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Bounds on a Distribution Function That Are Functions of Moments to Order Four [1]

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Explicit expressions are presented for bounds on a distribution function when moments to order four are known. These inequalities are given in a form suitable for applications.

1. Introduction and Statement of Problem

Tchebycheff [7] [2] in 1874 proposed a problem that can be stated as follows: Let $F(y)$ be an unknown distribution function over the closed interval [3] $[a, b]$, and satisfying the conditions

$$F(a-0)=0$$

$$m_j=\int_a^b y^j dF(y) \qquad (j=0, 1, \ldots, k).$$

If the moments m_j for $j=0,1, \ldots, k$ are known, then for a given value of x, $(a<x<b)$, what are the (sharp) upper and lower bounds on $F(x)$?

Tchebycheff presented without proof a solution to the above problem, which is sometimes called the reduced-moment problem. Proofs were later given by Markoff [1], Possé [2], and Stieltjes [5,6]. The book by Shohat and Tamarkin [4] gives an account of some of the modern-day treatments of the subject.

This paper presents the explicit expressions for solutions of the moment problem (often referred to as the Tchebycheff-Markoff inequalities) for the cases $k=2,3,4$. Inequalities that are functions of moments to order two were given by Tchebycheff [7] for distributions over the interval $[0,b]$. Inequalities that are functions of moments to order three were given by Possé [2]. Possé also solved the case of four moments for distributions over the interval $[a, \infty)$.

2. Explicit Expressions for Bounds

This section presents without proof the explicit expressions for bounds on a distribution function. Proofs may be found in [9]. These are derived as special cases of the Tchebycheff-Markoff inequalities.

In all that follows it will be assumed that

$$m_0=1, \quad m_1=0, \quad m_2=1. \tag{1}$$

This will result in no loss of generality, as any distribution function can be made to conform to these conditions by the use of a linear transformation. The assumption (1) implies that a, b satisfy the inequalities

$$\left.\begin{aligned} a<0<b \\ 1+ab\leq 0. \end{aligned}\right\} \tag{2}$$

This follows from the necessary conditions for the solution of the moment problem (cf. Shohat and Tamarkin [4]).

[1] A condensation of certain results obtained by the author in a thesis submitted to the University of North Carolina in June 1951 in partial fulfillment of the requirements for the Master of Arts degree.

[2] Figures in brackets indicate the literature references at the end of this paper.

[3] Throughout this paper it will be understood that a distribution function over the interval $[a, b]$ is one where the range of the random variable is $[a, b]$, and the end points will belong to that interval unless the end points are $-\infty$ or $+\infty$.

2.1. Bounds for Two Moments

Let $F(y)$ be a distribution function on $[a, b]$ with known moments

$$m_0=1, \quad m_1=0, \quad m_2=1$$

then for a given x, $(a<x<b)$,

$$0 \leq F(x) \leq \frac{1}{1+x^2} \qquad \text{if } a<x\leq -\frac{1}{b} \tag{3}$$

$$\frac{1+bx}{(a-b)(a-x)} \leq F(x) \leq 1-\frac{1+ax}{(b-a)(b-x)} \qquad \text{if } -\frac{1}{b}\leq x\leq -\frac{1}{a} \tag{4}$$

$$\frac{x^2}{1+x^2} \leq F(x) \leq 1 \qquad \text{if } -\frac{1}{a}\leq x<b. \tag{5}$$

For any distribution defined over $(-\infty, \infty)$ inequalities (3) and (5) hold for $x<0$ and $x>0$, respectively.

2.2. Bounds for Three Moments

Let $F(y)$ be a distribution function on $[a, b]$ with known moments

$$m_0=1, \quad m_1=0, \quad m_2=1, \quad m_3.$$

Let $g(x)=x^2-m_3x-1$, and define

$$z_1=\frac{m_3-(a+x)}{1+ax}$$

$$z_2=\frac{m_3-(b+x)}{1+bx}$$

$$w=\frac{m_3-(a+b)}{1+ab}$$

$$A(\alpha,\beta,\gamma)=\frac{1+\alpha\beta}{(\gamma-\alpha)(\gamma-\beta)},$$

then for a given x, $(a<x<b)$

$$0\leq F(x)\leq A(b,\ z_2,\ x) \qquad \text{if } x<0,\ g(x)\geq 0 \tag{6}$$

$$A(x,\ z_1,\ a)\leq F(x)\leq A(x,\ z_1,\ a)+A(a,\ z_1,\ x) \qquad \text{if } g(x)\leq 0,\ x\leq w \tag{7}$$

