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A Study of Absolute Standards of Mutual Inductance and in Particular the Three-Section National Bureau of Standards Type

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The results of a study of the number and location of the circles of zero field surrounding a multisection coil are presented. The configuration of the field surrounding the equatorial region of several three-section coils has been partially mapped out. One arrangement yields a design in which the mutual-inductance contribution of any secondary turn differs but little from that of the median turn. By locating the secondary symmetrically about the circles of zero field, the correction for winding distribution can be made small, and the effect of uncertainties in location of individual secondary turns minimized.

1. Introduction

Certain absolute measurements, as for example, the absolute measurement of resistance, depend upon the use of a mutual inductance whose value can be calculated from its geometrical dimensions and the number of turns on primary and secondary.

In order to minimize the effect of dimensional uncertainties the standard should be of fair size with relatively large separation between primary and secondary. Further, to obtain sufficient sensitiveness in the measurements in question, the value of the mutual inductance should be as great as of the order 10 millihenrys. These conditions demand a value of the product N_1N_2 of primary and secondary turns of the order of 100,000.

The dimensions of a single-layer coil, wound in a screw thread on a suitable cylindrical form, may be measured with great precision. With such a large requisite value of N_1N_2, however, it is impracticable that both the primary and secondary should be single-layer coils. Of necessity, one winding should be of the form of a multilayer coil, wound in a channel of square or rectangular cross section.

In a form of mutual-inductance standard designed by Albert Campbell in 1907 [1] [2] and used in absolute measurements at the National Physical Laboratory [2], the primary winding consists of two equal single-layer coils of radius a, wound on the same cylindrical form so that each coil has an axial length of $a/2$, with a gap between the adjacent coil ends of length a (see fig. 1, A). The two coils are joined in series, magnetically aiding. The secondary is a circular coil of square cross section, coaxial with the two parts of the primary winding and located in the median plane between their ends. In this place, the radial components of the magnetic field intensity, produced by the two primary windings separately, cancel for all values of radial distance. On the contrary, their axial components, which are equal, add, giving a resultant axial magnetic field intensity. With increasing radius this diminishes, becomes zero at a critical radius, which is about 1.46 a, and for still larger values of the radius, is in the opposite direction. The magnetic field intensity is, however, very small for moderate departures of radius from its critical value and for moderate displacements from the median plane. If, therefore, the central filament of the secondary coil has a radius equal to the critical value, all the secondary turns lie in regions of small magnetic field intensity, so that the mutual inductance of the primary and a secondary turn is nearly the same for all the secondary turns. The resultant mutual inductance is closely N_2M_0, where M_0 is the mutual inductance of the primary and the central filament, and the small error may be accurately determined.

In the Wenner method for the absolute measurement of resistance [3] a modified form of the Campbell type of standard was employed. In this method the secondary is periodically short-circuited, making it desirable for this winding to have a minimum of inductance and resistance. A coil arrangement was designed in which the number of secondary turns was only one-half the number on the primary. This allowed the use of wire of relatively large cross section in the

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[2] Figures in brackets indicate the literature references at the end of this paper.

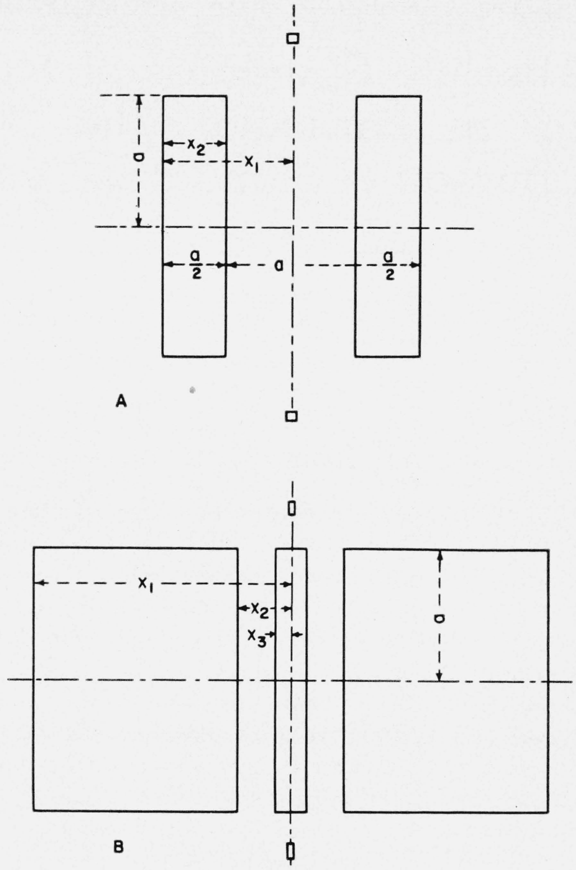

FIGURE 1. *Mutual inductance standards.*

A, Campbell type; B, National Bureau of Standards three-section type.

secondary. The necessary increase in the number of primary turns was obtained by the use of primary windings of greater axial length. As may be seen from figure 1, B, the primary winding consists of two end sections of equal length, and partly filling the space between them is a short middle section. The three sections of the primary are joined in series, aiding magnetically. The positions of zero magnetic field intensity were located experimentally by passing 60-cycle current through the primary winding and using a short, suspended, magnetic needle, whose period of torsional vibration was adjusted to the period of the current, to explore the field around the primary. With a chosen length of 5 cm for the middle section and gap lengths of 5.6 cm each, the field was explored for different lengths of the end sections. With relatively short end sections two zero points were found with equal radii and located at equal distances on both sides of the median plane. With longer end sections these two points of zero field were replaced by two zeros in the median plane, locating two circular filaments of differing radii along whose circumferences the magnetic field intensity is zero. The standard as used for the measurements had the following approximate dimensions:

Pitch of primary winding	0.2 cm.	Mean radius of primary	20.364 cm.
Length of middle section	5.0 cm.	Mean radius of secondary	26.348 cm.
Length of gaps	5.6 cm.	Number of primary turns	343
Length of end sections	31.8 cm.	Number of secondary turns	218
Approximate mutual inductance	10.897 mh.		

The radii of the circles of zero field intensity in the median plane were about 24.95 and 26.70 cm. The secondary turns were wound in a channel of rectangular cross section having dimensions of

2.542 cm radial, 0.82 cm axial. This arrangement of the standard will be designated as case A in what follows.

For comparison, the dimensions of the Campbell type of standard, used by the National Physical Laboratory, follow:

Pitch of primary winding	0.1 cm.
Length of primary coils	7.5 cm.
Length of gap between coils	15 cm.
Radius of primary	15 cm.
Approximate mutual inductance	10 mh.
Radius of secondary	21.9 cm.
Number of primary turns	150
Number of secondary turns	488

The secondary turns are wound in a channel of square cross section 1 cm on a side. It will be noted that the goal of a secondary of fewer turns, wound with larger wire, was reached in the NBS form.

In NBS Research Paper RP2029, page 296, the statement is made that consideration of the number and positions of possible points of zero field intensity is reserved for further study. As a result of correspondence with two of the authors of the paper on the absolute measurement of resistance by the Wenner method (James L. Thomas and Chester Peterson) the writer was encouraged by them to undertake a quantitative study of the NBS type of mutual-inductance standard, and they communicated curves of mutual inductance variations with secondary radius for two other arrangements with different lengths of end sections. These will be denoted as cases B and C. In comparison with case A, case B has two turns removed from the outer ends of each of the end sections, and case C has yet another turn removed from each of the outer ends of the end sections.

The manner in which the mutual inductance of a secondary turn in the median plane varies with the secondary radius is shown in figure 2. (These curves are based on data supplied by the

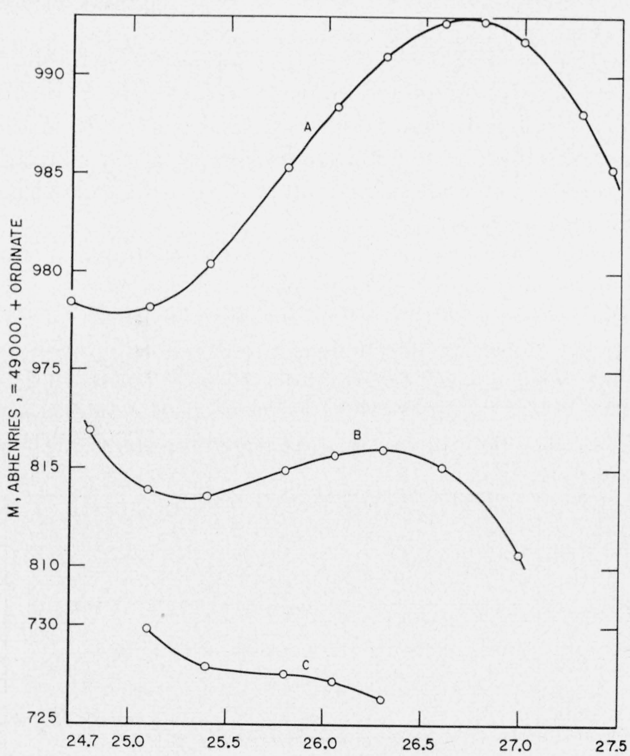

FIGURE 2. *Relation between mutual inductance and the radius of a single secondary turn for three arrangements of the National Bureau of Standards three-section type of standard.*

authors of RP2029 and are reproduced with their permission.) For case A the mutual inductance is a minimum for radius 24.95 cm and a maximum for radius 26.70 cm. For case B the minimum is at 25.32 cm and the maximum at 26.30 cm. The curve for case C shows neither maximum or minimum in the median plane.

