An Algorithm for Solving the Transportation Problem [1]

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This article describes a new computational scheme for solving the transportation problem (described below) in which combinatorial ideas, rather than the theory of linear inequalities, play the major role. Whether the algorithm is superior to other methods previously employed or proposed must await computational experience, but preliminary inspection is encouraging.

1. Introduction

The transportation problem (stated for integers) is the following: Let C be an $m \times n$ matrix of integers c_{ij}, and let $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ be positive integers such that $a_1 + \ldots + a_m = b_1 + \ldots + b_n$. We are required to find an $m \times n$ matrix $X = (x_{ij})$ such that [3]

(a) x_{ij} is a nonnegative integer,

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad (i=1, \ldots, m),$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad (j=1, \ldots, n),$$

and the linear form $l_c(x) = \sum_{i,j} c_{ij} x_{ij}$ is a minimum for all matrices satisfying (a) and (b).

Definitions: A matrix X satisfying (a) and (b) is called a *selection*. If X and Y are selections, we say $X(C) \preceq Y(C)$ (read "X is preferable to Y with respect to C") if $l_c(x) \leqq l_c(y)$. If $l_c(x) < l_c(y)$, we say $X(C) \prec Y(C)$ ("X is strongly preferable to Y with respect to C"). It is clear that the relation $\preceq$ is transitive. If X is preferable to all selections with respect to C (i. e., X solves the problem), it is said to be *optimal*.

If X is a selection, and i, j are indices such that $x_{ij} \geqq 1$, then c_{ij} is called an *X-selected element* of C. If $r_1, \ldots, r_m$, $s_1, \ldots, s_n$ are integers and $D = (d_{ij}) = (c_{ij} + r_i + s_j)$, then $C \sim D$ ("C and D are equivalent"). It is easy to see that $\sim$ is an equivalence relation. Further, one may prove that if $C \sim D$, then $X(C) \preceq Y(C)$ if and only if $X(D) \preceq Y(D)$, and $X(C) \quad Y(C)$ if, and only if, $X(D) \prec Y(D)$. Since throughout the computation subsequently described we work only with matrices equivalent to C, we shall henceforth write $X(C) \preceq Y(C)$ as $X \preceq Y$ (similarly, $X \prec Y$).

A *closed circuit* is a set of selected elements of the form $c_{i_1 j_1}, c_{i_1 j_2}, c_{i_2 j_2}, c_{i_2 j_3}, \ldots, c_{i_t j_t}, c_{i_t j_1}$, where $i_1, \ldots, i_t$ are distinct indices and $j_1, \ldots, j_t$ are distinct indices.

2. Summary of the Computation

The computation consists of two parts, which are performed alternately until an optimal selection is obtained. It will be seen that the algorithm itself yields the information that the optimal selection has been reached.

As a preliminary, we begin with some selection X. In part I, we find a matrix $D \sim C$ and a selection $Y \preceq X$ such that the Y-selected elements of D are 0. In part II, we discover if Y is optimal, and if not, obtain a selection $Z \prec Y$. Putting Z in place of X, we begin again with part I and continue. As there are only a finite number of selections, it is clear from the fact that $Z \prec Y \preceq X$, that this process terminates in a finite number of steps, and we thus arrive at an optimal selection.

3. Summary of Part I

With the pair of matrices C, X, we associate a pair of nonnegative integers (c_x, x) defined as follows: c_x is the number of nonzero X-selected elements of C, x is the number of nonzero elements of X.

If $c_x = 0$, we are finished with part I (set $D = C$, $Y = X$). If not, we construct, starting with a nonzero X-selected element of C a "tree I", which we discover to be either in case IA or case IB. If in case IA, we find a matrix C' such that $C' \sim C$ and the number of nonzero X-selected elements of C' is less than c_x. If in case IB, we find a selection X' preferable to X such that the set of nonzero elements of X' is a *proper* subset of the set of nonzero elements of X. Thus, after completing IA or IB, we have a matrix $C' \sim C$ and a selection $X' \preceq X$ such that

$$c'_{x'} \leqq c_x, \quad x' \leqq x, \tag{1}$$

[1] A code of this method in the case of the "personnel problem," using an 8×8 matrix, has recently been composed for SWAC (National Bureau of Standards, Western Automatic Computer). For a variant of the method in this case, which is also applicable to matrices consisting of nonintegers, see Theodore S. Motzkin, The assignment problem (Proceedings of the American Mathematical Society Sixth Symposium for Applied Mathematics, 1954).

