Journal of Research of the National Bureau of Standards Vol. 52, No. 4, April 1954 Research Paper 2491

New Formulas for Facilitating Osculatory Interpolation

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Hermite's n-point osculatory interpolation formula for equally spaced arguments at intervals of h, employing the function and its derivative is very much more accurate than the corresponding n-point Lagrangian formula and considerably more accurate than even the $2n$-point Lagrangian formula at intervals of h. Also it is specially suited for interpolation in many functions (e. g., Bessel, probability) that are tabulated with their derivative. To avoid the tremendous amount of labor in calculating the coefficients of f_i and f'_i in the forms that they are usually given, Hermite's formula is expressed as

$$f(x_0+ph)=\sum_i(\alpha_i f_i+\beta_i h f'_i)/\sum_i \alpha_i+R_{2n}(p),$$

where

$$\alpha_i\equiv a_i/(p-i)^2+b_i/(p-i),\ \beta_i\equiv a_i/(p-i),$$

and where

$$a_i=k(n)/\left\{\prod_{j=-[(n-1)/2]}^{[n/2]}{}'(i-j)\right\}^2,\ b_i=-2L_i^{(n)\prime}(i)a_i,\ L_i^{(n)}(p)$$

being

$$\prod_{j=-[(n-1)/2]}^{[n/2]}{}'(p-j)/\prod_{j=-[(n-1)/2]}^{[n/2]}{}'(i-j).$$

The constant $k(n)$, which may be picked arbitrarily, is here chosen to make a_i and b_i integers. The exact values of a_i and b_i are given for $n=2(1)11$, $i=-[(n-1)/2]$ to $[n/2]$ so that this formula can be applied exactly for any polynomial up to the 21st degree. A schedule gives approximate upper bounds for the coefficients of $f^{(2n)}(\xi)h^{2n}\sim\Delta^{2n}f(x)$ in $R_{2n}(p)$.

When a function $f(x)$ and its first derivative are known at n points x_i, $i=1, 2, \ldots, n$, a highly accurate interpolation formula due to Hermite is given by

$$f(x)=\sum_{i=1}^{n}\{L_i^{(n)}(x)\}^2\{1-2L_i^{(n)\prime}(x_i)(x-x_i)\}f(x_i)+\sum_{i=1}^{n}\{L_i^{(n)}(x)\}^2(x-x_i)f'(x_i)+R_{2n}(x), \tag{1}$$

where

$$L_i^{(n)}(x)=\prod_{j=1}^{n}{}'(x-x_j)/\prod_{j=1}^{n}{}'(x_i-x_j),\quad j=i \text{ is absent from } \Pi', \tag{2}$$

and

$$R_{2n}(x)=f^{(2n)}(\xi)\left\{\prod_{j=1}^{n}(x-x_j)\right\}^2/(2n)!,\quad \text{least } x_i\leq\xi\leq \text{greatest } x_i. \tag{3}$$

Thus (1) is exact whenever $f(x)$ is a polynomial of degree less than or equal to $2n-1$.

When the points x_i are equally spaced at intervals of h, it is customary to alter the notation in x_i, letting i run from $-[(n-1)/2]$ to $[n/2]$ instead of 1 to n, where $[m]$ denotes the largest integer not exceeding m. Then it is convenient to choose a variable p given by $x=x_0+ph$ and to let $x_i=x_0+ih$. Also, $f(x)=f(x_0+ph)\equiv fp\equiv f$, $f(x_i)\equiv f_i$, and $f'(x_i)\equiv f'_i$. Then (1), for $f(x)$ considered as a function of p, is expressible as

$$f(x_0+ph)=\sum_{i=-[(n-1)/2]}^{[n/2]}\{L_i^{(n)}(p)\}^2\{1-2L_i^{(n)\prime}(i)(p-i)\}f_i+\sum_{i=-[(n-1)/2]}^{[n/2]}\{L_i^{(n)}(p)\}^2(p-i)hf'_i+R_{2n}(p), \tag{4}$$

where now

$$L_i^{(n)}(p)=\prod_{j=-[(n-1)/2]}^{[n/2]}{}'(p-j)/\prod_{j=-[(n-1)/2]}^{[n/2]}{}'(i-j), \tag{5}$$

and

$$R_{2n}(p)=f^{(2n)}(\xi)h^{2n}\left\{\prod_{j=-[(n-1)/2]}^{[n/2]}(p-j)\right\}^2/(2n)!,\quad x_{-[(n-1)/2]}\leq\xi\leq x_{[n/2]}. \tag{6}$$