2. Method of Locating Points of Zero Field

2.1. Zeros in the Median Plane

For routine calculations of magnetic field intensity of solenoids, formulas previously published by the author [4] may be used; for definitive calculations the exact expressions in elliptic integrals must be used (see appendix for H_x). These formulas give the axial component of field H_x, produced by a current of 1 abampere flowing in a solenoid of radius a, winding density n turns per centimeter, and axial length x at a point at radius y in the end plane. Throughout this paper the symbol for current, I, will be omitted in all formulas for field strength, it being understood that all values of H are per unit of current. Thus, for a point at radius y in the median plane, the axial field due to one of the end sections is obtained by making two calculations by the basic formula, one for length x_1 and the other for length x_2 (see fig. 1, B) and subtracting the second from the first. The axial field intensity is the same for both end sections, and the values add in the resultant. The contribution due to the middle section is twice that obtained by making a calculation with the distance x_3 (fig. 1, B). Adding the resultants for the end sections and the middle section gives the resultant field, since, from symmetry, the radial components of the end sections cancel, and the radial component due to the middle section is zero.

The calculated values of $H_x/2n$ for the cases A, B, and C for points of different radii in the median plane are plotted in figure 3. Remembering the relation $(dM/dy)_{y=y_0}=2\pi y_0(H_x)_{y=y_0}$, it is seen that curve A checks the mutual-inductance curve figure 2, since the zero values of H_x occur at $y=24.95$ and $y=26.70$, the turning points of the mutual-inductance curve. The maximum value of H_x is found at $y=25.75$, which is the point where the mutual-inductance curve has its maximum positive slope. The curve B for case B (end sections shorter by two turns) is similar to that for case A; the zeros of H_x at $y=25.37$ and $y=26.25$ agree in position with the minimum and maximum points of the mutual-inductance curve in figure 2, and the much smaller value of the maximum of H_x is in line with the reduced slope and range of the mutual-inductance curve.

For case C, where three turns have been removed from each end section, the magnetic field intensity nowhere becomes zero in the median plane, although it is very small at $y=25.75$. This corresponds to the position of the minimum slope of the mutual-inductance curve in figure 2. There is no point of zero magnetic field intensity in the median plane for case C. It is, however, evident that, with a small fraction of a turn added to the end sections of case C, the curve of H_x would become tangent to the zero axis at about $y=25.75$, in which case there would be a single point of zero field intensity in the median plane at that radius.

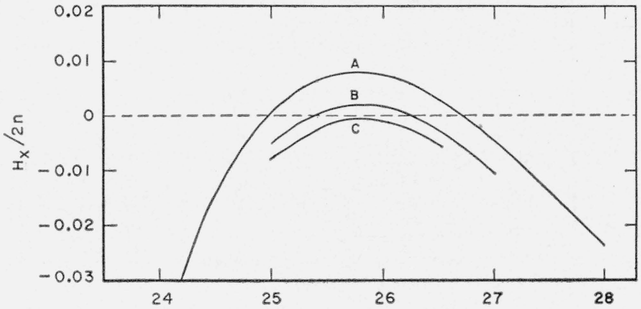

FIGURE 3. *Variation of the magnetic field intensity in the median plane for three arrangements of three-section standard.*

2.2. Zeros Not in the Median Plane

The study of the magnetic field intensity at points not in the median plane is complicated by the necessity of calculating the radial component, H_r, as well as the axial. Furthermore, the contribution of the end sections are not equal, and two distances x have to be considered for the middle section. Thus, for calculating H_x and H_r at a single point, the basic formulas have to be calculated for six values of x, both for H_x and H_r. For routine calculations, published formulas and tables may be used, but for final accurate values the elliptic integral formula for H_x given in the appendix has to be used. Then the radial component H_r may be made to depend on the formula for the calculation of the mutual inductance m_f of two coaxial circular filaments, one having the radius a of the primary and the other that of the circle through the point P in question. Two calculations have to be made for the two axial distances between P and the ends of the solenoid. If these distances are x_2 and x_3, with x_2 the smaller, then

$$H_r = \frac{n}{2\pi y}\left[m_{f_2} - m_{f_3}\right].$$

For calculating m_f the Maxwell elliptic integral formula, or any suitable series expansion of this, may be used, or alternatively, the tables [7] published by the author may be employed.

The required zeros of magnetic field intensity are evidently the intersections of the loci of $H_r=0$ and $H_x=0$. The determination of the loci is made by calculating the field component for chosen points, for example, for a given value of the displacement ξ from the median plane, and for a least three values of y. Thence the value of y for which the field component is zero is interpolated. From several interpolated points, so found, the curve of the locus may be drawn. When the approximate loci are not known beforehand, the exploration necessary can be quite time consuming. For greater accuracy in the location of a zero point, further calculations with the elliptic integral formulas may be made for the region of the graphically determined intersection of the loci. On account of the symmetry of the loci about the median plane, calculations have to be made for one half plane only.

In figure 4 are shown the loci and the zero points for cases A and C and for an arrangement D where the end sections are still shorter. The ratio of the length of an end section in comparison to the length of the middle section is 6.36 for case A; 6.28 for case C; and 5.00 for case D.

The curves shown make clear the changes brought about by changing the length of the end sections with respect to the length of the middle section. In case D, the zeros are at radius 24.7 and are displaced at equal distances $\xi=2.7$ cm on either side of the median plane. In case C, the two zeros are at radius about 25.75, but are located only about 0.2 cm either side of the median plane. In case A, the zeros are in the median plane with radii differing about 1.75 cm.

In all these cases the locus $H_r=0$ is seen to have two branches, one in the median plane, and the other, whose trace is practically the same curve in all these cases, intersects the former at about the same radius of 25.7 cm. It is, therefore, easy to visualize the changes in the position of the locus $H_x=0$ as the end sections are progressively lengthened in relation to the length of the middle section. With increase in this ratio, the intersections of the loci slide along the horizontal branch of the $H_r=0$ locus until they merge in a single point at about $y=25.75$ in the median plane, and, for still greater values of the ratio, two intersections again appear, now in the median plane, and these progressively separate, unsymmetrically with respect to $y=25.75$, as the ratio is still further increased. The spacings are, for example, -0.38 and $+0.50$ for case B; -0.8 and $+0.95$ for case A. The proportioning of the lengths of the sections is quite critical in its effect. The striking difference between case B and C is brought about by the subtraction of a single turn from the outer ends of the end sections, with the gap lengths of 5.6 cm, the length of middle section 5 cm, and the radius of primary 20.364 cm.

It is easy to show that decreasing the gap lengths, all else being unchanged, would shift the intersection of the two branches of the $H_r=0$ locus nearer the surface of the primary, and thus with it, all the zeros to smaller radii.

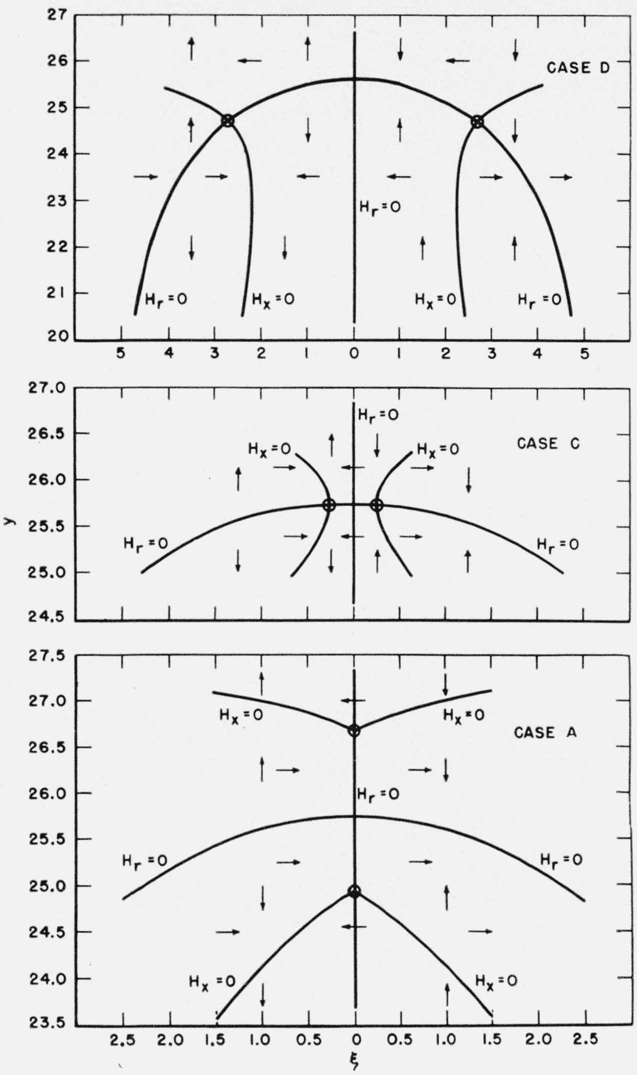

FIGURE 4. *Loci of zeros of the axial and radial field components.*

The negative slope of the mutual-inductance curves for the smaller values of the radius y in figure 2 raises the question whether there are still other zeros of field intensity, for the mutual inductance of a filament of radius $y=0$ is manifestly zero. The mutual inductance of a filament reaches in fact an absolute maximum for a radius equal to about the radius of the primary winding. In the median plane the component H_x reverses sharply in passing through the winding, so there is here a zero of H_x; but where no secondary can be usefully located.

3. Evaluation of the Correction for Secondary Distribution

Even with the secondary coil placed with its central filament coinciding with the circle of zero magnetic field intensity, the mutual inductance will vary from turn to turn. For very precise measurements, it is necessary to take these variations into account.

If m_0 is the mutual inductance of the primary on the central turn of the secondary, and N_2 is the number of turns on the secondary, the problem is to find the small correction factor δ in the equation

$$M=N_2m_0(1+\delta). \tag{1}$$

The general formulas developed for δ below yield a correction factor that takes into account, not only the distribution of the winding over the cross section, but also the effect of any residual value of the magnetic field intensity at the center of the winding.

3.1. General Correction Formulas

An obvious method of solution is to express the mutual inductance of the primary on any secondary turn by means of a Taylor's series development about the value for the central filament.