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[3] Other references to this problem and references to further literature are given by: George B. Dantzig, Maximimization of a linear function of variables subject to linear inequalities (Simplex Method), chapter XXI, and Application of the Simplex Method to a transportation problem, chapter XXIII, both included in Activity analysis of production and allocation, edited by T. C. Koopmans (John Wiley & Sons, Inc., New York, N. Y., 1951); Merrill M. Flood, On the Hitchcock distribution problem, Pacific J. Math. **3**, 369–386 (1953).

and strict inequality holds in at least one of the two.

If $c'_{x'}>0$, we begin again, obtaining C'', X'', and so on. By virtue of (1), this cannot continue indefinitely. But we cannot have $x^{(n)}=0$, for this is inconsistent with (b); therefore, for some n, $c_{x^{(n)}}=0$. Hence, the desired result of part I is achieved.

4. Summary of Part II

In part II, we begin with a pair of matrices D, Y such that $d_y=0$. Let e_D be the minimum of all elements of D that are *not* Y-selected. Let f_D be the number of elements of D equal to e_D.

If $e_D\geqq 0$, then, clearly, Y is optimal. Assume the contrary, therefore. Starting with an element $d_{i_1j_1}$ of D such that $d_{i_1j_1}=e_D$, we construct a "tree II." We examine this tree and discover whether we are in case IIA or case IIB. If in case IIA, we construct a matrix $D'\sim D$ such that the Y-selected elements of D′ are 0, $e_{D'}>e_D$ (and the behavior of f_D is unspecified), or

$$e_{D'}=e_D, \qquad f_{D'}<f_D, \tag{2}$$

and we put D' in place of D and begin again with part II. It is obvious from (2) that we cannot continue to be in case IIA indefinitely, for eventually we would obtain $e_{D^{(n)}}\geqq 0$, which implies that Y is optimal. Therefore, if Y is not optimal, we must at some stage enter case IIB. If in case IIB, we find a selection $Z\prec Y$, and go back to part I, with any matrix equivalent to C (C itself, if convenient) replacing C, and Z replacing X.

5. Detail of Part I

Assume $c_{i_0j_0}$ is an X-selected element of C that is not zero. We define, inductively, the following "family tree I":

1. The "founder" is $c_{i_0j_0}$ ("first generation").

2. Assume now that we have defined the kth generation, for $k=1, \ldots, s$. We now define the $(s+1)$th generation; i. e., the union, over all members of the sth generation of the "children" of each member of the sth generation. The "children" of such a member are: if s is odd, all *other* X-selected elements in the same row of C as the given member; if s is even, all *other* X-selected elements in the same column of C as the given member. If the $(s+1)$th generation is vacuous, the tree terminates with the sth generation, and we say we are in case IA; if the $(s+1)$th generation contains at least one member of a previous generation (possibly the sth) the tree terminates with the $(s+1)$th generation, and we say we are in case IB. If the tree does not terminate, we continue. The tree must terminate, of course, after a finite number of generations.

Case IA: In this case, we shall show how to construct numbers $r_1, \ldots, r_m$, $s_1, \ldots, s_n$ such that $C'=(c'_{ij})=(c_{ij}+r_i+s_j)\sim C$, and $c'_x<c_x$ (indeed, all the elements of C' corresponding to the elements of the tree will be zero).

Associated with each element of the tree, define inductively the following numbers $N(t)$:

$$N(c_{i_0j_0})=-c_{i_0j_0}.$$

If p is the parent of t, define

$$N(t)=-t-N(p).$$

We note that because we are in IA, rather than IB, every element has a unique parent. Now, let α be any row index of some element $c_{\alpha\beta}$ of an odd generation: for each such α, define $r_\alpha=N(c_{\alpha\beta})$. Observe that because we are in case IA, this definition is unambiguous. For suppose $c_{\alpha\beta_1}$ and $c_{\alpha\beta_2}$ were two elements of row α, each occurring in an odd generation, then it follows from the definition of the tree that since $c_{\alpha\beta_1}$ occurs in an odd generation, $c_{\alpha\beta_2}$ is its child. Similarly, $c_{\alpha\beta_1}$ is the child of $c_{\alpha\beta_2}$. This would mean that we were in case IB.

