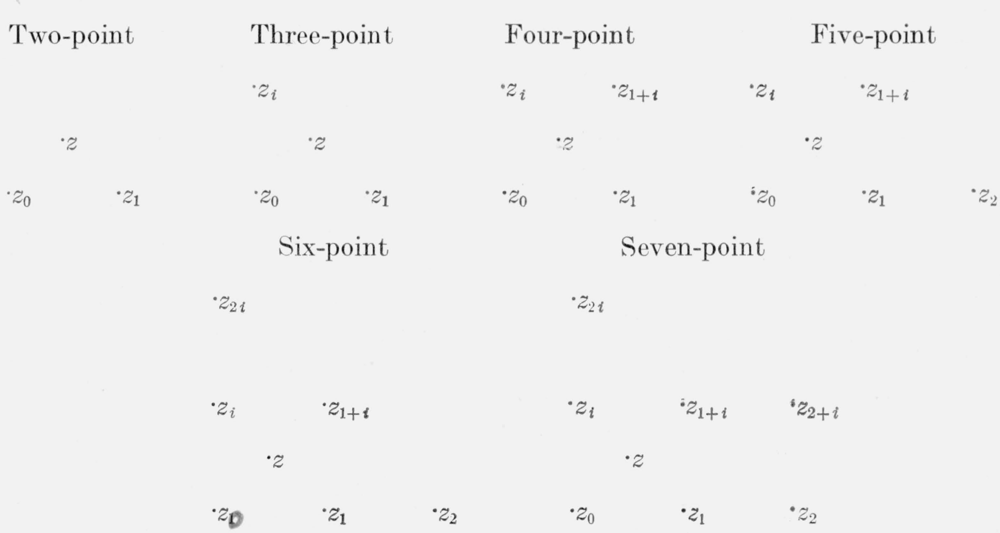
Tables of coefficients to facilitate osculatory n-point interpolation ($n=2(1)7$) in the complex plane are given.

The writer has described in a previous article [1] [2] a new method for osculatory interpolation that can be applied also in the complex plane when one has an analytic function $f(z)$, $z=x+iy$, which is known for arguments z_k at equal intervals along any straight line in the z-plane. However, for the tabular arguments $z_k=x_k+iy_k$ in the form of a Cartesian grid of length h, greater accuracy may be had by basing the Hermite osculatory interpolation formula upon values of $f(z_k)$ and $f'(z_k)$ at points z_k, which are closer together and not necessarily lying upon a straight line (except, of course, in the 2-point case). For a detailed discussion of the Hermite osculatory interpolation formula, the reader is referred to the previous article [1]. This present paper merely aims to supplement that article which was intended primarily for real functions, by giving here the corresponding auxiliary quantities, a_k and b_k, which are suited better for complex interpolation. Whereas the quantities, a_k and b_k, for real interpolation were tabulated up to the 11-point case (21st-degree accuracy), this present tabulation for complex interpolation does not go beyond the 7-point case (13th-degree accuracy), which is more than adequate for most of the practical problems that would arise.

To interpolate for $f(z)$, where $z=z_0+Ph$ and $P\equiv p+iq$ is now complex, we choose the $z_k=z_0+kh$ for k equal to certain small complex integers and such that z_0 will always be the lower left corner point of the configuration of points, z_0 and z_k. Also, in almost every example, $|p|\leq 1$, $|q|\leq 1$, so that the point z is usually not outside the square whose corners are z_0, z_1, z_i, and z_{1+i}. For each of the n-point formulas, $n=2(1)7$, the points z_k are chosen to lie in the following configurations:



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[2] Figures in brackets indicate the literature references at the end of this paper.

Hermite's n-point osculatory interpolation formula for $f(z) \equiv f$, in terms of $f(z_k) \equiv f_k$ and $f'(z_k) \equiv f'_k$, is expressible in the concise form

$$f \sim \sum_k (\alpha_k f_k + h\beta_k f'_k) / \sum_k \alpha_k, \tag{1}$$

where

$$\alpha_k \equiv a_k/(P-k)^2 + b_k/(P-k) \tag{2}$$

and

$$\beta_k \equiv a_k/(P-k), \tag{3}$$

the summation being taken for k corresponding to the n-points z_k of the configuration. The dependence of α_k, β_k, a_k, and b_k upon n is understood. The auxiliary quantities, a_k and b_k, are tabulated below for $n=2(1)7$ for each of the above n-point configurations of points z_k. The actual definitions of a_k and b_k are as follows:

$$a_k = C(n) \Big/ \left\{ \prod_j{}' (k-j) \right\}^2, \tag{4}$$

and

$$b_k = -2C(n) \left\{ \frac{d}{dP} \prod_j{}' (P-j) \right\}_{P=k} \Big/ \left\{ \prod_j{}' (k-j) \right\}^3, \tag{5}$$

where the product is taken for j corresponding to the $n-1$ points $z_j \neq z_k$ of the configuration, and where for each n, $C(n)$ is chosen as the least integer which allows the quantities, a_k and b_k, to appear as complex integers.