If m_0 is the mutual inductance of the central filament, the mutual inductance m of any other filament, whose coordinates, referred to the center point of the cross section are (x,η), is given by

$$m=m_0+x\left(\frac{dm}{dx}\right)_0+\eta\left(\frac{dm}{d\eta}\right)_0+\frac{1}{2!}\left[x^2\left(\frac{d^2m}{dx^2}\right)_0+2x\eta\left(\frac{d^2m}{dxd\eta}\right)_0+\eta^2\left(\frac{d^2m}{d\eta^2}\right)_0\right]$$
$$+\ldots+\frac{1}{n!}\left[x^n\left(\frac{d^nm}{dx^n}\right)_0+nx^{n-1}\eta\left(\frac{d^nm}{dx^{n-1}d\eta}\right)_0+\ldots+\eta^n\left(\frac{d^nm}{d\eta^n}\right)_0\right]+\ldots, \tag{2}$$

the coefficients of the differential coefficients being those of the binomial theorem. The subscript zero indicates that the differential coefficients are to be evaluated at the central point.

If the axial and radial dimensions of the rectangular cross section are, respectively, b and c, the total mutual inductance of the secondary is found by integrating (2) over the cross section, that is,

$$M=\frac{N_2}{bc}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{c}{2}}^{\frac{c}{2}} mdxd\eta.$$

As the differential coefficients in (2) relate to the central point, they are constants, and the integrations are readily performed, and if it be noted that differentiation with respect to η is the same as differentiation with respect to y, the value of δ, carried to include fourth-order differential coefficients, is evident by comparing the following equation with (1).

$$M=N_2m_0\left[1+\frac{1}{m_0}\left\{\frac{b^2}{24}\left(\frac{d^2m}{dx^2}\right)_0+\frac{c^2}{24}\left(\frac{d^2m}{dy^2}\right)_0+\frac{b^4}{1920}\left(\frac{d^4m}{dx^4}\right)_0+\frac{b^2c^2}{576}\left(\frac{d^4m}{dx^2dy^2}\right)_0+\frac{c^4}{1920}\left(\frac{d^4m}{dy^4}\right)_0+\ldots\right\}\right]. \tag{3}$$

Formulas for calculating the values of the differential coefficients by two separate methods are given in the appendix. It should be noted, however, that certain fundamental relations exist between these coefficients.

If H_r and H_x denote, respectively, the radial and axial components of the magnetic field intensity, per unit current in the primary coils, then $dm/dx=-2\pi yH_r$, and $dm/dy=2\pi yH_x$, and in addition, the condition that curl $H=0$ leads to the relation

$$\frac{dH_r}{dx}=\frac{dH_x}{dy}. \tag{3a}$$

From these are readily found the general relations

$$\left.\begin{aligned}
\frac{d^2m}{dy^2}&=\frac{1}{y}\frac{dm}{dy}-\frac{d^2m}{dx^2}\\
\frac{d^4m}{dy^4}&=\frac{1}{y^2}\frac{d^2m}{dx^2}-\frac{d^4m}{dx^4}-2\frac{d^4m}{dx^2dy^2}\\
\frac{d^4m}{dx^4}&=\frac{d^4m}{dy^4}-\frac{2}{y}\frac{d^3m}{dy^3}+\frac{3}{y^2}\frac{d^2m}{dy^2}-\frac{3}{y^3}\frac{dm}{dy}\\
\frac{d^4m}{dx^2dy^2}&=\frac{2}{y^3}\frac{dm}{dy}-\frac{2}{y^2}\frac{d^2m}{dy^2}+\frac{1}{y}\frac{d^3m}{dy^3}-\frac{d^4m}{dy^4}
\end{aligned}\right\} \tag{4}$$

Use of the first of eq (4) enables the second order terms δ_2 of the correction δ to be very simply expressed as

$$\left.\begin{aligned}\delta_2&=\frac{1}{m_0}\left[\frac{b^2-c^2}{24}\left(\frac{d^2m}{dx^2}\right)_0+\frac{c^2}{24y_0}\left(\frac{dm}{dy}\right)_0\right]\\&=\frac{1}{m_0}\left[\frac{b^2-c^2}{24}\left(\frac{d^2m}{dx^2}\right)_0+2\pi\ \frac{c^2}{24}\ (H_x)_0\right]\end{aligned}\right\} \tag{5}$$

The last term in (5) is the correction, usually relatively very small, for the residual vaue of the field H_x at the center of the cross section, due to imperfect centering. (Only when the mutual inductance of the central filament is a maximum or minimum will the value of be zero.)

Use of equations one and two of (4) enables eq (3) to be expressed in terms of variable x alone, except for the terms in dm/dy and d^4m/dx^2dy^2. There is found, however, no great simplification.

In a paper on the calculation of a Campbell form of standard at the National Physical Laboratory, Dye [5] has made use of a formula for the evaluation of the correction for the distribution of the winding, due to G. F. C. Searle. No statement as to the manner of derivation is given, and as printed it is not free from misprints. It may be shown to be equivalent to eq (3) as follows.

Substituting the first, third and fourth equations of (4) in (3) leads to the equation

$$\delta=\frac{1}{m_0}\left\{\frac{1}{y_0}\left(\frac{dm}{dy}\right)_0\left(\frac{b^2}{24}+\frac{b^2c^2}{288y_0^2}-\frac{b^4}{640y_0^2}\right)+\left(\frac{d^2m}{dy^2}\right)_0\left(\frac{c^2-b^2}{24}+\frac{b^4}{640y_0^2}-\frac{b^2c^2}{288y_0^2}\right)+\right.$$
$$\left.\left(\frac{d^3m}{dy^3}\right)_0\left(\frac{b^2c^2}{575y_0}-\frac{b^2}{960y_0}\right)+\left(\frac{d^4m}{dy^4}\right)_0\left(\frac{b^4}{1920}-\frac{b^2c^2}{576}+\frac{c^4}{1920}\right)+\ldots\right\}$$

and noting that, in Dye's nomenclature, $y_0\equiv A_0$, $b\equiv 2C$, and $c\equiv 2B$, and if we include the factor N_2 in each term, writing M in place of m to indicate this fact, we obtain Dye's expression free of misprints.

$$M=M_0+\frac{1}{A_0}\left(\frac{dM}{dA}\right)_0\left[\frac{C^2}{6}+\frac{20\,B^2C^2-9C^4}{360\,A_0^2}\right]+\left(\frac{d^2M}{dA^2}\right)_0\left[\frac{B^2-C^2}{6}+\frac{9\,C^4-20\,B^2C^2}{360\,A_0^2}\right]+$$
$$\left(\frac{d^3M}{dA^3}\right)_0\left[\frac{10\,B^2C^2-6\,C^4}{360\,A_0}\right]+\ldots \tag{6}$$

To compute this correction, Dye calculated dM/dA for the whole secondary for a number of values of A, using the Campbell formula [1], which involves an elliptic integral of the third kind. These values of dM/dA, plotted against A, gave a nearly straight line with, of course, the value of zero at the ordinate when the magnetic field intensity is zero.

Writing an equation for these calculated points in the form $dM/dA=-51.6\epsilon+9.0\epsilon^2$, in which ϵ represents the difference between any radius A and that for which dM/dA is zero, the other differential coefficients may readily be expressed, namely, $d^2M/dA^2=-51.6+18.0\epsilon$, and $d^3M/dA^3=18.0$

For the center point, $\epsilon_0=-0.0210$, so that $(dM/dA)_0=1.13$ μh/cm. Other data were, $a=15.008$, $A_0=21.90$, $2B=0.96$, $2C=1.00$, and the calculated values of the terms in (6) are $0.00209+0.16894+0.00046=0.1715$ μh, or, 17.2 ppm of the total mutual inductance of 10 mh.

The value of the correction for this standard may be checked by formula (5), for which the data become, $a=15.008$, $y=21.90$, $c=0.96$, $b=1.00$, $N_2=485$, and the winding density of the primary is $n=10$ turns per centimeter

From these there results $(d^2m/dx^2)_0=106.81$ and $(H_x)_0=0.00204$, and by formula (5) the correction is $\delta_2=16.922+0.024=16.946$ ppm. The value of $(H_x)_0$ here found is less than the value 0.0169 derived from the value of dM/dA quoted by Dye. This is explained by the fact that the radius of the secondary used was 21.8815 cm, whereas the Dye curve showed that the radius for zero field should be slightly greater than 21.90, which was the value for which the calculations here are made.

The fourth-order differential coefficients were calculated by both methods A and B, (appendix) which checked each other closely. The values found are $(d^4m/dx^4)_0 = -5.381$, $(d^4m/dx^2dy^2)_0 = +4.094$, and $(d^4m/dy^4)_0 = -2.584$. Using these values in formula (3), the fourth-order corrections terms are, in order, $-0.136+0.318-0.055=+0.127$ ppm, and adding these to the main terms, the correction becomes $16.946+0.127=17.07$ ppm, which checks Dye's value closely. In Dye's formula, the first term; in formula (5), the last term, take into account the effect of the residual magnetic field at the center of the cross section. It may be noted that, in each calculation, this amounts to only 12 parts in a thousand of the total correction for the cross section. The use of formula (3) with the equations in the appendix for calculating the differential coefficients, involves materially less work than the Dye procedure.

The adequacy of the general formula (3) may be tested by calculating the correction for distribution of the turns of the NBS mutual inductance standard, case A. For this we have available the very accurately determined value, used in the experiments for the measurement of the ohm, NBS Research Paper 2029. This value was obtained from thirteen values of m, calculated by the absolute elliptic integral formula for equally spaced turns in the median plane. The effect of the smaller (axial) dimension of the cross section was evaluated experimentally by directly measuring the difference of mutual inductance of two equal, flat coils, one placed in the median plane and the other at different axial displacements. The results of these tests, taken with the calculated values in the median plane led to an estimation of the correction for distribution of turns as -75.8 ppm, which may be considered as very accurate.