For every index $i=1, \ldots, m$, which is not a row index of an element of an odd generation, define $r_i=0$. Similarly, let β be any column index appearing in some element $c_{\alpha\beta}$ of an even generation: for each such β, define $s_\beta=N(c_{\alpha\beta})$. For each index $j=1, \ldots, n$, which is not a column index of an element of an even generation, define $s_j=0$.

It is now easily seen that C' has the desired property.

Case IB: In this case, we shall show how to find a selection $X'\preceq X$ such that $x'<x$. We first show how to find among the members of the tree a closed circuit. Let k be defined by the statement that the $(k+1)$th generation is the first one that contains an element e of the rth generation, where $r<k+1$, and let e_k be a member of the kth generation whose child is e_r. (Although the knowledge is not necessary for the subsequent argument, it is convenient to note that $k+1>3$.) In other words, it is at the $(k+1)$th generation that we discover that we are in case IB. Hence each member of generations $k, k-1, \ldots, 2$ has a unique parent in the preceding generation, for if a member of the tree has two parents in the preceding generation, then all three members are in the same row or column, one of the parents would be a child of the other parent, and we would have arrived at case IB earlier.

Now let e_{k-1} be the unique parent in the $(k-1)$th generation of e_k, e_{k-2} be the unique parent in the $(k-2)$th generation of e_{k-1}, etc. There are two possibilities: either (a) e_r is an ancestor of e_k, or (b) e_r is not an ancestor of e_k. If (a) holds, then clearly $e_k, e_{k-1}, e_{k-2}, \ldots, e_{r+1}, e_r$ is a closed circuit. If (b) holds, let s be defined by the statement that the sth generation is the last generation that contains a common ancestor e_s of both e_k and e_r. Let e'_{r-1} be the parent of e_r in the $(r-1)$th generation, e'_{r-2} the parent of e'_{r-1}, and so on until we reach e_s. From this "line of descent" $e_r, e'_{r-1}, e'_{r-2}, \ldots, e'_{s+1}, e_s$ and the previous $e_k, e_{k-1}, e_{k-2}, \ldots, e_{s+1}, e_s$, noting that

e'_{s+1}, e_s, e_{s+1} are in the same row or column, it is easy to see that

$$e_k,\ e_{k-1},\ e_{k-2},\ \ldots,\ e_{s+1},\ e'_{s+1},\ \ldots,\ e'_{r-2},\ e'_{r-1},\ e_r$$

is a closed circuit. For the definition of k implies that the same row or column index occurs only in consecutive elements of this cycle, and if we consider these elements as arranged in a cyclic order, the construction of the tree and definition of k imply that the same row or column index occurs only in consecutive elements.

Let us denote this closed circuit by

$$c_{i_1j_1}, c_{i_1i_2}, \ldots, c_{i_tj_1},$$

as explained in the introductory definitions.

Let

$$\Sigma_1 = c_{i_1j_1} + c_{i_2j_2} + \ldots + c_{i_tj_t},$$

$$\Sigma_2 = c_{i_1j_2} + c_{i_2j_3} + \ldots + c_{i_tj_1}.$$

Either $\Sigma_1 \leqq \Sigma_2$ or $\Sigma_1 > \Sigma_2$. Assume the former, and let $x_2 = \min(x_{i_1j_2}, \ldots, x_{i_tj_1})$.

Consider the new selection X'

$$x'_{i_1i_1} = x_{i_1i_1} + x_2, \ldots, x'_{i_ti_t} = x_{i_ti_t} + x_2,$$

$$x'_{i_1i_2} = x_{i_1i_2} - x_2, \ldots, x'_{i_ti_1} = x_{i_ti_1} - x_2.$$

For all other pairs of indices i, j, $x'_{ij} = x_{ij}$.

It is easy to verify that X' is a selection (i. e., satisfies (a) and (b), that $X' \leqq X$, and $x' < x$.

If $\Sigma_1 > \Sigma_2$, let $x_1 = \min(x_{i_1i_1}, \ldots, x_{i_ti_t})$. Consider the new selection X'.

$$X'_{i_1i_1} = x_{i_1i_1} - x_1, \ldots, x'_{i_ti_t} = x_{i_ti_t} - x_1$$

$$X'_{i_1i_2} = x_{i_1i_2} + x_1, \ldots, x'_{i_ti_1} = x_{i_ti_1} + x_1.$$

For all other pairs of indices i, j, $x'_{ij} = x_{ij}$. Then it is easy to verify that X' is a selection, $X' \prec X$, $x' < x$.