There are many advantages in the use of (1) or (4) over the ordinary Lagrangian interpolation formula given (for equal spacing) by

$$f(x_0+ph)=\sum_{i=-[(n-1)/2]}^{[n/2]} L_i^{(n)}(p)f_i+R_n(p), \tag{7}$$

where

$$R_n(p)=f^{(n)}(\xi)h^n \prod_{j=-[(n-1)/2]}^{[n/2]}(p-j)/n!, \quad x_{-[(n-1)/2]}\leq\xi\leq x_{[n/2]}. \tag{8}$$

Thus letting $\prod_{j=-[(n-1)/2]}^{[n/2]}(p-j)$ be denoted by $L^{(n)}(p)$, and recalling that for reasonably small h we have approximately

$$f^{(m)}(\xi)h^m\sim\Delta^m f, \tag{9}$$

where $\Delta^m f$ is the approximate mth difference of the tabulated $f(x)$, the remainder term for (7) is of the order of $\Delta^n f L^{(n)}(p)/n!$, whereas that for (4) is of the order of $\Delta^{2n}f\{L^{(n)}(p)\}^2/(2n)!$. Apart from the fact that $\Delta^{2n}f$ is usually very much smaller than $\Delta^n f$, the factor $\{L^{(n)}(p)\}^2/(2n)!$ is equal to the square of $L^{(n)}(p)/n!$ (where the latter is usually very small and less than unity so that its square is ever so much smaller) multiplied by the very small quantity $n!/(n+1)\ldots(2n)$. The user can appreciate the improvement by comparing the approximate upper bounds for the multiplier of $f^{(2n)}(\xi)h^{2n}$ in $R_{2n}(p)$ of (4) which are tabulated in the schedule at the end, with the approximate upper bounds for the multiplier of $f^{(n)}(\xi)h^n$ in $R_n(p)$ of (7), which are tabulated in [3, p. xvi][1] and from which this present schedule was calculated. Thus it will be apparent that (4) is a very much more accurate formula than (7). Of course, we are comparing (4), a confluent form of a $2n$-point formula, with (7), which is only an n-point formula.

But it is important to note than even if (4) is compared with formula (7) taken for $2n$ points at intervals of h, instead of n points, the remainder term would differ from that in (4) (apart from a different ξ in $f^{(2n)}(\xi)$) by the presence of the factor $L^{(2n)}(p)$ instead of the factor $\{L^{(n)}(p)\}^2$, which has a very much smaller upper bound than the $L^{(2n)}(p)$, showing that wherever it is possible to be used, the n-point Hermite osculating interpolation formula is much to be preferred, regarding accuracy, to a $2n$-point Lagrangian interpolation formula at the same interval h. This last statement becomes intuitively plausible when the osculating interpolation formula for n points at intervals of h is regarded as a confluent form of a $2n$-point Lagrangian formula whose $2n$ points lie within a range of nh so that the "average interval" between those $2n$ points is only half the interval of h for the usual $2n$-point Lagrangian formula. Thus the upper bound for the remainder term of the n-point osculating formula would be expected to be of the order of $(1/2^{2n})$th of the upper bound for the remainder term of the $2n$-point Lagrangian formula; actual estimates show it to be even considerably smaller. For example, the 2-, 3-, 4-, and 5-point osculating formulas have error terms whose upper bounds are around 1/16th, 1/110, 1/640, and 1/3000 of the respective upper bounds of the error terms in the 4-, 6-, 8-, and 10-point Lagrangian formulas.