The data for the calculation by formula (3) are $a=20.364$, $y_0=26.348$, $n=5$, $c=2.546$, $b=0.82$, $x_1=2.500$, $x_2=8.100$, and $x_3=39.90$. From the formulas in the appendix, we find $(H_x)_0=0.046$, $(d^2m/dx^2)_0=17.02$, $(d^4m/dy^4)_0=16.08$, $(d^4m/dx^2dy^2)_0=-16.91$, and $m_0=49{,}991$ abhenrys. Whence

$$\left.\begin{array}{l}
\left.\begin{array}{r}
\dfrac{1}{m_0}\left(\dfrac{b^2-c^2}{24}\right)\left(\dfrac{2^2m}{dx^2}\right)_0=-82.4_2 \\[2ex]
\dfrac{1}{m_0}\left[\dfrac{2\pi c^2}{24}(H_x)_0\right]=+1.5_6
\end{array}\right\}=-80.8_6 \\[4ex]
\left.\begin{array}{r}
\dfrac{1}{m_0}\left[\dfrac{b^4}{1920}\left(\dfrac{d_4m}{dx^4}\right)_0\right]=+0.0_8 \\[2ex]
\dfrac{1}{m_0}\left[\dfrac{b^2c^2}{576}\left(\dfrac{d^4m}{dx^2dy^2}\right)_0\right]=-2.5_6 \\[2ex]
\dfrac{1}{m_0}\left[\dfrac{c^4}{1920}\left(\dfrac{d^4m}{dy^4}\right)_0\right]=+7.0_4
\end{array}\right\}=+4.5_6
\end{array}\right\}=-76.3_0\text{ ppm},$$

which agrees well with the assumed value, and indicates that terms of order higher than the fourth are negligible.

3.2. Other Methods for Calculating the Correction

It is evident that the process of computation is very time consuming when fourth-order differential coefficients have to be taken into account, because calculations of each differential coefficient, involving separate calculations for each of the three distances x have to be made. It is, however, relatively simple to evaluate the second order terms in formula (5). The fourth order terms are, evidently, less important, the smaller the cross-sectional area. This suggests assuming the cross section to be divided into two or more equal subsections and applying correction formula (5) to each. Thus the distribution correction can be found for each subsection, referred to the mutual inductance of its central filament. The average of the corrected values for the subsections, referred to the value of the mutual inductance of the central filament of the whole cross section, will give the desired correction.

The details of this process may be illustrated for two subsections. For these we write

$$M_1=\frac{N_2}{2}(m_{01}+d_1),\ M_2=\frac{N_2}{2}(m_{02}+d_2),$$

and, therefore,

$$M=N_2\left[\frac{m_{01}+m_{02}}{2}+\frac{d_1+d_2}{2}\right]=N_2 m_0\left[1+\Delta+\frac{\Delta_1+\Delta_2}{2}\right],$$

where m_0 represents the mutual inductance of the central filament of whole section; m_{01} and m_{02}, the values for the central filaments of the subsections, and the total correction is

$$\delta=\Delta+\frac{\Delta_1+\Delta_2}{2},\ \text{with}\ \Delta=\left[\frac{m_{01}+m_{02}}{2}-m_0\right]\div m_0\ \text{and}\ \frac{\Delta_1+\Delta_2}{2}=\frac{1}{m_0}\left(\frac{d_1+d_2}{2}\right):$$

If the subsectional area is sufficiently small, only the second order terms formula (5) need be calculated, but the three mutual inductances m_{01}, m_{02}, and m_0 will have to be calculated by the precision elliptic integral formula.

As an example, assume the NBS secondary, case A, to be divided into two subsections, centered in the median plane. For each subsection $c=1.273$ and $b=0.82$. Taking the radii of the centers of subsections as y_1, and y_2, and $m_0=49991$, as before, the details of the calculation are as follows:

$y_1=$ 25.711	$y_2=$ 26.985	$\frac{m_{01}+m_{02}}{2}=49988.05$
$\left(\frac{d^2m}{dx^2}\right)_0=-1.276$	$\left(\frac{d^2m}{dx^2}\right)_0=$ 28.288	$m_0=\underline{49991.32}$
$(H_x)_0=$ 0.0794	$(H_x)_0=-0.03608$	Difference= -3.27
$d_1=$ 0.0849	$d_2=-$ 1.1376	$\Delta=$ -65.5 ppm
$\frac{d_1+d_2}{2}=-0.5263$	$m_{01}=$ 49984.20	$\frac{\Delta_1+\Delta_2}{2}=$ $\underline{-10.5}$
$\frac{\Delta_1+\Delta_2}{2}=-10.5$ ppm	$m_{02}=$ 49991.91	$\delta=$ -76.0 ppm

Evidently, in case only the corrected mutual inductance is required, it is only necessary to calculate M_1 and M_2 and to take their sum. This suggests still another procedure, using the sectioning principle.

The secondary may be replaced by two equivalent filaments, according to the method of Lyle [6], and the required mutual inductance, corrected for the distribution of the winding, is approximately the average of the precise values of the mutual inductances calculated for these equivalent filaments.

The radii of the equivalent filaments are given by $(r-D)$ and $(r+D)$, where

$$r=y_0+\frac{1}{24}(b^2/y_0)\quad\text{and}\quad D^2=\frac{c^2-b^2}{12}.$$

For case A, $r=26.34906$ and $D=0.695804$, so that the radii of the equivalent filaments and the mutual inductances, corresponding are

$$y_1 = r - D = 25.65326 \qquad m_1 = 49983.36$$
$$y_2 = r + D = 27.04487 \qquad m_2 = 49991.42$$
$$\text{Average} = 49987.39$$

$$m_0 = 49991.32$$
$$\text{Difference} = \quad -3.93 \text{ abhenrys.}$$
$$= \quad -78.6 \text{ ppm.}$$

Lyle's formulas neglect differential coefficients higher than the second, so as would be expected, this result for the undivided secondary reflects the neglect of fourth-order terms. However, the secondary may now be assumed divided into two subsections as in the previous example and the two Lyle filaments for each of the subsections considered.

This treatment yields the following results, the values Δm, referred to m_0, being given instead of actual values of m:

$$y_1 = 25.43151 \qquad \Delta m_1 = -10.5387$$

$$y_2 = 25.99368 \qquad \Delta m_2 = \quad -3.5815$$

$$y_3 = 26.70446 \qquad \Delta m_3 = \quad +1.5051$$

$$y_4 = 27.26663 \qquad \Delta m_4 = \quad -2.5946$$
$$\text{Average} = \quad -3.8024 \text{ abhenrys.}$$

The distribution correction resulting -76.06 ppm.

Evidently, by continued subdivision, with separate treatment of each subsection and combination of results, any desired accuracy may be attained. The labor, however, increases rapidly with the number of sections. The check furnished by the known value of the correction for the NBS coil, case A, indicates that the calculation using the second-order terms, using two subsections, or the calculation of the average of the mutual inductances of the four corresponding Lyle filaments is sufficient for practical purposes, even in the most precise work.

3.3. Examples of Special Cases

From formula (5) it is evident that for a square cross section the first term is zero, and if the central filament is accurately placed at the position of zero magnetic field intensity, the relatively small second term of (5) is zero also. That is, the distribution correction in such a case depends entirely on fourth- and higher-order terms.

In the case of the standard calculated by Dye, the correction depended largely upon the factor $(B^2 - C^2)$, which had a small value due to a small departure of the cross section from being a square. However, assuming an accurately square cross section, accurately centered on the radius of zero magnetic field intensity, the correction for the Campbell type of standard should be very small. Also, that arrangement of the three-section standard, denoted by the name, case C, may with proper design be made to depend upon the theory for a square cross section

For a more detailed study two cases were selected: (a) a Campbell standard with square cross section secondary; and (b), the arrangement in figure 4, case C. In order to obtain comparable results, cross sections of nearly equal areas were assumed in each case. Case A, already studied, has a cross-sectional area of 2.09 cm^2. For case C, a section with an axial dimension of 2 cm and a radial dimension of 1 cm, area 2 cm^2, was assumed, and for the Campbell type a section 1.5 cm square, giving an area of 2.25 cm^2 was assumed.

A detailed study of the Campbell case is covered at length in section 4. The correction for cross section is found to be about 0.70 ppm. A separate calculation by the general formula (3) gave correction for:

Residual field	$+0.013$
Fourth-order terms	$+0.232$
Total	$+0.245$ ppm.

For case C, using $c=1$, $b=2$, $y_0=25.75$, $(H_x)_0=-0.00304$, and $(d^2m/dx^2)_0=-0.7153$ there is found:

Second-order terms	-1.81
Fourth-order terms	$+1.15$
Total	-0.66 ppm.

On account of the symmetry of this case with respect to the median plane, a more accurate method of treatment is to consider either square half of the cross section. For these the residual field intensity at the center is $(H_x)_0=0.0234$, so that we have by (5) the correction $+0.123$. The fourth-order terms amount to -0.323, so that, referred to the center of the square, the correction is -0.20 ppm. But it is found that the mutual inductance of the filament at the center of a square is -0.42 ppm referred to the center of the whole section. Therefore, referred to this latter point, the correction is -0.20 plus -0.42, giving the result -0.62, which agrees well with the previous determination.

4. Detailed Study of Variations of Mutual Inductance From Turn to Turn Over the Secondary Cross Section

The variations in mutual inductance from turn to turn of the secondary coil can be obtained by direct calculation of the mutual inductance for filaments with regularly spaced values of the radius y and the displacement ξ from the median plane. As the value at each point is the resultant of calculations with six different values of x in the fundamental elliptic integral formula (for the NBS type) this is very time consuming, and on account of the near cancellation of terms, the resultant is appreciably affected by small errors in the individual terms.

It is more logical to make calculations of the sought-for changes ΔM directly, than to obtain these by subtraction of calculated values of M, and greater accuracy in the result is attainable. For this purpose, two series expressions for M are available, one suitable for larger values of x, due to Rosa [8], the other, suitable for smaller values of x, given by Dwight [9]. Noticing that for a point P having a displacement from the median plane, referred to a point having the same value of y, but situated in the median plane, a calculation of twice the value, using the longer distance x_1, has to be made for the latter, whereas, for the point P, the contribution of the left-end section is made with the distance $(x_1+\xi)$, and for the right-hand end section with the distance $(x_1-\xi)$. Combining these, a difference formula may be derived for the change $\Delta' M$, due to displacement, as far as distance x_1 is concerned. Similarly, the application of a difference formula for distances x_2 and x_3, enables $\Delta'' M$ and $\Delta''' M$ to be found. The desired total change is given by $\Delta M=\Delta' M-\Delta'' M+\Delta''' M$. This change ΔM gives the difference in mutual inductance of point P from that at the point in the median plane having the same radius.