Table of a_i and b_i

Two-point

a_0	1	b_0	2
a_1	1	b_1	-2

Three-point

a_0	-2	b_0	$-4+4i$
a_1	i	b_1	$1-3i$
a_i	$-i$	b_i	$3-i$

Four-point

a_0	i	b_0	$3+3i$
a_1	$-i$	b_1	$-3+3i$
a_i	$-i$	b_i	$3-3i$
a_{1+i}	i	b_{1+i}	$-3-3i$

Five-point

a_0	$125i$	b_0	$375+500i$
a_1	$-500i$	b_1	$-1500+500i$
a_2	$-20+15i$	b_2	$117-44i$
a_i	$80-60i$	b_i	$508+44i$
a_{1+i}	-250	b_{1+i}	$500-1000i$

Six-point

a_0	$-500i$	b_0	$-2000-2000i$
a_1	$1280+960i$	b_1	$1856-6208i$
a_2	$-30-40i$	b_2	$83+269i$
a_i	$-1280+960i$	b_i	$-6208+1856i$
a_{1+i}	$-2000i$	b_{1+i}	$6000+6000i$
a_{2i}	$30-40i$	b_{2i}	$269+83i$

Seven-point

a_0	$-80-60i$	b_0	$-648+64i$
a_1	$480-640i$	b_1	$-3264-2048i$
a_2	$30+40i$	b_2	$-3-329i$
a_i	$-320+240i$	b_i	$-1872+704i$
a_{1+i}	$-2000i$	b_{1+i}	$6000+2000i$
a_{2+i}	$80i$	b_{2+i}	$-240-448i$
a_{2i}	10	b_{2i}	$27+57i$

Formulas (1), (2), and (3) for complex osculatory interpolation, which utilize the Cartesian grid, will be especially useful in connection with such tables for complex arguments as (1) logarithm of the gamma function, log $\Gamma(z)$, together with its derivative the psi function, $\Psi(z)$,

(2) Bessel functions of the first or second kind giving $J_0(z)$ and $J_1(z)=-J_0'(z)$, $Y_0(z)$ and $Y_1(z)=-Y_0'(z)$, or linear combinations of them, (3) probability integral $\int_0^z e^{-u^2}\,du$ with its integrand, (4) miscellaneous tables of integrals of the more elementary functions where the first derivative or the integrand, although not tabulated, is easy to calculate, namely, the function $\int_z^\infty (e^{-u}/u)du$ and (5) tables of solutions of important linear differential equations, together with their first derivatives [2]. In all such tables, and in many others, the user will find these complex osculatory interpolation formulas to be particularly convenient. They are especially suitable in those cases where the grid length h is too large for sufficiently accurate complex interpolation, using either tables of complex interpolation coefficients [3, 4] or formulas corresponding to (1), (2), and (3) above [5], where the tables or formulas involve the use of only the functional values f_i.

Illustration.—To demonstrate the use of these tables in formulas (1) to (3), consider an example from [2], where the modified Hankel functions $h_j(z)\equiv(\frac{2}{3}z^{\frac{3}{2}})^{\frac{1}{3}}H_{\frac{1}{3}}^{(j)}(\frac{2}{3}z^{\frac{3}{2}})$, $j=1, 2$, and their first derivatives are tabulated over a Cartesian grid of length $h=0.1$ in the complex plane. Suppose that it is required to find h_1 $(1.24579\ 316+0.96155\ 803i)$, using the four-point osculatory interpolation formula and the tabulated values of h_1, h_1' at $z_0=1.2+0.9i$, $z_1=1.3+0.9i$, $z_i=1.2+1.0i$, $z_{1+i}=1.3+1.0i$ [2, pp. 21, 23]. Employing (2), (3) and a_i, b_i from these tables, since $P=0.45793\ 16+0.61558\ 03i$, one finds that

$$\begin{aligned}
\alpha_0&=i/(0.45793\ 16+0.61558\ 03i)^2+(3+3i)/(0.45793\ 16+0.61558\ 03i),\\
\alpha_1&=-i/(-0.54206\ 84+0.61558\ 03i)^2+(-3+3i)/(-0.54206\ 84+0.61558\ 03i),\\
\alpha_i&=-i/(0.45793\ 16-0.38441\ 97i)^2+(3-3i)/(0.45793\ 16-0.38441\ 97i),\\
\alpha_{1+i}&=i/(-0.54206\ 84-0.38441\ 97i)^2+(-3-3i)/(-0.54206\ 84-0.38441\ 97i),\\
\beta_0&=i/(0.45793\ 16+0.61558\ 03i),\ \beta_1=-i/(-0.54206\ 84+0.61558\ 03i),\\
\beta_i&=-i/(0.45793\ 16-0.38441\ 97i),\ \beta_{1+i}=i/(-0.54206\ 84-0.38441\ 97i),
\end{aligned}$$

from which one obtains.

