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Characteristics of Internal Solitary Waves[1]

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This is an application of the method of approximations initiated by Boussinesq to the disturbances of the interface points for waves of permanent form and the internal solitary wave. The system considered is a layer of liquid on another layer of greater density, the liquids of the layers being initially at rest and of constant total depth. The form of the wave is established. The dependence of wave velocity on wave height, on density differences and on layer thickness is determined.

1. Introduction

In investigations on the model laws of density currents, a project now being carried out at the National Hydraulics Laboratory, need arose for the consideration of the genesis and the damping of internal waves at the interface between layers of fresh and saline waters. Among the many possible modes of such disturbances, one may consider for the sake of simplicity the behavior of a single intumescence and of nearly sinusoidal progressive waves.

It has been our purpose to deal with these two types of internal wave motions experimentally. The experimental studies of solitary waves have been completed. As a natural guide in the study of the data, resort has been made to a theoretical analysis. The present paper gives the basis and the result of this analysis. Relations are here obtained giving the dependence of the velocity of wave propagation on wave heights, the form of the solitary wave, and the expressions for the velocity vector on the upper and the lower layers. Consideration of the experimental data, however, is reserved for a future occasion.

After putting forth the basic conditions for the analysis, the question of internal waves of infinitesimal height and negligible interfacial surface curvature is taken up in the first approximate solution. In the second approximate solution the characteristics of solitary waves are revealed. In general, there are seen to be some similarities between ordinary surface solitary waves and internal solitary waves. No attempt is made to extend the approximations to a third-order analysis as no special demand is made by experimental evidence for this.

2. Mathematical Formulation

A layer of lighter liquid of thickness H' and of density ρ' rests on a layer of a denser liquid of thickness H and of density ρ. The upper liquid at its free surface is exposed to air and the lower liquid rests on a rigid horizontal bed. Both liquids are initially at rest. The displacement of the interface with respect to its initial undisturbed position is denoted by h; the displacement of the free surface with respect to the level of the undisturbed free surface is denoted by h' (see fig. 1). Taking the origin, O, of rectangular axes at the undisturbed interface, the axis of x is drawn horizontally. The axis of z is drawn vertically and the positive branch points upwards. The velocity components along these axes are denoted by u, w, and u', w', the primed symbols referring to particle velocities in the upper liquid.

It will be assumed that the disturbances are produced in liquids initially at rest, so that the consequent flow is irrotational and admits the velocity potentials ϕ and ϕ'. The vorticity that is naturally present at the interface will be ignored. Since the type of disturbance visualized is translational, that is, the particle velocities in vertical planes normal to the direction of wave motion are nearly constant for each layer, it is appropriate to introduce the expressions of the velocity potentials

$$\phi=\left(\cos z\frac{\partial}{\partial x}\right)\phi_0+\left(\sin z\frac{\partial}{\partial x}\right)\theta_0 \tag{1}$$

for the lower layer, and

$$\phi'=\left(\cos z\frac{\partial}{\partial x}\right)\phi_0'+\left(\sin z\frac{\partial}{\partial x}\right)\theta_0' \tag{2}$$

for the upper layer. Here, ϕ_0, θ_0, ϕ_0', and θ_0' are functions of x and t alone.

Altogether there are six unknowns, h, h', ϕ_0, ϕ_0', θ_0, and θ_0', which are determined on the basis of the kinematic and dynamic conditions at the various boundaries. The pressure is atmospheric at the free surface. At the interface the pressure is continuous. A particle in the free surface remains in this surface. The particles in the upper surface of the lower liquid at the interface remain in this surface. Finally, the normal particle velocities vanish at the horizontal bottom.

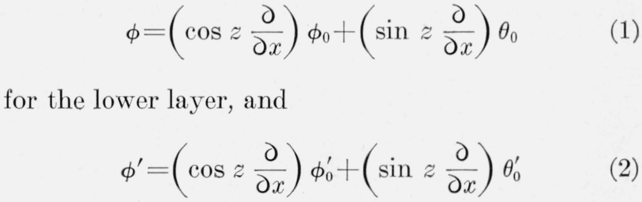

FIGURE 1. *Notation diagram.*

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The mathematical formulation of the dynamic boundary conditions requires that expressions be given for the pressures at all points in the two layers. Assuming that the atmoshperic pressure is reduced to zero,

$$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-gz-\frac{1}{2}(u^2+w^2)+\frac{\rho'}{\rho}gH' \tag{3}$$

for the lower liquid, and

$$\frac{p'}{\rho'}=\frac{\partial\phi'}{\partial t}-gz-\frac{1}{2}(u'^2+w'^2)+gH' \tag{4}$$

for the upper liquid. Translating now the statements of the boundary conditions above into mathematical relations, neglecting w^2 with respect to u^2, and w'^2 with respect to u'^2, they are

$$\frac{\partial\phi'}{\partial t}-gh'-\frac{1}{2}u'^2=0, \qquad z=H'+h', \tag{5}$$

$$w'-\frac{\partial h}{\partial t}-u'\frac{\partial h'}{\partial x}=0, \qquad z=H'+h', \tag{6}$$