$$A(x,\ b,\ z_2)\leq F(x)\leq A(x,\ b,\ z_2)+A(b,\ z_2,\ x) \qquad \text{if } g(x)\leq 0,\ x\geq w \tag{8}$$

$$1-A(a,\ z_1,\ x)\leq F(x)\leq 1 \qquad \text{if } g(x)\geq 0,\ x>0. \tag{9}$$

Inequalities (7) and (9) hold for any distribution $F(y)$ on $[a, \infty)$. Inequalities (6) and (8) hold for any distribution $F(y)$ on $(-\infty, b]$. Note that none of the inequalities (6) to (9) holds for distributions over $(-\infty, \infty)$.

2.3. Bounds for Four Moments

Let $F(y)$ be a distribution function on $[a, b]$ with moments

$$m_0=1, \quad m_1=0, \quad m_2=1, \quad m_3, \quad m_4.$$

Let

$$g(y,\gamma)=y^2+\left[\frac{\gamma-m_3+\gamma(\gamma m_3-m_4)}{1+\gamma(m_3-\gamma)}\right]y+\left[\frac{\gamma m_3-m_4+(m_3-\gamma)^2}{1+\gamma(m_3-\gamma)}\right].$$

$U(y)=g(y,a)$, $V(y)=g(y, b)$, $Z(y)=g(y, x)$, and let $u_1<u_2$, $v_1<v_2$, $z_1<z_2$ be the distinct zeros of $U(y)$, $V(y)$, $Z(y)$, respectively, then $a<v_1<u_1<v_2<u_2<b$.

Define

$$z_3=\frac{m_3(a+b+x)-m_4-ab-ax-bx}{abx+a+b+x-m_3}$$

$$A=\frac{m_4-m_3^2-1}{(1+x^2)(m_4-m_3^2-1)+(x^2-m_3x-1)^2}$$

$$B(\alpha,\beta,\gamma)=\frac{m_3-(\alpha+\beta+z_3)-\alpha\beta z_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-z_3)},$$

then for a given value of x, $(a<x<b)$,

$$0\leq F(x)\leq A \qquad \text{if } a<x\leq v_1. \tag{10}$$

$$B(b,x,a)\leq F(x)\leq B(b,x,a)+B(a,b,x) \qquad \text{if } v_1\leq x\leq u_1 \tag{11}$$

$$\frac{1+xz_2}{(z_1-x)(z_1-z_2)}\leq F(x)\leq\frac{1+xz_2}{(z_1-x)(z_1-z_2)}+A \qquad \text{if } u_1\leq x\leq v_2 \tag{12}$$

$$1-B(a,b,x)-B(a,x,b)\leq F(x)\leq 1-B(a,x,b) \qquad \text{if } v_2\leq x\leq u_2 \tag{13}$$

$$1-A\leq F(x)\leq 1 \qquad \text{if } u_2\leq x<b. \tag{14}$$

For any distribution defined over $(-\infty,\infty)$ inequalities (10), (12), and (14) hold, respectively, for $x<z_1$, $z_1<x<z_2$, $z_2<x$.

However, the ordering of x in relation to z_1, z_2 is equivalent to the following. Let $g(x)=x^2-m_3x-1$, then

$$x>0,\quad g(x)>0 \qquad \text{if, and only if, } z_2<x. \tag{15}$$

$$x<0,\quad g(x)>0 \qquad \text{if, and only if, } z_1>x \tag{16}$$

$$g(x)<0 \qquad \text{if, and only if, } z_1<x<z_2. \tag{17}$$

Using (15) to (17), the applications of the Tchebycheff-Markoff inequalities for the case where $F(y)$ is defined over $(-\infty,\infty)$ are made particularly easy.

3. Application of the Tchebycheff-Markoff Inequalities

Let $F(y)$ be a distribution function whose first four moments coincide with those of the standard normal distribution, i. e., $m_0=1$, $m_1=0$, $m_2=1$, $m_3=0$, $m_4=3$. The Tchebycheff-Markoff inequalities will be used to find bounds for $F(x)$ when $x=2, 3$.

