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Evaluation of the Exponential Integral for Large Complex Arguments

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Two methods of evaluating the exponential integral for (large) complex arguments are discussed. It is shown that the Laguerre quadrature method is more efficient than the asymptotic expansion. Examples are given to show the practicability of the Laguerre method.

1. Introduction

An extensive table of the exponential integral

$$E_1(z)=\int_z^\infty (e^{-u}/u)du \qquad z=x+iy$$

has been prepared by the National Bureau of Standards [1]; [1] the introduction to the table gives a precise definition of this function. This table covers the range $|x|\leq 20$, $|y|\leq 20$, with arguments variously spaced. In order to compute $E_1(z)$ outside this range, (or within this range, at points where interpolation is awkward, which are those with comparatively large argument), several methods can be considered. We examine two in detail and recommend the Laguerre quadrature, which is certainly practicable in the case of isolated arguments.

2. Asymptotic Series

The first of these is the use of the asymptotic series (see e. g., [1]),

$$E_1(z)\sim e^{-z}\left(\frac{1}{z}-\frac{1!}{z^2}+\frac{2!}{z^3}-\frac{3!}{z^4}+\cdots\right).$$

The remainder after n terms in the expansion for $e^zE_1(z)$ can be obtained by integration by parts and is

$$R_n=(-)^n n! e^z\int_z^\infty \frac{e^{-u}}{u^{n+1}}\,du.$$

If we put $u=z+\rho$, where ρ is real, then

$$R_n=(-)^n n!\int_0^\infty \frac{e^{-\rho}d\rho}{[(x+\rho)+iy]^{n+1}},$$

which can be estimated as

$$|R_n|\leq S_n=n![x^2+y^2]^{-\frac{1}{2}(n+1)} \qquad \text{if } x\geq 0$$

$$|R_n|\leq T_n=n!|y|^{-n-1} \qquad \text{if } x\leq 0,$$

from which the asymptotic character of the series is evident. In fact,

$$\frac{S_{n+1}}{S_n}=\frac{n+1}{(x^2+y^2)^{\frac{1}{2}}},$$

and so S_n decreases as n increases until n exceeds $(x^2+y^2)^{\frac{1}{2}}$. To obtain the least the least value of S_n we have to take $n\doteq(x^2+y^2)^{\frac{1}{2}}=|z|$. For this value of n, we find, using Stirling's formula, that S_n is about $(2\pi/n)^{\frac{1}{2}}e^{-n}$. In the second case, the least value of T_n is formally the same, but

[1] Figures in brackets indicate the literature references at the end of this paper.

now $n \doteq |y|$. The decrease in the effectiveness of this representation as z approaches the negative real axis is apparent from the estimates.

We note that the evaluation of the sums of the first n terms of the series for $e^z E_1(z)$, as a polynomial in z^{-1}, by successive multiplication by z^{-1} for instance, involves about $4n$ real multiplications, z being complex.

3. Laguerre Quadrature

The second method is the use of a Laguerre quadrature. We shall show that this is more efficient than the asymptotic series method just described.

We have

$$e^z E_1(z) = I_1 - iI_2,$$

where

$$I_1 = \int_0^\infty e^{-\rho} \cdot \frac{x+\rho}{(x+\rho)^2+y^2} \cdot d\rho, \quad I_2 = \int_0^\infty e^{-\rho} \cdot \frac{y}{(x+\rho)^2+y^2} \cdot d\rho.$$

It is well known [3] that, for any n,

$$I = \int_0^\infty e^{-t} \cdot f(t) \cdot dt \doteq \Sigma \lambda_i^{(n)} f(x_i^{(n)}) = Q,$$

where the $x_i^{(n)}$ are the zeros of the Laguerre polynomial $L_n(t)$, and the $\lambda_i^{(n)}$ are the corresponding Christoffel numbers. The $x_i^{(n)}$, $\lambda_i^{(n)}$ have been tabulated for $n=1(1)15$, $i=1(1)n$ by Salzer and Zucker [2].