$$\rho'\left(\frac{\partial\phi'}{\partial t}-gh-\frac{1}{2}u'^2\right)=\rho\left(\frac{\partial\phi}{\partial t}-gh-\frac{1}{2}u^2\right), \qquad z=h, \tag{7}$$

$$w-\frac{\partial h}{\partial t}-u\frac{\partial h}{\partial x}=0, \qquad z=h, \tag{8}$$

$$w'-\frac{\partial h}{\partial t}-u'\frac{\partial h}{\partial x}=0, \qquad z=H, \tag{9}$$

and

$$w=0, \qquad z=h. \tag{10}$$

3. Internal Waves of Infinitesimal Height

The functions θ_0 and θ_0' may be expressed in terms of ϕ_0 and ϕ_0', respectively, after noting the kinematic boundary conditions at the bottom and at the free surface. From eq 1, keeping only the first three terms,

$$\phi=\phi_0+z\frac{\partial\theta_0}{\partial x}-\frac{z^2}{2}\frac{\partial^2\phi_0}{\partial x^2}, \tag{11}$$

and therefore

$$\frac{\partial\phi}{\partial z}=\frac{\partial\theta_0}{\partial x}-z\frac{\partial^2\phi_0}{\partial x^2}. \tag{12}$$

The latter in view of eq 10 requires that

$$\frac{\partial\theta_0}{\partial x}=-H\frac{\partial^2\phi_0}{\partial x^2}. \tag{13}$$

Hence,

$$\phi=\phi_0-\left(Hz+\frac{z^2}{2}\right)\frac{\partial^2\phi_0}{\partial x^2}, \tag{14}$$

and therefore

$$\frac{\partial\phi}{\partial z}=-(H+z)\frac{\partial^2\phi_0}{\partial x^2}. \tag{15}$$

Similarly, from eq 2,

$$\phi'=\phi_0'+z\frac{\partial\theta_0}{\partial x}-\frac{z^2}{2}\frac{\partial^2\phi_0'}{\partial x^2}, \tag{16}$$

hence

$$\frac{\partial\phi'}{\partial z}=\frac{\partial\theta_0'}{\partial x}-z\frac{\partial^2\phi_0'}{\partial x^2}. \tag{17}$$

Taking the kinematic boundary condition of the upper surface, eq 6, and neglecting the product term which is a small quantity,

$$\frac{\partial\theta_0'}{\partial x}-(H'+h')\frac{\partial^2\phi_0'}{\partial x^2}=-\frac{\partial h'}{\partial t}. \tag{18}$$

And as h' may be neglected in comparison with H',

$$\frac{\partial\theta_0'}{\partial x}=-\frac{\partial h'}{\partial t}+H'\frac{\partial^2\phi_0'}{\partial x^1}, \tag{19}$$

hence

$$\phi'=\phi_0'-z\frac{\partial h'}{\partial t}+\left(zH'-\frac{z^2}{2}\right)\frac{\partial^2\phi_0}{\partial x^2}, \tag{20}$$

and therefore

$$\frac{\partial\phi'}{\partial x}=-\frac{\partial h'}{\partial t}+(H'-z)\frac{\partial^2\phi_0'}{\partial x^2}. \tag{21}$$

For the points at the interface, that is $z=h$, one may write from eq 14 and 15, neglecting the terms involving z and its square,

$$\frac{\partial\phi}{\partial t}=\frac{\partial\phi_0}{\partial t} \tag{22}$$

and

$$\frac{\partial\phi}{\partial z}=-w=-H\frac{\partial^2\phi_0}{\partial x^2}. \tag{23}$$

From eq 20 and 21

$$\frac{\partial\phi'}{\partial t}=\frac{\partial\phi_0'}{\partial t} \tag{24}$$

and

$$\frac{\partial\phi'}{\partial z}=-w'=-\frac{\partial h'}{\partial t}+H'\frac{\partial^2\phi_0}{\partial x^2}. \tag{25}$$

These imply that wave height h is small in comparison with H or H' and also that the curvature of the wave surface is small.

Taking next the boundary conditions, eq 5, 7, 8, and 9, neglecting in them terms consisting of squares or of products, and using the relations from eq 22 to 25, inclusive,

$$\frac{\partial \phi_0'}{\partial t} = gh', \tag{26}$$

$$\rho'\left(\frac{\partial \phi_0'}{\partial t} - gh\right) = \rho\left(\frac{\partial \phi_0}{\partial t} - gh\right), \tag{27}$$

$$H\frac{\partial^2 \phi_0}{\partial x^2} = \frac{\partial h}{\partial t}, \tag{28}$$

$$-H\frac{\partial^2 \phi_0}{\partial x^2} = -\frac{\partial h'}{\partial t} + H'\frac{\partial^2 \phi_0'}{\partial x^2}. \tag{29}$$

Rearranging the terms in eq 27 and then differentiating twice with respect to t, we have

$$\frac{\partial^3 \phi_0'}{\partial t^3} = \frac{\rho}{\rho'}\frac{\partial^3 \phi_0}{\partial t^3} - \frac{\rho - \rho'}{\rho'}g\frac{\partial^2 h}{\partial t^2}. \tag{30}$$