This completes the discussion of part I.

6. Detail of Part II

Assume $d_{i_1i_1}$ is an element of D that is *not* Y-selected, and such that $0 > d_{i_1i_1} = e_D =$ minimum of all elements of D not selected by Y. We construct inductively, the following "family tree II".

1. The founder is $d_{i_1i_1}$.

2. Assume now that we have defined the kth generation, for $k = 1, \ldots, s$. We now define the $(s+1)$th generation, the union, over all members of the sth generation of the children of each member of the sth generation. These are the children: If s is odd, $s > 1$ and there exists a Y-selected element of D whose column index is j_1, then this element is its child, the tree terminates, and we say we are in case IIB. If no such element exists or if $s = 1$, then the children consist of all Y-selected elements in the same row as the parent member, omitting, however, those whose column index is the same as that of a member of generations $1, \ldots, s$. If s is even, the children of a member are all nonpositive elements of D in the same column as the member, omitting, however, those whose row index is the same as that of a member of generations of $1, \ldots, s$.

Eventually, the tree must terminate, and if it does not terminate in case IIB, we say we are in case IIA.

Case IIA: In this case, we shall show how to construct numbers $r_1, \ldots, r_m$, $s_1, \ldots, s_n$ such that $D' = (d'_{ij}) = (d_{ij} + r_i + s_j) \sim D$, and D' has the properties (2).

Let α be any row index appearing in some number $d_{\alpha\beta}$ of an odd generation. For all such α, define $r_\alpha = +1$. For any index $i = 1, \ldots, m$ not included in $[\alpha]$, define $r_i = 0$. Similarly, let β be any column index appearing on some member $d_{\alpha\beta}$ of an even generation. For all such β, define $s_\beta = -1$. For any index $j = 1, \ldots, n$ not included in $\{\beta\}$, define $s_j = 0$.

We now show that D' has the properties (2). First, the Y-selected elements of D' are 0: For if d_{ij} is such an element, and if i resp. j appears as a row resp. column index of a member of an odd resp. even generation, then j resp. i appears as a column resp. row index of a member of an even resp. odd generation; hence, $r_i + s_j = 0$. Second, if $d_{ij} < 0$ and $s_j = -1$, then by our construction $r_i = 1$; hence the minimum of the elements of D' is not less than the minimum of the elements of D. In addition, because we are in case IIA, $d'_{i_1i_1} = d_{i_1i_1} + 1 > e_D$. These two statements imply the remainder of (2).

Case IIB: Because we are in case IIB, it is clear how to find a closed circuit in the tree containing $d_{i_1j_1}$. Let us denote this circuit by

$$d_{i_1j_1}, \ldots, d_{i_tj_1}.$$

Then $d_{i_1j_2}, d_{i_2j_3}, \ldots, d_{i_tj_1}$ are Y-selected elements of D. Consider the new selection Z:

$$Z_{i_1j_1} = y_{i_1j_1} + 1, \ldots, z_{i_tj_t} = y_{i_tj_t} + 1,$$

$$Z_{i_1j_2} = y_{i_1j_2} - 1, \ldots, z_{i_tj_1} = y_{i_tj_1} - 1.$$

For all other pairs of indices i,j, define $z_{ij} = y_{ij}$.

Then it is easy to verify that Z is a selection. Further, because all elements of the circuit are nonpositive, $d_{i_1j_2} = \cdots = d_{i_tj_1} = 0$, and $d_{i_1j_1} < 0$, it follows that $Z \prec X$.