Likewise, these two basic series formulas may be applied to find the difference in mutual inductance between a chosen point in the median plane, having a radius y, and the reference central filament of radius y_0. These latter calculations enable us to refer the calculated ΔM for all points to the reference value of mutual inductance of the central filament.

Details of these difference formulas are given in the appendix. Even using the difference equations, the results have still to be found by combining terms that partly cancel, and careful checking is necessary to detect errors. The work is necessarily arduous.

5. Distribution of Flux Over the Cross Section

5.1. Campbell Type

For purposes of comparison, a primary having the same winding density and about the same radius as the National Bureau of Standards type is assumed, the details of its design being

Radius of primary $=a=20$ cm.
Winding density, $n=5$ turns per centimeter.
Axial length of each primary section $=10$ cm.
Mean radius of the secondary $=1.4582\ a=29.164$ cm.

With this secondary radius, the secondary turn at the center of the cross section lies very closely in the circle of magnetic field intensity zero.

There are 100 primary turns and 594 secondary turns; the mutual inductance would be about the same as that of the National Bureau of Standards mutual-inductance standard.

Assuming for the secondary coil a square cross section, 1.5 cm on a side, 49 values of mutual-inductance values, referred to the value at the center of the cross section, were calculated, using the difference formulas of the appendix. The grid of these values, which are spaced at intervals of 0.25 cm of Δy and 0.25 cm of distance ξ from the median plane are shown in figure 5. The maximum deviation from the value at the center is about 850 ppm. The values

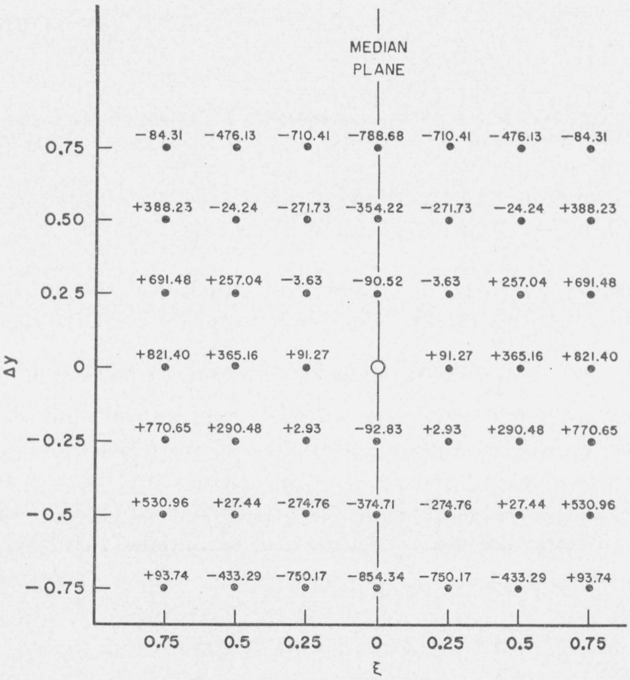

FIGURE 5. *Calculated values of variations of mutual inductance from turn over the cross section—Campbell type.*

of ΔM are symmetrically arranged with respect to the median plane and are mostly plus; those in the median plane are nearly symmetrical about the mean radius and are negative. (It was noted, during the calculations that, for a given radius, ΔM is very closely proportional to ξ^2. This is true also for the other cases investigated.)

Wilmotte [10] has derived expressions for the change δM in mutual inductance for points of the cross section of the Campbell type of standard, referred to the value at the central point of the section. His eq (4) expresses this as a function of the coordinates x and y, referred to the center point as origin, and differential coefficients of the radial component of the magnetic field. This expression, which has recently been brought to the attention of the author, in form seems to be similar to the formula (5) of this paper.

Wilmotte's eq (5) expresses δM in polar coordinates ($\delta r,\theta$), referred to the center of the cross section as origin. The principal term of this shows that δM is proportional to $\delta r \cos 2\theta$. An inspection of figure 5 shows that this law is here approximately obeyed. The slight lack of symmetry about the mean radius in figure 5 is in line with the second term of Wilmotte's formula (5), which is proportional to $\sin^3\theta$, and involves higher-order differential coefficients of the radial field component. This term would add to the principal term amounts of opposite sign for points of greater radii than the radius of the center point than it would for points of smaller radius than this value. It may therefore be concluded that formally, at least, the results of this investigation confirm Wilmotte's equation.

To obtain the average deviation of the mutual inductance of a secondary turn from that of the central turn, the deviations have to be integrated over the cross section and the result divided by the area of the cross section. This integration may be accomplished by means of a suitable mechanical integration formula, such as Simpson's rule. Using this with the values of ΔM of figure 5, there is found a correction for cross section of +0.71 ppm.

A check on this value is provided by the well-known quadrature formula derived from the Taylor's series formula (4). This gives the average mutual inductance of an array of filaments distributed over a rectangular section $2\Delta y$ by 2ξ in the form

$$M=\frac{N_2}{6}(M_1+M_2+M_3+M_4+2M_0), \tag{7}$$

in which

M_0 = the mutual inductance of the central filament,
M_1 = the mutual inductance of the filament at $(\Delta y,0)$,
M_2 = the mutual inductance of the filament at $(-\Delta y,0)$,
M_3 = the mutual inductance of the filament at $(0,\xi)$,
M_4 = the mutual inductance of the filament at $(0,-\xi)$.

Replacing these by $M_0=\mu+\delta_0$, $M_1=\mu+\delta_1$, etc., in which μ is the mutual inductance of the reference filament, which can be outside the area considered, we have the formula

$$M=N_2\mu[1+\tfrac{1}{6}(\Delta_1 M+\Delta_2 M+\Delta_3 M+\Delta_4 M+2\Delta_0 M)], \tag{8}$$

which expresses the average mutual inductance for any subdivision of the secondary cross section in terms of the mutual inductance μ of the central filament of the secondary and the deviations ΔM of the grid of values. Applying this quadrature formula to the nine component square sections of which the secondary may be considered to consist, and remembering that each section consists of only one-ninth of the total secondary turns N_2, we have to average the following nine values:

$$\begin{array}{ccc} -25.18, & -355.18, & -25.18, \\ +365.08, & -0.13, & +365.08, \\ +28.04, & -374.38, & +28.04, \end{array}$$

which yields the total correction for cross section of +0.69 ppm, which checks the other integration method. The correction was calculated also from the Taylor's series formula, with the result:

Correction for residual magnetic field at the center	+0.013
Fourth-order terms	+0.232
Total ppm	+0.25

The reason for this surprisingly small correction may be made clearer in what follows.

In the median plane the mutual inductance is a maximum for the central filament. For a given radius, the mutual inductance is a minimum in the median plane. Thus, the point of zero magnetic field intensity is located in the center of a saddle-like region of flux distribution. This is made clear by figure 6, which is a plot of the loci of points having the same values of ΔM. These curves are obtained from graphical interpolation from the grid of calculated values of ΔM in figure 5. In figure 6 each curve is labelled with the corresponding value of ΔM in ppm. Each curve is the locus of filaments linked with the same flux. That is, each

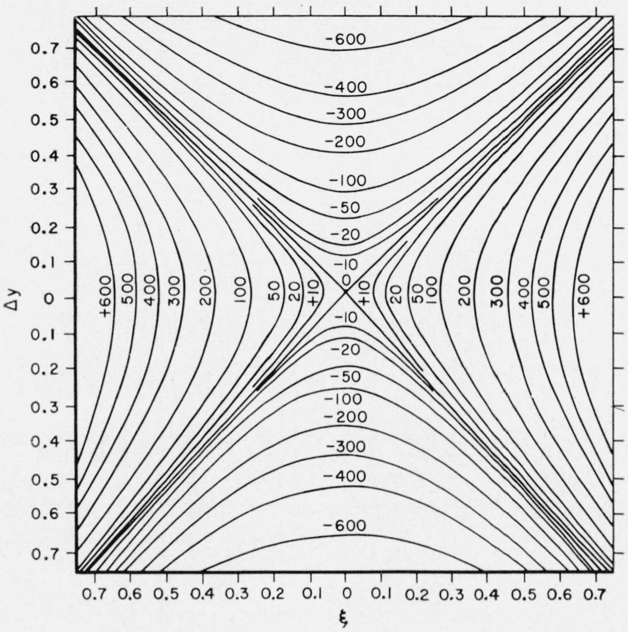

FIGURE 6. *Loci of turns of equal ΔM over the cross section of the Campbell type.*

locus traces a line of magnetic flux, and the figure gives a picture of the flux distribution over the cross section of the secondary. Assuming the positive direction of a flux line along the axis of the primary as from left to right, the direction of the lines in the lower quarter of figure 6 is from left to right, and of those in the upper quarter from right to left. Those in the right-hand quarter are directed inward toward the axis, and those in the left-hand quarter outward from the axis.

Figure 6 makes clear that the very small total resulting correction for cross section is the result, not of small deviations of mutual inductance of the filaments, but from the very close balance of positive and negative deviations.

5.2. National Bureau of Standards Type, Case C

The treatment of this case follows the same lines as for the Campbell type. A grid of 45 calculated values of ΔM spaced at 0.25-cm intervals of Δy and ξ over the cross section, 2 cm by 1 cm in dimensions, is shown in figure 7. These values of ΔM are distributed symmetrically about the median plane, as would be expected, but about the radius of the center of the cross-section (25.75 cm) positive values on one side are nearly offset by negative values on the other. Furthermore, excepting for the corner points, the actual magnitudes of the ΔM are less than 100 ppm, compared with values 5 or 6 times as great in the Campbell type.

For a given radius, ΔM, is closely proportional to ξ^2, a fact that does not appear in the figure, where all values are referred to the value for the point at the center of the cross section.