Differentiating eq 27 twice with respect to x,

$$\frac{\partial^3 \phi_0'}{\partial t \partial x^2} = \frac{\rho}{\rho'}\frac{\partial^3 \phi_0}{\partial t \partial x^2} - \frac{\rho - \rho'}{\rho}g\frac{\partial^2 h}{\partial x^2}. \tag{31}$$

Multiplying eq 29 by g, introducing the value of h' from eq 26, and differentiating the result with respect to t

$$-gH\frac{\partial^3 \phi_0}{\partial x^2 \partial t} - gH'\frac{\partial^3 \phi_0'}{\partial x^2 \partial t} = -\frac{\partial^3 \phi_0'}{\partial t^3}.$$

This reduces, after eliminating ϕ_0' by means of eq 30 and 31, to

$$-gH\frac{\partial^3 \phi_0}{\partial x^2 \partial t} - gH'\frac{\rho}{\rho'}\frac{\partial^3 \phi_0}{\partial x^2 \partial t} + g^2H'\frac{(\rho - \rho')}{\rho}\frac{\partial^2 h}{\partial x^2} = -\frac{\rho}{\rho'}\frac{\partial^3 \phi}{\partial t^3} + \frac{\rho - \rho'}{\rho'}g\frac{\partial^2 h}{\partial t^2}.$$

Differentiating the latter with respect to t, and then eliminating h by means of eq 28, there results, finally,

$$\frac{\partial^4 \phi_0}{\partial t^4} - g(H + H')\frac{\partial^4 \phi_0}{\partial x^2 \partial t^2} + g^2H'H\frac{\rho - \rho'}{\rho}\frac{\partial^4 \phi_0}{\partial x^2} = 0, \tag{32}$$

which is the differential equation of wave motion in a liquid system consisting of two strata of different densities for small disturbances.

Putting $\rho - \rho' = \Delta\rho$,

$$\omega_1^2\omega_2^2 = g^2H'H\frac{\Delta\rho}{\rho}, \tag{33}$$

and

$$\omega_1^2 + \omega_2^2 = g(H + H'), \tag{34}$$

the wave equation, eq 32, may be rewritten

$$\left(\frac{\partial^2}{\partial t^2} - \omega_1^2\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \omega_2^2\frac{\partial^2}{\partial x^2}\right)\phi_0 = 0. \tag{35}$$

Two types of waves are possible. In one the waves have the velocity of propagation $\pm\omega_1$; in the other, the velocity $\pm\omega_2$. The plus sign refers to waves moving in the direction of x negative and the minus sign, in the opposite direction. Furthermore, these waves, of infinitesimal wave height and of negligible surface curvature, travel without distortion of form.

The discrete values of the velocities may be shown from eq 34, neglecting the higher powers of the density differences, to be

$$\omega_1^2 = g(H + H')\left(1 - \frac{H'H}{(H + H')^2}\frac{\Delta\rho}{\rho}\right) \tag{36}$$

and

$$\omega_2^2 = \frac{gH'H}{H + H'}\frac{\Delta\rho}{\rho}\left(1 + \frac{H'H}{(H + H')^2}\frac{\Delta\rho}{\rho}\right). \tag{37}$$

Of these, the second refers to disturbances of the interface, that is, to internal waves. The first refers to ordinary waves; in this expression one notices the effect of the nonhomogeneity of fluid on the velocity of propagation of the ordinary waves. Nonhomogeneity reduces the value of the velocity of propagation. For a given relative density difference the reduction is greatest when the layers are of equal depth. For the purposes of the present work, the secondary effects of the relative density differences will be ignored. Accordingly the velocity of propagation of internal waves of infinitesimal height, replacing subscript 2 by 0, will be given as

$$\omega_0^2 = \frac{gH'H}{H + H'}\frac{\Delta\rho}{\rho}. \tag{38}$$

The next question to be considered is in regard to the particle velocities in the two layers. One commences the analysis with the lower layer. As was noted above, the internal waves of infinitesimal height progress without change of form. This fact is equivalent to the equality of the operators

$$\frac{\partial}{\partial t} = \mp\omega\frac{\partial}{\partial x}. \tag{39}$$

The negative sign is chosen for waves moving in the direction of x positive and the positive sign for waves moving in the direction of x negative. For the subsequent analysis it will be assumed that the internal waves move in the direction of x positive, and thus

$$\frac{\partial}{\partial t} = -\omega\frac{\partial}{\partial x}. \tag{40}$$

One now naturally selects that kinematic condition of the interface stating that a particle of the lower liquid once on the interface remains on the interface. This condition is given by eq 28, which now may be written as

$$H\frac{\partial^2 \phi_0}{\partial x^2} = -\omega\frac{\partial h}{\partial x},$$

and, as in terms of the potential, the particle velocity u_0 is

$$u_0 = -\frac{\partial \phi_0}{\partial x}, \tag{41}$$

we have

$$H\frac{\partial u_0}{\partial x} = \omega \frac{\partial h}{\partial x}.$$

Integrating the latter equation with respect to x and observing that u_0 vanishes when h vanishes, then

$$u_0 = \omega \frac{h}{H}. \tag{42}$$

Accordingly, particle velocities in the lower layer are proportional to the velocity of wave propagation and vary with the wave-element height of the internal wave. When these heights are positive, that is when the wave elements are elevated, particle motion in the lower layer is in the direction of wave motion.