A second important advantage in (4) is that it is suited for use with very many tables where the derivative of the function is tabulated alongside of the function itself. For example, it is useful in tables of Bessel functions of the first and second kind [4, 5, 6] which give $J_1(x)=-J_0'(x)$, $Y_1(x)=-Y_0'(x)$, and probability functions [7], and in numerous tables of more elementary functions and their integrals, such as tables of sine, cosine, or exponential integrals [8, 9], where the derivative is very easy to obtain.

However, the use of (1) or (4) in the form usually presented [1, 2] requires a considerable amount of computational labor which mounts considerably as the number of points x_i increases. It is the purpose of this present article to provide a means of using (4) with a small fraction of the labor involved in the direct calculation of the coefficients of f_i and f_i'. The idea

[1] Figures in brackets indicate the literature references at the end of this paper.

behind this method goes back to a scheme first used by W. J. Taylor for calculating Lagrangian interpolation coefficients for functions tabulated at real equidistant arguments [10], and which was generalized by the present writer for functions tabulated at nonequidistant arguments [11], and also for complex arguments whether in Cartesian [12] or polar form [13, 14], and finally even for functions that are interpolable by expressions that are not transformable into polynomials [15]. Recently, the writer in looking for some way to reduce the amount of work in using (4), observed that Taylor's idea could be extended also to the calculation of osculating interpolation coefficients. In place of extensive tables of the $(2n-1)$th degree polynomial coefficients of f_i and f_i' in (4), one requires for each separate n only some fixed quantities a_i and b_i, which are exact integers and are tabulated below. To see this, one merely expresses (4) as

$$f(x_0+ph)=\{L^{(n)}(p)\}^2 \sum_{i=-[(n-1)/2]}^{[n/2]} \left\{\left(\frac{A_i^2}{(p-i)^2}-\frac{2\,L_i^{(n)\prime}(i)\,A_i^2}{(p-i)}\right)f_i+\frac{A_i^2}{(p-i)}\,hf_i'\right\}+R_{2n}(p), \tag{10}$$

where [2]

$$A_i\equiv 1\Big/\prod_{j=-[(n-1)/2]}^{[n/2]}{}' (i-j). \tag{11}$$

Now the right member of (4) or (10) without the $R_{2n}(p)$ gives the expression for a $(2n-1)$th degree polynomial, which, with its derivative, assumes preassigned values of f_i and f_i' at $x=x_i$, and moreover that polynomial is *uniquely determined by the* f_i *and* f_i'. For proof of uniqueness see [1, p. 85–86], where T. Fort gives a demonstration of the unique existence of a more general osculating formula. His proof is practically complete save for the explicit indication that the mode of representation of any $(mn-1)$th degree polynomial which is given at the bottom of page 85 is always possible (which is fairly obvious). Now we make use of this uniqueness of representation by putting $f(x)\equiv 1$ into (10), so that both f_i' and $R_{2n}(p)$ are zero, $f_i=1$, and we get

$$\{L^n(p)\}^2=\frac{1}{\displaystyle\sum_{i=-[(n-1)/2]}^{[n/2]}\left(\frac{A_i^2}{(p-i)^2}-\frac{2L_i^{(n)\prime}(i)A_i^2}{p-i}\right)}. \tag{12}$$

Thus from (10) and (12),

$$f(x_0+ph)=\frac{\displaystyle\sum_{i=-[(n-1)/2]}^{[n/2]}\left\{\left(\frac{a_i}{(p-i)^2}+\frac{b_i}{(p-i)}\right)f_i+\frac{a_ih}{(p-i)}\,f_i'\right\}}{\displaystyle\sum_{i=-[(n-1)/2]}^{[n/2]}\left(\frac{a_i}{(p-i)^2}+\frac{b_i}{(p-i)}\right)}+R_{2n}(p), \tag{13}$$

where a_i and b_i are given by

$$a_i=k(n)A_i^2, \tag{14}$$

$$b_i=k(n)\{-2L_i^{(n)\prime}(i)A_i^2\}, \tag{15}$$

and where $k(n)$ is any suitably chosen constant of proportionality that depends only upon n. In the present case the $k(n)$ was chosen as to give exact integral values for a_i and b_i instead of rational fractional values.