In figure 7 are sketched in the loci $H_r=0$ and $H_x=0$. Along the former $dM/d\xi=0$ and along the latter $dM/dy=0$. These facts are checked in a general way in the figure.

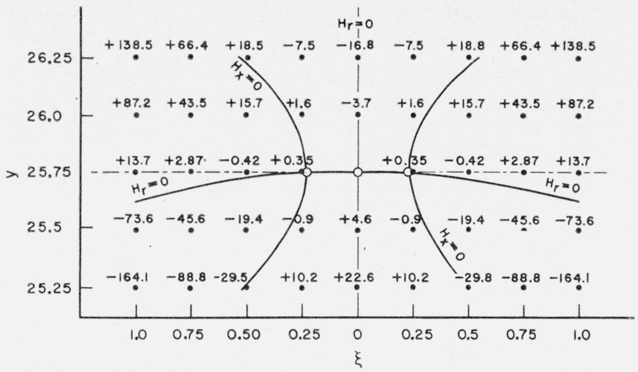

FIGURE 7. *Calculated values of mutual-inductance variations over the cross section—three-section type, case C.*

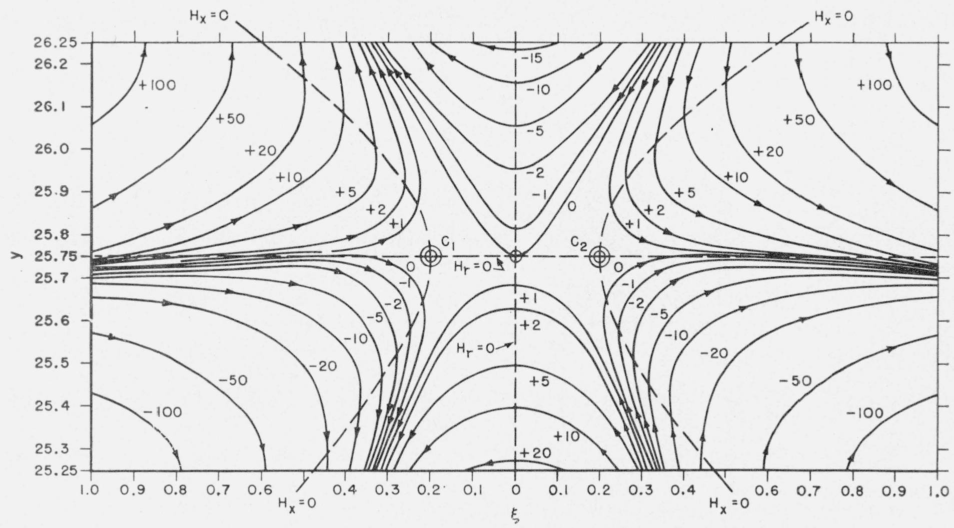

FIGURE 8. *Curves of equal ΔM over the cross section for case C.*

Averaging the values of figure 7 over the cross section by Simpson's rule, the correction found is -0.44 ppm, and by applying the quadrature formula to the four square sections into which either half of the total cross section may be divided, the result is -0.38 ppm. Either half of the cross section may be replaced by the Lyle filament to which the square section is equivalent. This has a radius of $25.75[1+(1/24)(1/25.75)^2]$, and the coordinate $\xi=0.5$. It is therefore, closely coincident with the filament, (25.75, 0.5) in figure 7, for which $\Delta M=-0.42$. Thus this value may be taken as the value of the correction given by the Lyle method. Thus, the conclusion is amply checked that, for case C, the correction for cross section is not far from -0.4 ppm.

Figure 8, which is derived from the values of figure 7, gives a picture of the distribution of ΔM over the cross section and the shape of the flux lines over the whole region. This figure demonstrates in a striking way the smallness of the deviations in mutual inductance of the filaments of the secondary. The directions along the flux lines are indicated by arrows on the loci of equal values of ΔM, and also there are indicated the loci of $H_r=0$ and $H_x=0$. At their intersections with the curves for $H_x=0$, the ΔM should have a radial direction, and this is substantiated by the curves as well as can be expected. C_1 and C_2 are the points of zero magnetic field intensity.

It may be noted that, in passing in a circuit around the center of the cross section, there are six regions of the field in which ΔM is alternately plus and minus, whereas for the Campbell type there are four. This suggests that with a primary of n sections there would be $2n$ such regions.

5.3. National Bureau of Standards Type, Case A

For comparison with case C, figure 9 has been plotted for case A. The data for this was obtained for the median plane from the curve of mutual inductance for points in the median plane, figure 2, curve A. Values of the difference in mutual inductance at the boundaries

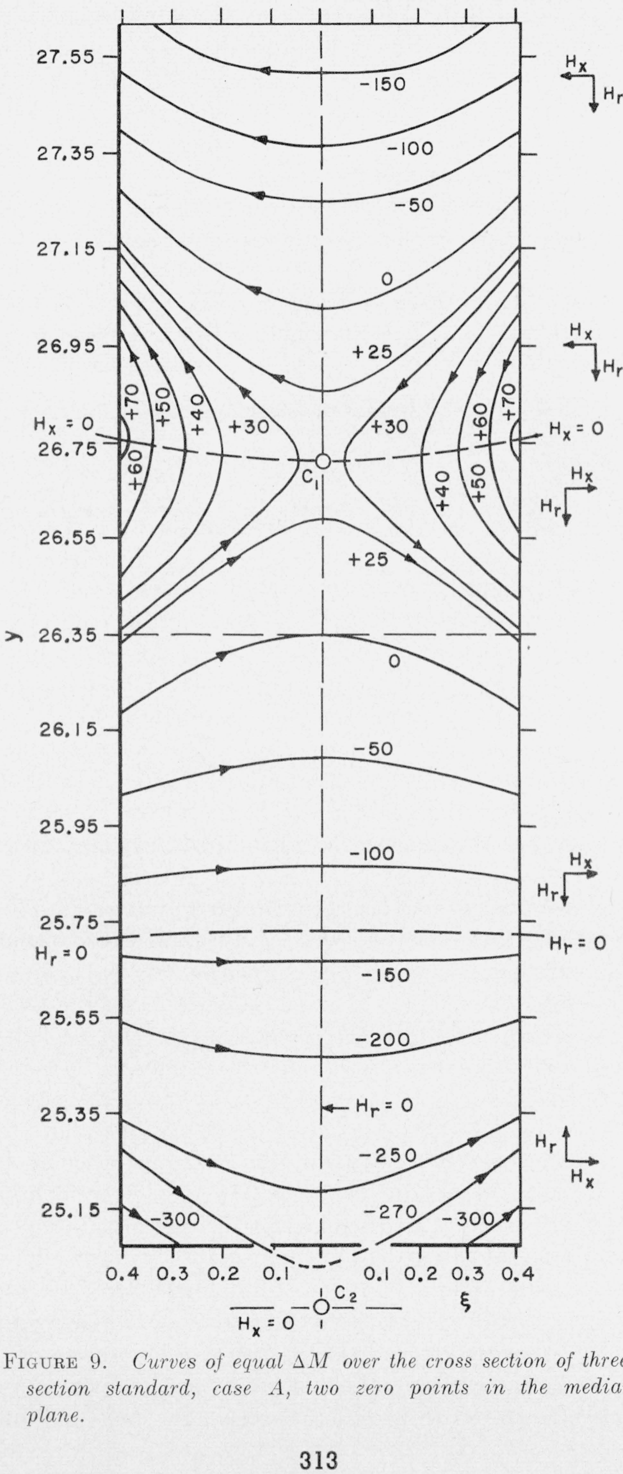

FIGURE 9. *Curves of equal ΔM over the cross section of three-section standard, case A, two zero points in the median plane.*

$\xi = \pm 0.41$ and for the median plane, at the same radius, were found for three values of the radius, namely, in the center and at the extreme corners. These values are relatively small, ranging from +59 ppm for the larger radius, +29 for the mean radius, and −47 for the smallest radius. From these, other values on the boundary were derived by graphical interpolation. For interior points, the values were obtained from the law of proportionality to ξ^2, which applies in all these cases (see fig. 19, RP2029). The grid so obtained checked the axial correction of +7.2 ppm given on p. 316, RP2029, to one-tenth part in a million.

In figure 9 are sketched also the loci $H_r=0$ and $H_x=0$. The points of zero field intensity are indicated by C_1 and C_2. The directions of the flux at different parts of the field are indicated by arrows on the lines of flux mapped by the loci of equal values of ΔM, and also in the margin of the figure are indicated the directions of the axial and radial components of the field in different regions. The corresponding diagram to the left of the cross section are not indicated, but they are at once evident if it is remembered that the radial component reverses in passing the median plane, and the axial component is continuous.

The range of variation of mutual inductance over the cross section in only about 200 ppm on either side of the average, which is nearly as good as for case C, and 4 or 5 times better than the Campbell type, but the marked lack of symmetry about the radius of the center of the cross section ○ is evident. The center of gravity of the values (the point where $M=-75.8$ ppm) lies at about $y=25.97$ cm in the median plane, that is, about 0.35 cm below the center of the cross section.

6. Conclusions

1. The inherent balancing out of the plus and minus variations of the mutual inductance from turn to turn over a square cross section has been noted and illustrated. This is in line with the well-known fact that, in mutual-inductance calculations, the geometric mean distance of a square area from an external point is very closely equal to the distance from the center of the square to the external point.

2. In the Campbell type of standard, for this reason, the correction for cross section is surprisingly small, even with rather large variations of the mutual inductance among the turns over the cross section. However, in order to be assured of this satisfactory balance, uncertainties in the actual placing of the turns must be minimized. The example here illustrated assumed a cross section greater than is necessary. The NPL standard employs a square cross section of secondary only 1 cm on a side, and, for such a coil, the extreme variations ΔM would be of the order of less than half those in the example above. Where the use of a finer wire on the secondary offers no disadvantage, the Campbell type has the advantage of simplicity.