In terms of the velocity potential the particle velocity in the upper layer is

$$u_0' = -\frac{\partial \phi_0'}{\partial x}. \tag{43}$$

To obtain the values of u_0' one now considers the other kinematic boundary condition of the interface. The reference is to eq 29, which when treated in an analogous manner yields

$$H'u_0' + Hu_0 = \omega h'. \tag{44}$$

Now, when in eq 26 the partial differentiation with respect to t is replaced by the partial differentiation with respect to x, the result is

$$\omega u_0' = gh'. \tag{45}$$

Eliminating h between eq 44 and 45, and introducing the value of ω from eq 38, one obtains the relation

$$H'u_0'\left(1 - \frac{\Delta\rho}{\rho}\frac{H}{H+H'}\right) + Hu_0 = 0, \tag{46}$$

and this connects the particle velocities in the two layers. As the term containing the relative density difference is small, it will be neglected, and therefore

$$\frac{u_0'}{u_0} = -\frac{H}{H'}. \tag{47}$$

Thus the motion of the particles in the upper layer is oppositely directed to the motion of the particles in the lower layer.

It will be instructive to show to what extent the internal waves do affect the free surface. This will be better understood if the ratio h'/h is investigated. The combinations of the expressions in eq 42 and 45 gives

$$\frac{h'}{h} = \frac{\omega^2}{gH}\frac{u_0'}{u_0},$$

which, in view of eq 38 and 39, reduces to

$$\frac{h'}{h} = -\frac{\Delta\rho}{\rho}\frac{H}{H+H'}. \tag{48}$$

The interpretation is that the displacement of the surface is directed oppositely to that of the interface. Furthermore, the disturbances of the free surface are very much reduced in comparison with the displacements of the internal waves.

Finally, the question of the energy of internal waves of infinitesimal heights may be considered. The energy of waves is in one part potential and in another part kinetic. Evaluating the potential energy with respect to the undisturbed configuration of the two layers, the appropriate expression is

$$E\rho = g\frac{\Delta\rho}{2}\int_0^\lambda h^2 dx + g\frac{\rho'}{2}\int_0^\lambda h'^2 dx,$$

where λ is the effective wave length, that is the length along which h^2 is measurable. Expressing h' in terms of h through eq 48,

$$E\rho = g\frac{\Delta\rho}{2}\left[1 + \frac{\Delta\rho}{\rho}\left(\frac{H}{H+H'}\right)^2\right]\int_0^\lambda h^2 dx. \tag{49}$$

The kinetic part of the energy in terms of the particle velocities in the two layers is

$$E_k = \frac{\rho'}{2}\int_0^\lambda u_0'^2 H' dx + \frac{\rho}{2}\int_0^\lambda u_0^2 H dx.$$

Expressing u_0' in terms of u_0 with the use of eq 46, expressing u_0 in terms of h and ω with the use of eq 42, and introducing the more exact values of ω from eq 37, the final value of kinetic energy is

$$E_k = g\frac{\Delta\rho}{2}\left[1 + \frac{\Delta\rho}{\rho}\left(\frac{H}{H+H'}\right)^2\right]\int_0^\lambda h^2 dx. \tag{50}$$

Comparison of the expressions for the two energies shows that, in internal waves of infinitesimal wave height when the upper surface is free, the potential and kinetic energies are of like value. When the relative density-difference term is ignored the energy of internal waves is simply

$$E = g\Delta\rho\int_0^\lambda h^2 dx. \tag{51}$$

The form of the internal waves together with the difference of the density of the two layers is sufficient for the evaluation of the energy.

Some of these properties of internal waves of large wave length and of infinitesimal wave height are well known. A short derivation of these results, for example, by the method of Lord Rayleigh is given by Thorade [1].

4. The Internal Solitary Wave

In general when the wave-element heights of the internal waves are finite, a deformation of the wave surface occurs during the travel of the waves even when the two liquids are assumed to be perfect and ideal. The extent of the deformation depends on the height h and on the surface curvature $\partial^2 h/\partial x^2$. The corresponding mode of the deformation in the ordinary translation waves, as first discussed by Boussinesq, is well known [2, 3]. A similar theory can be worked out also in the cases of internal waves, and this theory could form a basis for the study of internal waves which travel without deformation. A second procedure is to assume the existence of internal solitary waves, positive or negative, which travel long distances without deformation in the absence of viscosity. These intumescences will be referred to as internal solitary waves, and the analysis to be followed below will determine the form of the waves.

Ordinarily in establishing the character of the progressive waves of the permanent type resort is made to an artifice [4]. By superposing on the flow a current of the magnitude equaling the velocity of propagation of the wave and moving in the direction opposite to that of the wave, the system is reduced to a steady state. Another method, and this will be used here, is that the time differentiations will be changed into space differentiations for the wave traveling in the direction of x positive using

$$\frac{\partial}{\partial t}=-\omega\frac{\partial}{\partial x}, \tag{40a}$$

where ω is the velocity of progression of the wave of permanent type. In the present case ω is the velocity of progression of the solitary wave.