It is simplest to think of the approximation to $f(x)$ in the concise form

$$f\sim\frac{\Sigma(\alpha_if_i+\beta_ihf_i')}{\Sigma\alpha_i}, \tag{16}$$

where

$$\alpha_i\equiv a_i/(p-i)^2+b_i/(p-i), \tag{17}$$

and

$$\beta_i\equiv a_i/(p-i). \tag{18}$$

[2] The dependence upon n of A_i, as well as of a_i and b_i given below, is not indicated, so as to avoid cumbersome notation.

In using (16), (17), and (18) with a desk calculator, it is easiest to first divide a_i by $p-i$ to get β_i, which is next both multiplied by hf_i' and increased by b_i. The latter, or β_i+b_i, is again divided by $(p-i)$ to give α_i, from which one obtains both $\alpha_i f_i$ and $\Sigma\alpha_i$ and finally $\Sigma(\alpha_i f_i+\beta_i hf_i')/\Sigma\alpha_i$.

The computation of the quantities a_i and b_i was quite straightforward. Since $A_i=(-1)^{[n/2]-i}\binom{n-1}{i+[(n-1)/2]}\Big/(n-1)!$, instead of A_i^2, the proportional quantities $\binom{n-1}{i+[(n-1)/2]}^2$ were calculated. Then they were multiplied by the $-2L_i^{(n)\prime}(i)$, which were calculated by differentiating the explicit polynomial expressions $L_i^{(n)}(p)$ and then setting $p=i$. All fractions in $-2\binom{n-1}{i+[(n-1)/2]}^2 L_i^{(n)\prime}(i)$ were cleared by multiplication of these quantities, as well as the $\binom{n-1}{i+[(n-1)/2]}^2$, by some suitable integer, for each n, to yield the exact integral values for a_i and b_i, which are tabulated below. The a_i and b_i were checked by both recomputation and by use in an example for every n where the answer was known exactly and where the computation by (13) (or the equivalent (16), (17), and (18)) doing the work in decimal form to avoid too much labor, gave agreement to 10 significant figures.

The schedule giving the approximate upper bounds for the coefficients of $f^{(2n)}(\xi)h^{2n}\sim\Delta^{2n}f$ in the error term $R_{2n}(p)$, (see (4) with (6), or (13)), namely, the quantities $\{L^{(n)}(p)\}^2/(2n)!$, was calculated from the approximate upper bounds for $L^{(n)}(p)/n!$ given in a schedule in [3, p. xvi], by squaring the entries in the latter and multiplying by $n!/(n+1)\ldots(2n)$. Thus, since the $L^{(n)}(p)/n!$ was tabulated only approximately, in some cases to only one significant figure, some of the upper bounds given here for $\{L^{(n)}(p)\}^2/(2n)!$ are not at all precise. For example, if we take a rounded 0.01 and square it to obtain 0.0001, the true value of that square may be only one-fourth as large or over twice as large, depending upon whether the 0.01 was rounded from 0.0051 or 0.0149. Hence in the schedule below, in some cases for the larger number of points, only the order of magnitude of an upper bound for the $\{L^{(n)}(p)\}^2/(2n)!$ is indicated. But more precise determination will hardly be needed there due to the extreme accuracy that is surely indicated even within the range of the uncertainty.

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[10] W. J. Taylor, Method of Lagrangian curvilinear interpolation, J. Research NBS **35,** 151–155 (1945) RP1667.