7. An Example

Consider the transportation problem with

$$C = \begin{pmatrix} 2595 \\ 8358 \\ 7314 \\ 5972 \end{pmatrix},$$

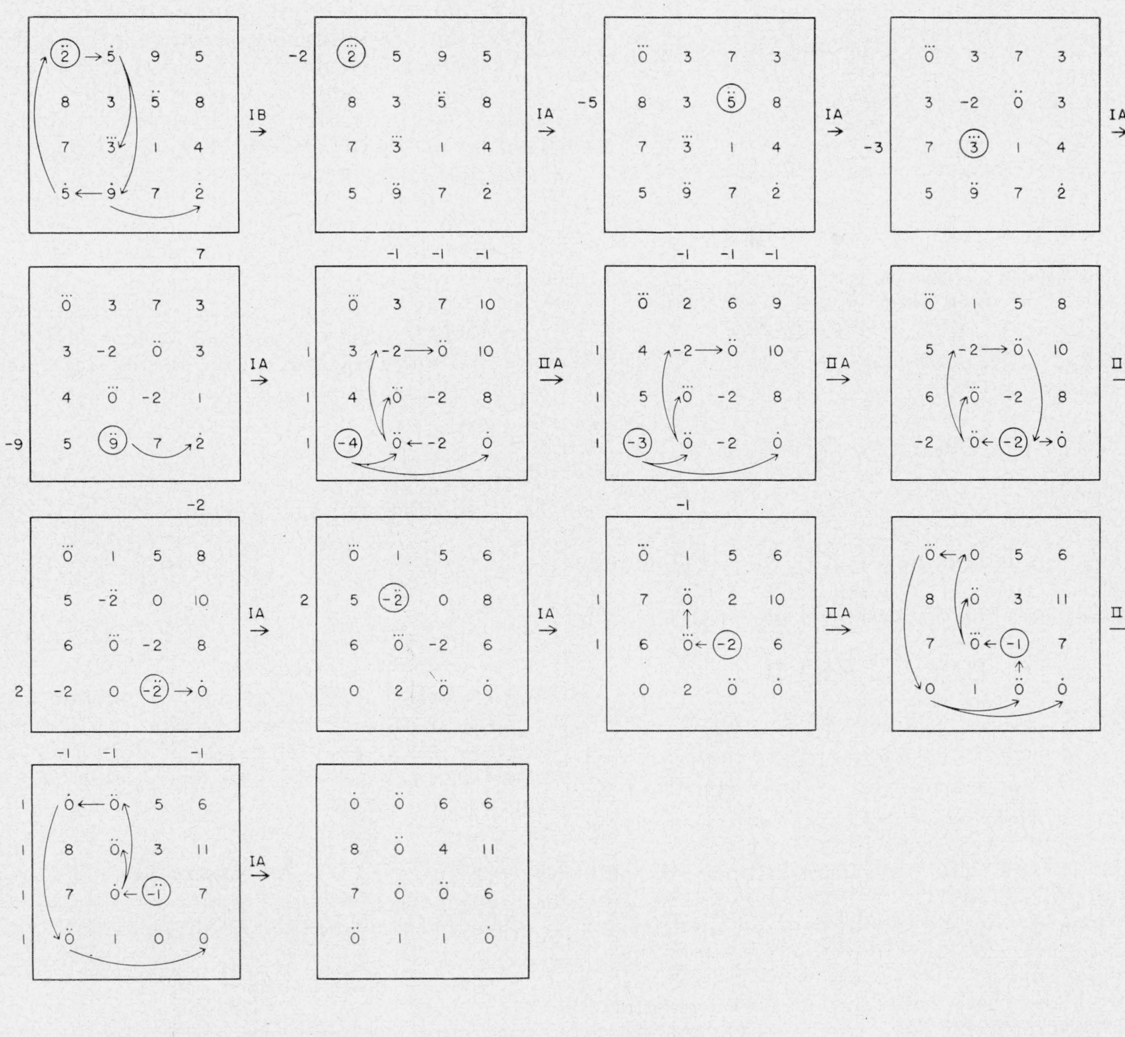
and $a_i = (3,2,3,3)$, $b_j = (3,5,2,1)$. For purposes of illustration, we assume an initial selection matrix,

$$X=\begin{pmatrix}2100\\0020\\0300\\1101\end{pmatrix}.$$

It is convenient to indicate the selection by dots placed above the elements of the matrix C, as follows,

$$\begin{pmatrix}\ddot{2} & \dot{5} & 9 & 5\\ 8 & 3 & \ddot{5} & 8\\ 7 & \dddot{3} & 1 & 4\\ \dot{5} & \dot{9} & 7 & \dot{2}\end{pmatrix}$$

The steps in the solution and the operations associated with cases IA, IB, IIA, and IIB will be clear from the following: [4]



The last matrix is a solution. No closed circuit of zeros exists. Hence the solution is unique. For purposes of illustration, we have followed the method explicitly. If one permits some flexibility in choosing the sequence of operations, much faster convergence may be obtained.

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[4] Numbers on borders at matrices are quantities added to the indicated row or column. Circles drawn about an element indicate the first element of the tree.