The expression for the vector potential in the lower layer to the second approximation, is, from eq 1,

$$\phi=\phi_0+z\frac{\partial\theta_0}{\partial x}-\frac{z^2}{2}\frac{\partial^2\phi_0}{\partial x^2}-\frac{z^3}{6}\frac{\partial^3\theta}{\partial x^3}. \tag{52}$$

Evaluating the last term on the right-hand side, since it is the term of smallest value, from the first approximative value,

$$\frac{\partial\theta}{\partial x}=-H\frac{\partial^2\phi_0}{\partial x^2}, \tag{13a}$$

the vector potential now is

$$\phi=\phi_0+z\frac{\partial\theta_0}{\partial x}-\frac{z^2}{2}\frac{\partial^2\phi_0}{\partial x^2}+\frac{Hz^3}{6}\frac{\partial^4\phi_0}{\partial x^4},$$

and, therefore,

$$\frac{\partial\phi}{\partial z}=\frac{\partial\theta_0}{\partial x}-z\frac{\partial^2\phi_0}{\partial x^2}+\frac{Hz^2}{2}\frac{\partial^4\phi_0}{\partial x^4}.$$

In view of the bottom kinematic boundary condition, eq 10,

$$\frac{\partial\theta_0}{\partial x}=-H\frac{\partial^2\phi_0}{\partial x^2}-\frac{H^3}{2}\frac{\partial^4\phi_0}{\partial x^4}, \tag{53}$$

which than is the second approximative value of $\partial\theta_0/\partial x$. Thus, the resulting form of the vector potential is

$$\phi=\phi_0-Hz\frac{\partial^2\phi_0}{\partial x^2}-\frac{H^3 z}{2}\frac{\partial^4\phi_0}{\partial x^4}-\frac{z^2}{2}\frac{\partial^2\phi_0}{\partial x^2}+\frac{Hz^3}{6}\frac{\partial^4\phi_0}{\partial x^4}, \tag{54}$$

and, therefore,

$$\frac{\partial\phi}{\partial z}=-H\frac{\partial^2\phi_0}{\partial x^2}-z\frac{\partial^2\phi_0}{\partial x^2}-\frac{H^3}{2}\frac{\partial^4\phi_0}{\partial x^4}+\frac{Hz^2}{2}\frac{\partial^4\phi_0}{\partial x^4}. \tag{55}$$

This is the expression for the velocity potential in the lower layer to the second approximation involving the single unknown function ϕ_0. When the height of the internal solitary wave is smaller than the depth H of the lower layer, the expression for the vertical component of the particle velocities at the interface may be simplified by neglecting z^2 in eq 55. Mathematically the basis of the approximation is

$$\frac{h^2}{2}\frac{\partial^4\phi_0}{\partial x^4}\ll H\frac{\partial^2\phi_0}{\partial x^2}. \tag{56}$$

Thus, neglecting z^2,

$$\frac{\partial\phi}{\partial x}=-H\frac{\partial^2\phi_0}{\partial x^2}-z\frac{\partial^2\phi_0}{\partial x^2}-\frac{H^3}{2}\frac{\partial^4\phi_0}{\partial x^4}. \tag{57}$$

Now, the last two terms on the right-hand side being the smallest in value, substitutions may be made for them from the first approximative values,

$$\left.\begin{aligned}\frac{\partial\phi_0}{\partial x}&=-u_0=-\omega_0\frac{h}{H},\\ \frac{\partial^2\phi_0}{\partial x^2}&=-\frac{\omega_0}{H}\frac{\partial h}{\partial x},\\ \frac{\partial^3\phi_0}{\partial x^3}&=-\frac{\omega_0}{H}\frac{\partial^2 h}{\partial x^2},\\ \frac{\partial^4\phi_0}{\partial x^4}&=-\frac{\omega_0}{H}\frac{\partial^3 h}{\partial x^2}.\end{aligned}\right\} \tag{58}$$

Introducing these in eq 55 and putting $z=h$, the vertical component of the particle velocity at the interface, sign reversed, is

$$\frac{\partial\phi}{\partial z}=-H\frac{\partial^2\phi_0}{\partial x^2}+\frac{\omega_0^h}{H}\frac{\partial h}{\partial x}+\frac{\omega_0 H^2}{2}\frac{\partial^3 h}{\partial x^3},\qquad z=h. \tag{59}$$

Similarly the expression for the vector potential in the upper liquid, from eq 1, is

$$\phi=\phi_0'+z\frac{\partial\theta_0'}{\partial x}-\frac{z^2}{2}\frac{\partial^2\phi_0'}{\partial x^2}-\frac{z^3}{\partial}\frac{\partial^3\theta_0'}{\partial x^3}. \tag{60}$$

The last term on the right-hand side being the smallest term, the value of θ_0' may be taken from the first approximation,

$$\frac{\partial\theta_0'}{\partial x}=-\frac{\partial h'}{\partial t}+H'\frac{\partial^2\phi_0'}{\partial x^2}. \tag{19a}$$