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Table of a_i and b_i

Two-point

a_0	1	b_0	2
a_1	1	b_1	−2

Three-point

a_{-1}	1	b_{-1}	3
a_0	4	b_0	0
a_1	1	b_1	−3

Four-point

a_{-1}	3	b_{-1}	11
a_0	27	b_0	27
a_1	27	b_1	−27
a_2	3	b_2	−11

Five-point

a_{-2}	6	b_{-2}	25
a_{-1}	96	b_{-1}	160
a_0	216	b_0	0
a_1	96	b_1	−160
a_2	6	b_2	−25

Six-point

a_{-2}	30	b_{-2}	137
a_{-1}	750	b_{-1}	1625
a_0	3000	b_0	2000
a_1	3000	b_1	−2000
a_2	750	b_2	−1625
a_3	30	b_3	−137

Seven-point

a_{-3}	10	b_{-3}	49
a_{-2}	360	b_{-2}	924
a_{-1}	2250	b_{-1}	2625
a_0	4000	b_0	0
a_1	2250	b_1	−2625
a_2	360	b_2	−924
a_3	10	b_3	−49

Eight-point

a_{-3}	70	b_{-3}	363
a_{-2}	3430	b_{-2}	9947
a_{-1}	30870	b_{-1}	48363
a_0	85750	b_0	42875
a_1	85750	b_1	−42875
a_2	30870	b_2	−48363
a_3	3430	b_3	−9947
a_4	70	b_4	−363

Nine-point

a_{-4}	140	b_{-4}	761
a_{-3}	8960	b_{-3}	28544
a_{-2}	1 09760	b_{-2}	2 08544
a_{-1}	4 39040	b_{-1}	3 95136
a_0	6 86000	b_0	0
a_1	4 39040	b_1	−3 95136
a_2	1 09760	b_2	−2 08544
a_3	8960	b_3	− 28544
a_4	140	b_4	−761

Ten-point

a_{-4}	1260	b_{-4}	7129
a_{-3}	1 02060	b_{-3}	3 50649
a_{-2}	16 32960	b_{-2}	35 69184
a_{-1}	88 90560	b_{-1}	109 65024
a_0	200 03760	b_0	80 01504
a_1	200 03760	b_1	−80 01504
a_2	88 90560	b_2	−109 65024
a_3	16 32960	b_3	−35 69184
a_4	1 02060	b_4	−3 50649
a_5	1260	b_5	−7129

Eleven-point

a_{-5}	1260	b_{-5}	7381
a_{-4}	1 26000	b_{-4}	4 60900
a_{-3}	25 51500	b_{-3}	62 14725
a_{-2}	181 44000	b_{-2}	275 61600
a_{-1}	555 66000	b_{-1}	407 48400
a_0	800 15040	b_0	0
a_1	555 66000	b_1	−407 48400
a_2	181 44000	b_2	−275 61600
a_3	25 51500	b_3	−62 14725
a_4	1 26000	b_4	−4 60900
a_5	1260	b_5	−7381

Schedule of approximate upper bounds for $\{L^{(n)}(p)\}^2/(2n)!$

(Figures in parentheses indicate the number of zeros between the decimal point and the first significant digit.)

Range of p	$0<p<1$	$-1<p<0$; $1<p<2$	$-2<p<-1$; $2<p<3$	$-3<p<-2$; $3<p<4$	$-4<p<-3$; $4<p<5$
Two-point	.0026				
Four-point	.(5)82	.(4)25			
Six-point	.(7)26	.(7)55	.(6)62		
Eight-point	.(10)94	.(9)15	.(9)85	.(7)20	
Ten-point	.(12)34	.(12)49	.(11)18	.(10)18	.(9)78
Range of p	$0<\|p\|<1$	$1<\|p\|<2$	$2<\|p\|<3$	$3<\|p\|<4$	$4<\|p\|<5$
Three-point	.00021				
Five-point	.(6)57	.(5)38			
Seven-point	.(8)18	.(8)62	.(6)11		
Nine-point	.(11)60	.(10)15	.(9)12	.(8)40	
Eleven-point	.(13)6	.(13)6	.(12)2	.(11)6	.(9)1

WASHINGTON, May 12, 1953.