$$\begin{aligned}
\alpha_0&=7.09824\ 5538-1.29187\ 9023i, & \beta_0&=1.04576\ 6212+0.77794\ 7889i,\\
\alpha_1&=6.63654\ 0823+0.51581\ 3695i, & \beta_1&=-0.91498\ 3814+0.80571\ 7487i,\\
\alpha_i&=9.82415\ 3465-1.10147\ 8051i, & \beta_i&=1.07536\ 0453-1.28099\ 9733i,\\
\alpha_{1+i}&=8.43081\ 1380+1.81986\ 9385i, & \beta_{1+i}&=-0.87048\ 2793-1.22746\ 3667i,
\end{aligned}$$

Then for (1) we employ the following tabulated values of h_1 and h_1':

$$\begin{aligned}
f_0&=0.19018\ 585-0.19313\ 840i, & f_0'&=0.14952\ 614+0.33229\ 461i,\\
f_1&=0.20311\ 754-0.15966\ 418i, & f_1'&=0.10901\ 235+0.33659\ 231i,\\
f_i&=0.15891\ 089-0.17847\ 850i, & f_i'&=0.14376\ 041+0.29375\ 232i,\\
f_{1+i}&=0.17143\ 575-0.14888\ 108i, & f_{1+i}'&=0.10669\ 347+0.29764\ 346i,
\end{aligned}$$

and also

$$\sum_k \alpha_k=31.98975\ 121-0.05767\ 399i,$$

so that we have

$$\begin{aligned}
h_1(1.24579\ 316+0.96155\ 803i)\equiv f\sim[\alpha_0 f_0+\alpha_1 f_1+\alpha_i f_i\\
+\alpha_{1+i}f_{1+i}+0.1\{\beta_0 f_0'+\beta_1 f_1'+\beta_i f_i'+\beta_{1+i}f_{1+i}'\}]/\sum_k \alpha_k,
\end{aligned}$$

whose terms are

$$
\begin{aligned}
&[(7.09824\ 5538-1.29187\ 9023i)(0.19018\ 585-0.19313\ 840i)\\
&\quad+(6.63654\ 0823+0.51581\ 3695i)(0.20311\ 754-0.15966\ 418i)\\
&\quad+(9.82415\ 3465-1.10147\ 8051i)(0.15891\ 089-0.17847\ 850i)\\
&\quad+(8.43081\ 1380+1.81986\ 9385i)(0.17143\ 575-0.14888\ 108i)\\
&\quad+0.1\{(1.04576\ 6212+0.77794\ 7889i)(0.14952\ 614+0.33229\ 461i)\\
&\quad+(-0.91498\ 3814+0.80571\ 7487i)(0.10901\ 235+0.33659\ 231i)\\
&\quad+(1.07536\ 0453-1.28099\ 9733i)(0.14376\ 041+0.29375\ 232i)\\
&\quad+(-0.87048\ 2793-1.22746\ 3667i)(0.10669\ 347+0.29764\ 346i)\}]\\
&\quad\div(31.98975\ 121-0.05767\ 399i),
\end{aligned}
$$

or after multiplying,

$$
\begin{aligned}
&[(1.10047\ 4414-1.61664\ 0896i)+(1.43035\ 4817-0.95484\ 7040i)\\
&\quad+(1.36457\ 4820-1.92843\ 7032i)+(1.71628\ 6592-0.94319\ 7631i)\\
&\quad+0.1\{(-0.10213\ 8505+0.46382\ 6021i)+(-0.37094\ 2846-0.22014\ 3359i)\\
&\quad+(0.53089\ 0903+0.13173\ 2581i)+(0.27247\ 1703-0.39005\ 5868i)\}]\\
&\quad\div(31.98975\ 121-0.05767\ 399i)\\
&=(5.64471\ 8767-5.44458\ 6662i)/(31.98975\ 121-0.05767\ 399i),
\end{aligned}
$$

or finally,

$$0.17676\ 025-0.16987\ 916i,$$

which is correct to within a unit in the last place, as was seen from the independent calculation of $h_1(1.24579\ 316+0.96155\ 803i)$ from the Taylor series around $z_0=1.2+0.9i$.

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[1] H. E. Salzer, New formulas for facilitating osculatory interpolation, J. Research NBS **52**, 211 (1954) RP2491.

[2] Harvard Computation Laboratory, Tables of the modified Hankel functions of order one-third and of their derivatives (Harvard University Press, Cambridge, Mass., 1945).

[3] A. N. Lowan and H. E. Salzer, Coefficients for interpolation within a square grid in the complex plane. J. Math. Phys. **23**, 156 (1944).

[4] H. E. Salzer, Coefficients for complex quartic, quintic, and sextic interpolation within a square grid, J. Math. Phys. **27**, 136 (1948).

[5] H. E. Salzer, Formulas for complex Cartesian interpolation of higher degree, J. Math. Phys. **28**, 200 (1949).

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