It is permissible at this time to affect a small modification in the analysis, and this consists of neglecting the term $\partial h'/\partial t$. The assumption implies that

$$-\frac{\partial h'}{\partial t}\ll H'\frac{\partial^2\phi_0'}{\partial x},$$

and may be verified as follows. In view of eq 40a,

$$-\frac{\partial h'}{\partial t}=\omega_0\frac{\partial h}{\partial x}.$$

Also, in view of eq 43, 45, and 38,

$$H'\frac{\partial^2\phi_0'}{\partial x^2}=-H'\frac{\partial u_0'}{\partial x}=\frac{gH'}{\omega_0}\frac{\partial h'}{\partial x}=\frac{\omega_0(H+H')}{H}\frac{\rho}{\Delta\rho}\frac{\partial h'}{\partial x}.$$

Thus, since

$$\Delta\rho H\ll(H+H')\rho.$$

it is permissible to neglect $\partial h'/\partial t$ and write for the first approximative value of $\partial\theta'/\partial x$

$$\frac{\partial\theta'}{\partial x}=H'\frac{\partial^2\phi_0'}{\partial x^2}. \tag{19b}$$

Introducing this is eq 55,

$$\phi'=\phi_0'+z\frac{\partial\theta_0}{\partial x}-\frac{z^2}{2}\frac{\partial^2\phi_0'}{\partial x^2}-\frac{z^3H'}{6}\frac{\partial^4\phi_0'}{\partial x^4}, \tag{61}$$

and, therefore,

$$\frac{\partial\phi'}{\partial z}=\frac{\partial\theta_0'}{\partial x}-z\frac{\partial^2\phi_0'}{\partial x^2}-\frac{z^2H'}{2}\frac{\partial^4\phi_0'}{\partial x^4}. \tag{62}$$

Since $\partial h'/\partial t$ may be neglected and also $u'\partial h'/\partial t$ which is still smaller, the kinematic boundary condition for the upper surface, eq 6 reduces to

$$w'=\frac{\partial\phi'}{\partial z}=0,\qquad z=H'+h', \tag{63}$$

and this requires that

$$\frac{\partial\theta_0'}{\partial x}=H'\frac{\partial^2\phi_0'}{\partial x^2}+\frac{H'^3}{2}\frac{\partial^4\phi_0'}{\partial x^4}, \tag{64}$$

which is the second approximative value of $\partial\theta_0'/\partial x$. Substituting in eq 61,

$$\phi'=\phi_0'+zH'\frac{\partial^2\phi_0'}{\partial x^2}-\frac{z^2}{2}\frac{\partial^2\phi_0'}{\partial x^2}+\frac{2H'^3}{2}\frac{\partial^4\phi_0'}{\partial x^4}-\frac{z^3H'}{6}\frac{\partial^4\phi_0'}{\partial x^4} \tag{65}$$

and

$$\frac{\partial\phi'}{\partial z}=H'\frac{\partial^2\phi_0'}{\partial x^2}-z\frac{\partial^2\phi_0'}{\partial x^2}+\frac{H'^3}{2}\frac{\partial^4\phi_0'}{\partial x^4}-\frac{z^2H'}{2}\frac{\partial^4\phi_0'}{\partial x^4}. \tag{66}$$

When the latter expression is applied to the region close to the interface, h^2 being smaller than H'^2, the term involving z^2 may be neglected. Thus,

$$\frac{\partial\phi'}{\partial z}=H'\frac{\partial^2\phi_0'}{\partial x^2}-z\frac{\partial^2\phi_0'}{\partial x^2}+\frac{H'^3}{2}\frac{\partial^4\phi_0'}{\partial x^4}. \tag{67}$$

Now the last two terms on the right-hand side being the smallest in value, substitution may be made for them from the first approximate values

$$\left.\begin{aligned}
\frac{\partial\phi_0'}{\partial x}&=-u_0'=\omega_0\frac{h}{H'},\\
\frac{\partial^2\phi_0'}{\partial x^2}&=\frac{\omega_0}{H'}\frac{\partial h}{\partial x},\\
\frac{\partial^3\phi_0'}{\partial x^3}&=\frac{\omega_0}{H'}\frac{\partial^2 h}{\partial x^2},\\
\frac{\partial^4\phi_0'}{\partial x^4}&=\frac{\omega_0}{H'}\frac{\partial^3 h}{\partial x^3}.
\end{aligned}\right\} \tag{68}$$

Hence,

$$\frac{\partial\phi'}{\partial z}=H'\frac{\partial^2\phi_0'}{\partial x^2}-\frac{\omega_0^h}{h'}\frac{\partial h}{\partial x}+\frac{\omega_0 h'}{2}\frac{\partial^3 h}{\partial x^3},\qquad z=h. \tag{69}$$

Consider next the two kinematic boundary conditions for the interface given by eq 8 and 9. These may be written, in view of the rule of eq 40a,

$$w=-\frac{\partial\phi}{\partial z}=-\omega\frac{\partial h}{\partial x}+u\frac{\partial h}{\partial x},\qquad z=h, \tag{70}$$

$$w'=-\frac{\partial\phi'}{\partial z}=-\omega\frac{\partial h}{\partial x}+u'\frac{\partial h}{\partial x},\qquad z=h, \tag{71}$$