Bounds using two moments: Since $x>0$, inequality (5) is applicable, and we have

$$.8000\leq F(2)\leq 1$$

$$.9000\leq F(3)\leq 1.$$

Bounds using four moments: Since $x>0$, $g(2)>0$, $g(3)>0$, inequality (14) is applicable. Substituting the appropriate values, we have

$$1-\frac{2}{2(1+x^2)+(x^2-1)^2}\leq F(x)\leq 1 \text{ for } x=2,3$$

therefore,

$$.8947\leq F(2)\leq 1$$

$$.9777\leq F(3)\leq 1.$$

Note that there are no inequalities applicable using only moments to order three.

4. Appendix

Statements and proofs of the Tchebycheff-Markoff inequalities can be found in Shohat and Tamarkin [4], Uspensky [8], and Royden [3]. This section contains a statement of the Tchebycheff-Markoff inequalities as the above sources do not give the theorem in full generality, and it is not readily available in the literature.

Before stating the theorem it will be convenient to define the following: Let $T_n(y)$, $U_n(y)$, $V_n(y)$, $W_n(y)$ be polynomials of degree n defined by

$$\int_a^b T_n(y)\theta_{n-1}(y)dF(y)=0 \tag{18}$$

$$\int_a^b U_n(y)\theta_{n-1}(y)(y-a)dF(y)=0 \tag{19}$$

$$\int_a^b V_n(y)\theta_{n-1}(y)(b-y)dF(y)=0 \tag{20}$$

$$\int_a^b W_n(y)\theta_{n-1}(y)(y-a)(b-y)dF(y)=0, \tag{21}$$

where $\theta_{n-1}(y)$ is any polynomial of degree $\leq n-1$, and the coefficient for y^n in $T_n(y)$, $U_n(y)$ $V_n(y)$, $W_n(y)$ is unity.

Tchebycheff-Markoff Inequalities: Let $F(y)$ be any distribution function on $[a,b]$ with moments $m_0, m_1, \ldots, m_k$

$$m_j=\int_a^b y^j dF(y) \qquad (j=0,1, \ldots, k),$$

and let x be a given number $(a<x<b)$, then

$$\sum_{y_i<x}\frac{p(y_i)}{q'(y_i)}\leq F(x)\leq \sum_{y_i<x}\frac{p(y_i)}{q'(y_i)}+\frac{p(x)}{q'(x)},$$

where

$$p(z)=\int_a^b \frac{q(z)-q(y)}{z-y}\,dF(y),$$

and $y_1<y_2<\ldots<x<\ldots$ are the zeros of the polynomial $q(y)$ of degree r defined by

$$q(y)=(y-x)w(y) \qquad \text{if } k=2n,\ U_n(x)V_n(x)>0, \tag{22}$$

$$q(y)=(y-b)(y-a)(y-x)w(y) \qquad \text{if } k=2n,\ U_n(x)V_n(x)<0, \tag{23}$$

$$q(y)=(y-a)(y-x)w(y) \qquad \text{if } k=2n-1,\ T_n(x)W_{n-1}(x)>0, \tag{24}$$

$$q(y)=(y-b)(y-x)w(y) \qquad \text{if } k=2n-1,\ T_n(x)W_{n-1}(x)<0, \tag{25}$$

where $r=n+2$ for eq (23), $r=n+1$ for (22), (24), (25), and $w(y)$ is determined by

$$\int_a^b q(y)y^i dF(y)=0 \qquad \begin{array}{l} i=0,\ 1, \ldots, n-1 \text{ for } (22) \\ i=0,\ 1, \ldots, n-2 \text{ for } (23),\ (24),\ (25). \end{array}$$

COROLLARY: *For eq (22) the inequalities hold for any distribution over* $(-\infty, \infty)$ *with moments* $m_0, m_1, \ldots, m_{2n}$. *The inequalities for eq (24) hold for any distribution over* $[a, \infty)$ *with moments* $m_0, m_1, \ldots, m_{2n-1}$. *The inequalities for eq (25) hold for any distribution over* $(-\infty, b]$ *with moments* $m_0, m_1, \ldots, m_{2n-1}$.

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