Moreover, it is known [3] that for some ξ, $0 \leq \xi \leq \infty$,

$$R = [I - Q] = (n!)^2 \frac{f^{(2n)}(\xi)}{(2n)!}.$$

We shall now estimate this error. It is convenient to handle the two integrals separately. We calculate the $2n$th derivatives of

$$f_1(t) = \frac{x+t}{(x+t)^2+y^2}, \quad f_2(t) = \frac{y}{(x+t)^2+y^2}$$

and find

$$R_1 = (n!)^2 r_1^{-2n-1} \cos(2n+1)\theta_1,$$

$$R_2 = -(n!)^2 r_2^{-2n-1} \sin(2n+1)\theta_2,$$

where the $\xi_i = \xi_i(x, y)$ satisfy $0 \leq \xi_i \leq \infty$, $r_i = [(x+\xi_i)^2 + y^2]^{\frac{1}{2}}$, and θ_i is defined by $\cos\theta_i = (x+\xi_i)/r_i$, $\sin\theta_i = y/r_i$.

Hence

$$|R_i| \leq \frac{(n!)^2}{[x^2+y^2]^{n+\frac{1}{2}}} \qquad x \geq 0,$$

$$|R_i| \leq \frac{(n!)^2}{|y|^{2n+1}} \qquad x \leq 0.$$

The difficulties that were noticed in section 2, as z approaches the negative real axis, are still present. As before, there is an optimum value of n, which is about $|z|$ in the first case, $|y|$ in the second. With this choice of n, the value of our bounds for the R_i are about $2\pi e^{-2n}$ in each case.

The important fact to notice is that our estimate for the error committed by using an n-point Laguerre quadrature is approximately the square of that for the estimate of the error committed by neglecting all terms in the asymptotic series after the nth. This, together with the fact that about the same number ($4n$) of real multiplications are required in both methods, indicates that the Laguerre method is more efficient.

4. Examples

In order to demonstrate the practicability of the Laguerre method, and to show that our conclusions are not dependent on the crudity of our appraisals of the remainders, we give three examples in detail. We choose $n=5$, which is optimum in one but not in the others. For completeness, we repeat (from [2]) the values of $x_i^{(5)}$, $\lambda_i^{(5)}$.

$x_i^{(5)}$	$\lambda_i^{(5)}$
0. 26356 03197 18	0. 52175 56105 83
1. 41340 30591 07	. 39866 68110 83
3. 59642 57710 41	. 07594 24496 817
7. 08581 00058 59	. 00361 17586 7992
12. 64080 08442 76	. 00002 33699 723858

The superscripts (5) will be dropped from now on.

We shall work to about 10 decimals throughout our examples. We begin with a case of z real: $z=10$ and follow with $z=1+10i$, and then $z=-10+5i$; it is only in the last case that we are, with $n=5$, near the optimum number of terms.

(a) $z=10$.

In this case $f_2=0$, and we have only to deal with

$$I_1=\int_0^\infty e^{-t}\cdot\frac{1}{(10+t)}\cdot dt \doteqdot \sum\lambda_i\cdot\frac{1}{10+x_i}=Q_1.$$

A complete worksheet for this calculation follows. The sum Q_1 is cumulated on the machine, there being no reason to record individual products.

$\alpha_i=10+x_i$	α_i^{-1}
10. 26356 03197	0. 09743 20771
11. 41340 30591	. 08761 62872
13. 59642 57710	. 07354 87412
17. 08581 00059	. 05852 81002
22. 64080 08443	. 04416 80490

$$Q_1=\sum\lambda_i\alpha_i^{-1}=0.\,09156\ 33319$$

The correct value of $e^{10}E_1(10)$ is 0. 09156 3334. The actual error is about 2×10^{-9}, whereas our estimate is 144×10^{-9}. Using five terms of the asymptotic series, we obtain the value 0.09164. There is thus an actual error of about 8×10^{-5}, compared with an estimated error of 12×10^{-5}.

(b) $z=1+10i$.

In this case we have

$$e^{1+10i}E_1(1+10i)=I_1-iI_2,$$

where

$$I_1=\int_0^\infty e^{-t}\cdot\frac{1+t}{(1+t)^2+100}\cdot dt\doteqdot\sum\lambda_i\cdot\frac{1+x_i}{(1+x_i)^2+100}=\sum(1+x_i)\left\{\frac{\lambda_i}{(1+x_i)^2+100}\right\}$$

and

$$I_2=\int_0^\infty e^{-t}\cdot\frac{10}{(1+t)^2+100}\cdot dt\doteqdot\sum\lambda_i\cdot\frac{10}{(1+x_i)^2+100}=\sum 10\left\{\frac{\lambda_i}{(1+x_i)^2+100}\right\}.$$

A complete work sheet for this calculation follows. It is convenient to compute the common factors γ_i, in braces { } above, and then obtain the sums $\sum \alpha_i \gamma_i$ and $\sum 10\gamma_i$ by accumulation on the machine.