The last terms on the right-hand sides of these two equations being the smallest, the values of u and u' in them may be taken from the first approximate values

$$u=\omega_0\frac{h}{H}\quad\text{and}\quad u'=-\omega_0\frac{h}{H}.$$

Thus,

$$w=-\frac{\partial\phi}{\partial z}=-\omega\frac{\partial h}{\partial x}+\frac{\omega_0 h}{H}\frac{\partial h}{\partial x},\qquad z=h, \tag{72}$$

and

$$w'=-\frac{\partial\phi'}{\partial z}=-\omega\frac{\partial h}{\partial x}-\frac{\omega_0 h}{H}\frac{\partial h}{\partial x},\qquad z=h. \tag{73}$$

Substituting in eq 72 the value of w from eq 59, and in eq 73 the value of w' from eq 69, the kinematic boundary conditions reduce to

$$H\frac{\partial^2\phi_0}{\partial x^2}=-\omega\frac{\partial h}{\partial x}+2\frac{\omega_0 h}{H}\frac{\partial h}{\partial x}+\frac{\omega_0 H^2}{2}\frac{\partial^3 h}{\partial x^3},\qquad(74)$$

and

$$-H'\frac{\partial^2\phi_0'}{\partial x^2}=-\omega\frac{\partial h}{\partial x}-2\frac{\omega_0 h}{H'}\frac{\partial h}{\partial x}+\frac{\omega_0 H'^2}{2}\frac{\partial^3 h}{\partial x^3}.\qquad(75)$$

Multiplying eq 74 by ρ/H, integrating with respect to x, and putting the constant of integration equal to zero since there is no disturbance at infinity, one obtains

$$\rho\frac{\partial\phi_0}{\partial x}=-\rho\omega\frac{h}{H}+\rho\omega_0\left(\frac{h}{H}\right)^2+\rho\frac{\omega_0 H}{2}\frac{\partial^2 h}{\partial x^2}.\qquad(76)$$

Multiplying eq 75 by ρ'/H', integrating with respect to x, and again putting the constant of integration equal to zero,

$$-\rho'\frac{\partial\phi_0'}{\partial x}=-\rho'\omega\frac{h}{H'}-\rho'\omega_0\left(\frac{h}{H'}\right)^2+\rho'\frac{\omega_0 H'}{2}\frac{\partial^2 h}{\partial x^2}.\qquad(77)$$

Adding eq 76 and 77,

$$-\rho'\frac{\partial\phi_0'}{\partial x}+\rho\frac{\partial\phi_0}{\partial x}=-\omega\left[\frac{\rho'}{H'}+\frac{\rho}{H}\right]h+\omega_0\left[\frac{\rho}{H^2}-\frac{\rho'}{H'^2}\right]h^2$$
$$+\frac{\omega_0}{2}[\rho'H'+\rho H]\frac{\partial^2 h}{\partial x^2}.\qquad(78)$$

The dynamic condition for the interface given by eq 7 may be written in the form

$$\rho'\frac{\partial\phi'}{\partial t}-\rho\frac{\partial\phi}{\partial t}=\rho' gh-\rho gh+\frac{\rho' u'^2}{2}-\frac{\rho u^2}{2}.\qquad(79)$$

The last two terms on the right-hand side are small quantities and these may be evaluated by the first approximative values of the particle velocities $u'=-\omega_0 h/H'$ and $u=\omega_0 h/H$.

Hence,

$$\rho'\frac{\partial\phi'}{\partial t}-\rho\frac{\partial\phi}{\partial t}=(\rho'-\rho)\,gh+\frac{\omega_0^2}{2}\left[\rho'\left(\frac{h}{H'}\right)^2-\rho\left(\frac{h}{H}\right)^2\right].\qquad(80)$$

In view of the rule given by eq 40a,

$$-\rho'\omega\frac{\partial\phi'}{\partial x}+\rho\omega\frac{\partial\phi}{\partial x}=-(\rho-\rho')gh$$
$$+\frac{\omega_0^2}{2}\left[\rho'\left(\frac{h}{H'}\right)^2-\rho\left(\frac{h}{H}\right)^2\right].\qquad(81)$$

Expressing the value of the density difference $\rho-\rho'=\Delta\rho$ in terms of ω_0^2, using eq 38, and dividing by ω,

$$-\rho'\frac{\partial\phi'}{\partial x}+\rho\frac{\partial\phi}{\partial x}=-\frac{\rho\omega_0^2}{\omega}\left[\frac{1}{H'}+\frac{1}{H}\right]h$$
$$+\frac{\omega_0^2}{2\omega}\left[\rho'\left(\frac{h}{H'}\right)^2-\rho\left(\frac{h}{H}\right)^2\right].\qquad(82)$$

Keeping only the first two terms of the right-hand members of eq 54 and 65,

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi_0}{\partial x}-Hh\frac{\partial^3\phi_0}{\partial x^3},\qquad z=h,$$

and

$$\frac{\partial\phi'}{\partial x}=\frac{\partial\phi_0'}{\partial x}-H'h\frac{\partial^3\phi_0'}{\partial x^3},\qquad z=h.$$