3. For a given area of secondary cross section, the three-section standard has been proved to have much smaller variations ΔM over the cross section than the Campbell type. This is true both for case A and case C. However, case A shows a lack of symmetry of the distribution about the central point, and the balancing of values of opposite signs is not taken advantage of. The correction for cross section may be calculated to better than 1 part in a million, but its actual value appears large in comparison with the other cases.

An adjustment of the shape ratios of the three-section standard so as to attain the special case of a single circle of zero magnetic field intensity in the median plane, and the use of a secondary of square cross section, accurately centered in this zero region, would seem to be an optimum arrangement, but the adjustment would be very critical. A good approximation to this is offered by the arrangement case C. In the example of this given above, there is a secondary that is essentially a combination of two equal square cross sections of equal mean radius, and the symmetry about the median plane leads to a total correction for cross section which is the same as that for either half of the cross section. Furthermore, each square component is centered in a region of nearly zero magnetic field intensity. The range of the values of ΔM is in general much smaller than the extreme values of about ±150 ppm, and it is probable that the secondary could actually be wound to have a correction for cross section not exceeding a part in a million. A refinement on this case would be to slightly adjust the dimensions so that the two zero points would be located exactly at the centers of the component squares that form the cross section, but this would also be a critical adjustment.

4. A thorough study of the distribution of the ΔM over the cross section by calculating the values for a grid of points is very time consuming. The abbreviated methods already discussed will, in general, be used. The Taylor's series formula should be applied to subsections of dimensions sufficiently small to require the calculation of second-order terms only. However, if the field intensity H_x is not zero at the center of the subsection, its value must be calculated and correction made. For case A, two subsections are found to suffice. Since, however, calculations of $(H_x)_0$ and $(d^2m/dx^2)_0$ are necessary for each, the calculation of M directly for equivalent Lyle filaments is to be recommended. Especially is this true, if the given cross section may be divided into square subsections. For each of these a single Lyle filament is required and the average of the mutual inductance calculated for each of these, averaged to find the final corrected value of the mutual inductance. It is only necessary to be sure that subsections are of small enough area to ensure that fourth-order differentials are negligible.

With case A it has been shown that division into two subsections, not squares, calculating and averaging the mutual inductances of the four corresponding Lyle filaments gives a very accurate result. It is interesting in this connection to note that the cross section in this case may be divided quite closely into three square subsections. If this were accurately the case, only three Lyle filaments would be necessary, and the subsections would be so small as to make certain that fourth-order differentials would be negligible.

For case C, we have an especially favorable case with the cross-sectional dimensions chosen, i. e., $b=2$, $c=1$, in that the Lyle filaments for the whole cross section are the same as the Lyle filaments for the two equal square subsections. The residual field is very small over each subsection, and higher-order differentials are negligible. The calculation of the mutual inductance of the single Lyle filament for a square subsection gives a very accurate determination of the corrected mutual inductance per turn. The final value to be used for the mutual inductance of the whole secondary is merely N_2 times this value.

7. Appendix

7.1. Evaluation of Differential Coefficients

Methods for calculating the differential coefficients that appear in the general Taylor's series formula for the evaluation of the correction for the distribution of the winding may be based on two separate methods of approach, (a) on the formula for the radial component, per unit current, of the magnetic field intensity of a circular filament exerted at a point P distant y_0 from the axis and x from the plane of the filament, and (b) on the formula for the mutual inductance of two coaxial circular filaments, m_f, one of them of radius a and the other y_0, the distance between their planes equal to x. Since in each case computations have to be made for the three distances x_1, x_2, and x_3, the basic elliptic integral formula has to be used in both methods, no single series development being sufficiently convergent for all three values of x.

a. Method A

Placing H_0 for the radial component of the field, due to a unit current, in a circular filament of radius a at a point P distance y_0 from the axis and x from the plane of the filament

$$\left(\frac{d^2m}{dx^2}\right)_0=-2\pi y_0(2n)\,[H_{01}-H_{02}+H_{03}]. \tag{A1}$$

Calculations made for the three distances, x_1, x_2, and x_3 are combined as shown; n equals the winding density of the primary winding, and the factor 2 takes account of the symmetry about the median plane.

H_0 is calculated by the Russell elliptic integral formula [11]

$$H_0=\frac{2x}{r_1y}\left[\frac{2ay}{r_2^2}E-(K-E)\right], \tag{A2}$$

in which K and E are the complete elliptic integrals of the first and second kinds to modulus k.

$$k^2=\frac{4ay}{r_1^2},\ k'^2=1-k^2=\frac{r_2^2}{r_1^2},\ r_1^2=(y+a)^2+x^2,\ r_2^2=(y-a)^2+x^2. \tag{A3}$$

The coefficient $(d^2m/dy^2)_0$ is derived from the relation

$$\left(\frac{d^2m}{dy^2}\right)_0=2\pi(H_x)_0-\left(\frac{d^2m}{dx^2}\right)_0, \tag{A4}$$

in which $(H_x)_0$ is the residual axial component, per unit current, of the magnetic field at the center of the cross section.

$$H_x=n\left[\frac{4ax}{r_1(a+y)}K+\frac{2c'x}{r_1}(K-\Pi)\right]\text{with } c'=\frac{y-a}{y+a},\ c^2=1-c'^2. \tag{A5}$$

In this expression, Π is an elliptic integral of the third kind to modulus k and parameter c. Numerical values of $K-\Pi$ may be computed by the methods illustrated in tables 5, 6, and 7 of reference [3], or alternatively, they may be calculated by use of the relation $K-\Pi=[c(y+a)/c'kx]\,\psi$, where ψ is Legendre's elliptic function defined in eq (23) of reference [4]; values of ψ, K, and E, may be calculated by use of tables given in the latter reference. Using eq (A5) H_x must be evaluated for the three distances x, combined as in eq (A1), and multiplied by 2.

For the fourth-order differential coefficients we have

$$\left.\begin{aligned}\left(\frac{d^4m}{xd^4}\right)_0&=-2\pi(2n)\left[y_0\left(\frac{d^2H_0}{dx^2}\right)_0\right]\\ \left(\frac{d^4m}{dx^2dy^2}\right)_0&=-2\pi(2n)\left[2\left(\frac{dH_0}{dy}\right)_0+y_0\left(\frac{d^2H_0}{dy^2}\right)_0\right]\\ \left(\frac{d^4m}{dy^4}\right)_0&=2\pi(2n)\left[3\left(\frac{dH_0}{dy}\right)_0+y_0\left(\frac{d^2H_0}{dy^2}\right)_0\right].\end{aligned}\right\} \tag{A6}$$

Each differential coefficient in (A6) has to be calculated for each of the distances x and the results combined as in (A1). The common factor $2n$ takes into account the winding density of the primary and the symmetry about the median plane.

The differential coefficients of H_0 were obtained by differentiation of the elliptic integral formula (A2). This offers no especial difficulty, but the result may be transformed in a number of ways in each case. The forms which follow are believed to have some advantages.

$$\frac{dH_0}{dy}=\frac{2x}{r_1y^2}\left[\mu(K-E)-\frac{2ay}{r_2^2}\nu E\right], \tag{A7}$$

where

$$\left.\begin{aligned}\mu&=1+\frac{y^2}{r_1^2}+\frac{2ay^2(y-a)}{r_1^2r_2^2}\\ \nu&=1-\frac{y(y+a)}{r_1^2}+\frac{2y(y-a)}{r_2^2}\end{aligned}\right\} \tag{A8}$$

$$y\frac{d^2H_0}{dy^2}=\frac{2x}{r_1y^2}\left[\rho(K-E)-\frac{2ay}{r_2^2}\sigma E\right], \tag{A9}$$

with

$$\left.\begin{aligned}\rho&=-2-\frac{y^2}{r_1^2}+\frac{y^2[(y-a)^2+2a^2]}{r_1^2r_2^2}-\frac{2y^2(y+a)^2}{r_1^4}-\frac{2y^2(y-a)^2}{r_2^4}\\ \sigma&=-2-\frac{y^3}{r_1^2}+\frac{3ay}{r_2^2}+\frac{4y^2(y+a)^2}{r_1^4}-\frac{y^2(y+a)(3a-y)}{r_1^2r_2^2}-\frac{8y^2(y-a)^2}{r_2^4}\end{aligned}\right\} \tag{A10}$$

and

$$y\,\frac{d^2H_0}{dx^2}=\frac{4x(y-a)^2}{r_1r_2^4}\left[\psi(K-E)-\frac{8ay}{r_2^2}\,\varphi E\right], \tag{A11}$$

in which

$$\left.\begin{aligned}\psi&=1-\frac{1}{2}\,\frac{(y^2+a^2)}{(y-a)^2}\cdot\frac{r_2^2}{r_1^2}+\frac{(y+a)^2}{(y-a)^2}\cdot\frac{r_2^4}{r_1^4},\\ \varphi&=1-\frac{1}{8}\,\frac{[(y+a)^2+4ay]}{(y-a)^2}\cdot\frac{r_2^2}{r_1^2}-\frac{1}{2}\,\frac{(y+a)^2}{(y-a)^2}\cdot\frac{r_2^4}{r_1^4}\end{aligned}\right\}. \tag{A12}$$

The modulus of the elliptic integrals and the equations for r_1^2 and r_2^2 are the same as in the formulas (A2) and (A5) above.

b. Method B

In this method of approach, the base is the formula for m_f, the mutual inductance of two coaxial circular filaments of radii a and y, with a distance x between their planes.

$$\left(\frac{d^2m}{dx^2}\right)_0=2n\left[\left(\frac{dm_f}{dx}\right)_1-\left(\frac{dm_f}{dx}\right)_2+\left(\frac{dm_f}{dx}\right)_3\right], \tag{A13}$$

which indicates that calculations have to be made for the three distances x_1, x_2, and x_3 and the results combined as in (A13). The common factor $2n$ takes into account the winding density of the primary winding and the symmetry about the median plane.