$\alpha_i = 1 + x_i$	$\beta_i = 100 + \alpha_i^2$	$\gamma_i = \lambda_i / \beta_i$
1.26356 0320	101.59658 47	0.00513 55625
2.41340 3059	105.82451 43	.00376 72444
4.59642 5771	121.12712 99	.00062 69648
8.08581 0006	165.38032 35	.00002 18391
13.64080 0844	286.07144 76	.00000 00817

$$\mathscr{R}e^{1+10i}E_1(1+10i) \doteq \sum \alpha_i \gamma_i = 0.01864\ 0471$$

$$\mathscr{I}e^{1+10i}E_1(1+10i) \doteq -\sum 10\gamma_i = -0.09551\ 6925$$

The correct value of $e^{1+10i}E_1(1+10i)$ is $0.01864\ 049 - 0.09551\ 688i$. The actual error is of the order of 5×10^{-8}, whereas our estimate is about 144×10^{-9}. Using five terms of the asymptotic series gives

$$0.018595 - 0.095600i.$$

There is thus an actual error of about 1×10^{-4}, compared with an estimated error of about 1.2×10^{-4}.

(c) $z = -10 + 5i$

$\alpha_i = -10 + x_i$	$\beta_i = 25 + \alpha_i^2$	$\gamma_i = \lambda_i / \beta_i$
−9.73643 9680	119.79825 76	0.00435 52855
−8.58659 6941	98.72964 70	.00403 79645
−6.40357 4229	66.00576 29	.00115 05428
−2.91418 9994	33.49250 33	.00010 78378
2.64080 0844	31.97382 91	.00000 07309

$$\mathscr{R}e^{-10+5i}E_1(-10+5i) \doteq \sum \alpha_i \gamma_i = -0.08475\ 7264$$

$$\mathscr{I}e^{-10+5i}E_1(-10+5i) \doteq -\sum 5\gamma_i = -0.04826\ 1807$$

The correct value of $e^{-10+5i}E_1(-10+5i)$ is

$$-.08475\ 749 - 0.04826\ 039i.$$

The actual error is of the order of 2×10^{-6}, whereas our estimate is about 3×10^{-4}. Using five terms of the asymptotic series, we obtain the value

$$-0.084855 - 0.048277i.$$

There is thus an actual error of about 1×10^{-4}, compared with an estimated error of about 8×10^{-3}.

5. Remarks

1. A third method of attack is conceivable: The integration in the complex plane of the differential equation for $E_1(z)$, taking z^{-1} as the independent variable, extending the ideas of Fox and Miller [4].

2. It is conceivable that the Laguerre method would be effective in cases when the integrand is not precisely of the form $e^{-t}f(t)$. A case of importance might be that of the generalized exponential integral $\int^{\infty} (e^{-w}/w)du$, where $w = (a^2 + u^2)^{\frac{1}{2}}$. In such a case it would be convenient to follow the device of A. Reiz [5] and use the quadrature in the form

$$\int_0^{\infty} F(t)dt \doteq \sum \mu_i^{(n)} F(x_i^{(n)}),$$

where $\mu_i^{(n)} = \lambda_i^{(n)} \exp x_i^{(n)}$; the $\mu_i^{(n)}$ have been tabulated by Salzer and Zucker [2].

3. The table of Mashiko [6] gives $E_1(z)$, or rather auxiliary functions from which this can be obtained, for $z=re^{i\theta}$ when $r^{-1}=0.01(0.01)0.2$ and $\theta=0(2°)60°(1°)90°$.

6. References

[1] National Bureau of Standards, Tables of $E_1(z)$, (publication pending).

[2] H. E. Salzer and Ruth Zucker, Table of the zeros and weight factors of the first fifteen Laguerre polynomials, Am. Math. Soc. Bul. **55,** 1004–1012 (1949).

[3] G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Coll. Publications **23,** 369 (1939).

[4] L. Fox and J. C. P. Miller, Table-making for large arguments. The exponential integral, MTAC **5,** 163–167 (1951).

[5] A. Reiz, On the numerical solution of certain types of integral equations, Arkiv for Mat., Astron. Fysik, **29** (A); No. 29 (1943).

[6] M. Mashiko, Tables of generalized exponential-, sine-, and cosine-integrals, Numerical Computation Bureau, Report No. 7 (Tokyo,1953).

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