In view of the approximative values from eq 58 and eq 68,

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi_0}{\partial x}+\omega_0 h\frac{\partial^2 h}{\partial x^2},\qquad z=h,$$

$$\frac{\partial\phi'}{\partial x}=\frac{\partial\phi_0'}{\partial x}+\omega_0 h\frac{\partial^2 h}{\partial x^2},\qquad z=h.$$

Thus, if $(\omega_0 h)\,\partial^2 h/\partial x^2$ is neglected, eq 82 reduces to

$$-\rho'\frac{\partial\phi_0'}{\partial x}+\rho\frac{\partial\phi_0}{\partial x}=-\rho\frac{\omega_0^2}{\omega}\left[\frac{1}{H'}+\frac{1}{H}\right]h$$
$$+\frac{\omega_0^2}{2\omega}\left[\rho'\left(\frac{h}{H'}\right)^2-\rho\left(\frac{h}{H}\right)^2\right],\qquad z=h\qquad(83)$$

Equating the right-hand members of eq 78 and 82, since the left-hand members are the same quantities, and collecting the terms,

$$\left[\omega\left(\frac{\rho'}{H'}+\frac{\rho}{H}\right)-\rho\frac{\omega_0^2}{\omega}\left(\frac{1}{H'}+\frac{1}{H}\right)\right]h$$
$$+\left[\frac{\omega_0^2}{2\omega}\left(\frac{\rho'}{H'^2}-\frac{\rho}{H^2}\right)+\omega_0\left(\frac{\rho'}{H'^2}-\frac{\rho}{H^2}\right)\right]h^2$$
$$-\frac{\omega_0}{2}[\rho'H'+\rho H]\frac{\partial^2 h}{\partial x^2}=0.\qquad(84)$$

If we divide by $\omega\rho$ and ignore the terms containing $\Delta\rho/\rho$, the final result is

$$\left[\left(\frac{\omega}{\omega_0}\right)^2-1\right]\left[\frac{1}{H'}+\frac{1}{H}\right]h-\frac{3}{2}\left[-\frac{1}{H'^2}+\frac{1}{H^2}\right]h^2$$
$$-\frac{1}{2}[H'+H]\frac{\partial^2 h}{\partial x^2}=0.$$

This simplifies, after dividing by $(H'+H)/HH'$, to

$$\left[\left(\frac{\omega}{\omega_0}\right)^2-1\right]h-\frac{3}{2}\left[\frac{H'-H}{H'H}\right]h^2-\frac{1}{2}[H'H]\frac{\partial^2 h}{\partial x^2}=0.\qquad(85)$$

Multiplying by $\partial h/\partial x$, integrating once with respect to x, and putting the constant of integration equal to zero since all disturbances vanish at infinity, we obtain

$$\left(\frac{\omega^2}{\omega_0^2}-1\right)h^2-\left(\frac{H'-H}{H'H}\right)h^3-\frac{1}{2}H'H\left(\frac{\partial h}{\partial x}\right)^2=0. \quad (86)$$

Let the crest height of the wave be h_1. As this is a maximum, $\partial h/\partial x$ vanishes at this point. Hence

$$\left(\frac{\omega^2}{\omega_0^2}-1\right)h_1^2-\left(\frac{H'-H}{H'H}\right)h_1^3=0,$$

or

$$\frac{\omega^2}{\omega_0^2}-1=\frac{H'-H}{H'H}h_1, \quad (87)$$

or

$$\omega=\omega_0\sqrt{1+\left(\frac{H'-H}{H'H}\right)h_1}. \quad (88)$$

This is the law of velocity of propagation of an internal solitary wave. The magnitude of ω_0, the velocity of propagation when the wave is exceedingly small, is given in eq 38. Substituting in eq 86 from eq 88,

$$h_1h^2-h^3-\frac{1}{2}\frac{H'^2H^2}{H'-H}\left(\frac{\partial h}{\partial x}\right)^2=0,$$

and the solution of the equation is

$$h=h_1\operatorname{sec}^2\mathrm{h}\,\alpha\left(\frac{x}{H}-\frac{\omega t}{H}\right), \quad (89)$$

where

$$\alpha=\sqrt{\frac{1}{2}\left(\frac{H'-H}{H'}\right)\frac{h_1}{H}}. \quad (90)$$

This gives the form of internal solitary waves.

Examination of eq 86 reveals that the relation of the thickness of the two layers has an important bearing on the formation of internal solitary waves. When the depth of the upper layer is greater than the depth of the lower layer, $H'>H$, the internal solitary wave is of positive type, that is, h is positive. When the depth of the lower layer is greater than the depth of the upper layer, the internal solitary wave is of negative height, that is, h is negative everywhere. When the two layers are of the same depth and the difference of the densities is very small, the formation of a solitary wave is excluded by this analysis.

The consideration of a fairly large number of experiments tending to verify the above theoretical results, first in regard to the dependence of velocity of propagation on wave height, and second in regard to the form of internal solitary waves, will be reserved for another occasion.

5. References

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