The fourth order differential coefficients are

$$\left.\begin{aligned}\left(\frac{d^4m}{dx^4}\right)_0&=2n\left[\left(\frac{d^3m_f}{dx^3}\right)_1-\left(\frac{d^3m_f}{dx^3}\right)_2+\left(\frac{d^3m_f}{dx^3}\right)_3\right],\\ \left(\frac{d^4m}{dx^2dy^2}\right)_0&=2n\left[\left(\frac{d^3m_f}{dy^2dx}\right)_1-\left(\frac{d^3m_f}{dy^2dx}\right)_2+\left(\frac{d^3m_f}{dy^2dx}\right)_3\right]\end{aligned}\right\} \tag{A14}$$

and from these and the general relation

$$\left(\frac{d^4m}{dy^4}\right)=\frac{1}{y^2}\left(\frac{d^2m}{dx^2}\right)-\frac{d^4m}{dx^2}-2\,\frac{d^4m}{dx^2dy^2} \tag{A15}$$

is derived the value of $(d^4m/dy^4)_0$.

Starting with the Maxwell elliptic integral formula for the mutual inductance of two coaxial circular filaments in its integral form

$$m_f=4\pi ay\int_0^{\pi}\frac{\cos\theta\,d\theta}{(a^2+y^2+x^2-2ay\cos\theta)^{\frac{1}{2}}} \tag{A16}$$

the required differentiations are made under the integral sign. In the results occur integrals of the form

$$I_n=\int_0^{\frac{\pi}{2}}\frac{d\varphi}{[1-k^2\sin^2\varphi]^{n/2}},$$

which are related by a recursion formula. We have, in fact,

$$I_3=E/k'^2,\quad I_5=\frac{2(2-k^2)I_3-K}{3k'^2},\quad I_7=\frac{4(2-k^2)I_5-3I_3}{5k'^2}. \tag{A17}$$

Introducing then the nomenclature

$$\rho_3 = p^2 I_3 - q^2 K, \quad \rho_5 = p^2 I_5 - q^2 I_3, \quad \rho_7 = p^2 I_7 - q^2 I_5, \tag{A18}$$

in which $p^2 = (2-k^2)/k^2$, and $q^2 = 2/k^2$; the differential coefficients may then be written in the forms convenient for calculation

$$\frac{d^2m}{dx^2} = -\frac{8\pi ayx(2n)}{r_1^3}\,[p^2 I_3 - q^2 K] = -\frac{8\pi ayx(2n)\,\rho_3}{r_1^3} \tag{A19}$$

$$\frac{d^4m}{dx^4} = \frac{72\pi ayx(2n)}{r_1^5}\left[\rho_5 - \frac{5}{3}\frac{x^2}{r_1^2}\,\rho_7\right] \tag{A20}$$

$$\frac{d^4m}{dx^2dy^2} = \frac{72\pi ayx(2n)}{r_1^5}\left[\rho_5 - \frac{2}{3}\frac{a}{y}(p^2\rho_5 - q^2\rho_3) - \frac{5}{3}\rho_7\left(\frac{y^2}{r_1^2} - \frac{2ay}{r_1^2}p^2 + \frac{a^2}{r_1^2}p^4\right) - \frac{10}{3}\rho_5 q^2\left(\frac{ay}{r_1^2} - p^2\frac{a^2}{r_1^2}\right) - \frac{5}{3}\frac{a^2}{r_1^2}q^4\rho_3\right]. \tag{A21}$$

In all these elliptic integral equations the modulus and nomenclature is the same as above. For each differential coefficient calculations for all three values of x have to be made and combined in the standard way.

Results of calculations of numerical problems give results by the two methods, show that methods A and B closely agree in all cases.

7.2. Difference Formulas

These give the difference in mutual inductance ΔM between the value for a circular filament in the median plane and two equal solenoids of length x, and the value when the filament is displaced a distance ξ toward one solenoid and the same distance away from the other.

For the longer distances, x_1, Rosa's formula [8] is made the base

$$M = 2\pi^2 a\alpha\rho n x Q, \tag{a}$$

in which $\alpha = a/y$, and $\rho^2 = y^2/(y^2+x^2)$. In the median plane $x = x_1$ for both sections: for a displacement ξ one becomes $(x_1+\xi)$ and the other $(x_1-\xi)$.

Adopting the nomenclature

$$\begin{array}{ccc} x_1 - |\xi| & x_1 & x_1 + |\xi| \\ Q + q_2 & Q & Q + q_1 \\ \rho + r_2 & \rho & \rho + r_1 \end{array}$$

$$\frac{\Delta_1 M}{M} = \frac{r_1}{\rho}\left[1 + \left|\frac{\xi}{x_1}\right|\right] + \frac{r_2}{\rho}\left[1 - \left|\frac{\xi}{x_1}\right|\right] + \frac{q_1}{Q}\left[1 + \left|\frac{\xi}{x_1}\right| + \frac{r_1}{\rho}\left(1 + \left|\frac{\xi}{x_1}\right|\right)\right] + \frac{q_2}{Q}\left[1 - \left|\frac{\xi}{x_1}\right| + \frac{r_2}{\rho}\left(1 - \left|\frac{\xi}{x_1}\right|\right)\right],$$

with

$$Q = 1 + \frac{3}{8}\alpha^2\rho^4\left\{1 - \frac{5}{6}\alpha^2\rho^2\xi_2 + \frac{35}{48}\alpha^4\rho^4\xi_4 - \frac{21}{32}\alpha^6\rho^6\xi_6 + \frac{77}{128}\alpha_8\rho^8\xi_8 - \frac{143}{256}\alpha^{10}\rho^{10}\xi_{10} + \ldots\right\},$$

in which

$$\xi_2 = 1 - \frac{7}{4}\rho^2, \quad \xi_4 = 1 - \frac{9}{2}\rho^2 + \frac{33}{4}\rho^4$$

$$\xi_6 = 1 - \frac{33}{4}\rho^2 + \frac{143}{8}\rho^4 - \frac{715}{64}\rho^6$$

$$\xi_8 = 1 - 13\rho^2 + \frac{195}{4}\rho^4 - \frac{1165}{16}\rho^6 + \frac{4199}{128}\rho^8$$

$$\xi_{10} = 1 - \frac{75}{4}\rho^2 + \frac{425}{4}\rho^4 - \frac{8072}{32}\rho^6 + \frac{33915}{128}\rho^8 - \frac{52003}{512}\rho^{10}$$

$Q+q_1$ and $Q+q_2$ are derived from the expression for Q by replacing x_1 by $(x_1+\xi)$ and $(x_1-\xi)$, respectively, and ρ by $(\rho+r_1)$ and $(\rho+r_2)$.

Finally, $\Delta_1 M$ is obtained from $\Delta_1 M/M$, using equation (a) to calculate M.

For shorter distances x_2 (or x_3), the basic formula, adapted from Dwight [9], is

$$M=2\pi n x\sqrt{ay}\,G, \tag{b}$$

in which

$$G=\left(\lambda+\frac{\beta^2}{4}\lambda_1-\frac{3}{64}\beta^4\lambda_2+\frac{5}{256}\beta^6-\ldots\right)\log_e\frac{16}{\gamma^2+\beta^2}+\nu+\frac{\beta^2}{4}\nu_1\gamma^2+\frac{5}{64}\beta^4\nu_2-\frac{31}{768}\beta^6+\ldots-2-2\mu\frac{\gamma}{\beta}\tan^{-1}\frac{\beta}{\gamma}$$

with

$$\gamma^2=\frac{(y-a)^2}{4ay}, \qquad \beta^2=\frac{x^2}{4ay}$$

$$\lambda=1+\frac{3}{4}\gamma^2-\frac{15}{64}\gamma^4+\frac{35}{256}\gamma^6-\ldots, \qquad \nu=\frac{1}{2}\gamma^2+\frac{15}{64}\gamma^4-\frac{151}{768}\gamma^6+\ldots$$

$$\lambda_1=1-\frac{5}{8}\gamma^2+\frac{35}{64}\gamma^4-\ldots, \qquad \nu_1=1-\frac{209}{192}\gamma^2+\ldots$$

$$\lambda_2=1-\frac{7}{4}\gamma^2+\ldots, \qquad \nu_2=1-\frac{43}{20}\gamma^2+\ldots$$

$$\mu=1+\frac{1}{2}\gamma^2-\frac{1}{8}\gamma^4+\frac{1}{16}\gamma^6-\ldots.$$

The distances $x=x_2$, $x=x_2+|\xi|$, and $x=x_2-|\xi|$ are substituted for x to find G, $G+g_1$, and $G+g_2$, and

$$\frac{\Delta_2 M}{M}=\frac{g_1\left[1+\left|\frac{\xi}{x_2}\right|\right]+g_2\left[1-\left|\frac{\xi}{x_2}\right|\right]}{G}$$

from whence, using $x=x_2$ in (b) to calculate M, we find $\Delta_2 M$.

To calculate the difference between the mutual inductance for a filament with radius y, situated in the median plane and the mutual inductance of the central point with radius y_0, we have for the longer distances x_1,

$$\frac{\Delta M}{M_0}=\frac{1}{2}\delta^2-\frac{1}{8}\delta^4+\frac{1}{16}\delta^6-\frac{5}{128}\delta^8+\ldots+\frac{q}{Q_0}\sqrt{1+\delta^2},$$

in which $\delta^2=y_0^2-y^2)/(y^2+x_1^2)$ and Q_0 is calculated from the formula for Q, with $x=x_1$, and with $y=y_0$ in the equation for ρ, and (Q_0+q) is the corresponding value from the formula for Q with $x=x_1$ and y used in the equation for ρ.

For the shorter distances x_2 (or x_3), the formula becomes

$$\frac{\Delta M}{M_0}=\frac{\epsilon}{2}-\frac{\epsilon^2}{8}+\frac{\epsilon^3}{16}-\frac{5}{128}\epsilon^4\ldots+\frac{g}{G_0}\sqrt{1+\epsilon},$$

in which $\epsilon=(y-y_0)/y_0$ and G_0 is derived from the above formula for G with $x=x_2$ and $y=y_0$, and g is the increment in this when y_0 is replaced with y.

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