Journal of Research of the National Bureau of Standards Vol. 51, No. 2, August 1953 Research Paper 2439

Penetration of X- and Gamma Rays to Extremely Great Depths[1]

U. Fano [2]

Earlier work on the asymptotic trend of the X-ray intensity at great distances from a source is reviewed and completed in various aspects. The asymptotic law is shown to be the same as in the "straight-ahead" approximation (which disregards deflections) whether the primary energy is higher or lower than the energy of minimum absorption, provided a constant is replaced by the eigenvalue of a suitable Wick equation. The penetration in directions oblique to the source direction hardly ever attains its asymptotic trend when the source energy is lower than the energy of minimum absorption. This situation raises a difficult problem regarding the penetration law in the range of great depths where the asymptotic trend is being approached very slowly.

1. Introduction

The deep penetration of X-rays, or gamma rays, in an infinite, homogeneous medium are discussed. Much fundamental understanding of this phenomenon derives from the work of Wick [1] [3] on the analogous phenomenon of neutron penetration. Vice versa, the present work might contribute additional clarification to neutron problems as well as some background for still unsolved problems of charged-particle penetration.

This paper constitutes a final report of developments since 1948. It includes a review of preliminary reports [2, 3, 4] and new material that was required to complete the initial program. Some concepts and techniques developed in the course of the study but not utilized in the eventual solution are nevertheless reported briefly.

The assumption of an infinite, homogeneous medium disregards the effect of boundaries and inhomogeneities, which constitutes a separate, still largely unsolved, problem.

The very deep penetration of X-rays depends primarily on the course of multiple Compton scattering under conditions where photoelectric absorption and pair production are comparatively unimportant. At the low-energy end of the spectrum, for example, below 50 kev, where the energy shift of the scattered photons can be disregarded or treated as a small correction, the diffusion of photons has been studied by Chandrasekhar [5], but without specific reference to very deep penetrations. (Notice that photons below 50 kev disappear rapidly by photoelectric effect, in most materials.)[4] At the high-energy end, where large amounts of X-rays are regenerated by electrons, a complete study of X-ray penetration would require a treatment of the whole cascade shower process. The X-ray regeneration remains moderate for secondary electrons up to 10 Mev in lead, and up to 100 Mev in light materials, and may be treated, if necessary, in this energy range as a secondary source. The present paper disregards the regeneration of X-rays by electrons and treats pair production as a mechanism of outright absorption. Nevertheless, its results apply to the penetration of the tail end of showers that is controlled by photons below 10 Mev in lead and below 100 Mev in light elements.

We shall deal primarily with photons from 50 kev to 50 Mev which experience concurrent processes of energy degradation and multiple scattering terminating in outright absorption. Repeated Compton scattering of the "primary" radiation emitted by a source gives rise to "secondary" X-rays of lower energy traveling in all directions. In a medium of low atomic weight, a photon may be scattered 5 or 10 times, on the average, before eventual absorption by photoelectric effect. The main complication in our study arises from the generation of this complex radiation.

---

[1] Work supported by the Office of Naval Research and by the Atomic Energy Commission Reactor Division.

[2] Appendix C is by L. V. Spencer.

[3] Figures in brackets indicate the literature references at the end of this paper.

[4] This consideration applies also to the further development of Chandrasekhar's method that was carried out by O'Rourke [21] in connection with the X-ray penetration problem.

This complication and the complicated dependence of the cross sections on the photon energy preclude a fully analytical treatment of the problem. Therefore, the analytical treatment that is developed in this paper does not lend itself readily to direct numerical applications. However, its results provide one of the footings for a practical method of numerical calculation [6] whose range of application extends to very large distances from the source. The analytical laws of intensity variation versus depth also serve to extrapolate the results of numerical calculations to still greater distances from the source.

Throughout the present investigation, much reliance has been placed on gaining an initial qualitative understanding of the factors that control the process of degradation, scattering, and penetration of X-rays. Such an understanding enables one to adapt the mathematical procedure to the conditions affecting each specific problem. In this respect our work departs from other work in the same field that has been characterized by the initial adoption of less flexible procedures. For example, the X-ray distribution has been resolved into "orders of scattering" (that is, into once-, twice-, thrice-, . . ., scattered photons) [7, 8, 9]. The expansion converges poorly where the photons experience protracted scattering and its application becomes cumbersome. However, at least in some instances, adequate calculations of penetration have been made [8]. Semiempirical improvements to the procedure have also been introduced [9]. Alternatively one may schematize the scattering and degradation process in such a way that the corresponding basic equations are sufficiently simplified to allow analytic solution [7, 9, 10, 11]. These simplifications, however, seem generally to involve considerable limitations in the applications of the theory and may delete some characteristic features of the phenomenon. Finally, calculations by the "Montecarlo" sampling methods are possible [12] but do not yet appear convincingly successful, at least in their application to very deep penetrations, which requires the use of "biased sampling".

Our treatment will be limited to X-ray distributions having a plane symmetry, that is, which vary in one direction of space only and are generated by sources having the same symmetry. The study of spherically symmetrical distributions is essentially equivalent to that of plane-symmetrical ones. The relationship between these types of distributions is well known [1] and will be indicated in Appendix A. We shall also assume that the source is concentrated on a single plane, which implies no additional restriction owing to the linearity of the problem. Notice that plane symmetry may always be attained by considering, instead of the X-ray flux at each point and in each direction, only the integral or the average flux over all points of a plane and over all directions forming the same angle with the plane. The reason is that this plane integral flux generated by a localized source is equal to the localized flux generated by a plane distribution of sources.

The most general type of source may be regarded as an aggregate of point monodirectional sources. The study of X-ray penetration under conditions of plane symmetry may serve as a first step for the treatment of point monodirectional sources. This approach has been exploited in the study of X-ray penetration to small or moderate distances from a source [13].

2. Survey of the Problem

This section contains a qualitative analysis of the deep penetration of X-rays, with the purpose of pointing out the specific problems that require a detailed mathematical treatment.

2.1. Transient Processes and Equilibrium States

The layers of material near an X-ray source are traversed almost exclusively by primary radiation because it takes a certain thickness of material to generate a substantial amount of secondary radiation. Therefore, one expects the intensity of secondary radiation to build up rapidly from one layer to the next near the source in a sort of transient process. Vice versa, at greater depth within a material one may hope to find some sort of steady state in which the secondary radiation is present in substantial amounts. The nature of this steady state is the main object of this investigation.

A typical steady state in which a softer radiation continually arises from a harder one is commonly called a state of radiative "equilibrium." The X-ray penetration problem does not lead to a typical state of equilibrium, but it is helpful to review first the properties of such a state. Equilibrium involves, in general, three different features, namely: (1) The ratio of the intensity of the secondary to the intensity of the primary radiation approaches asymptotically a maximum value, as the depth of penetration increases. (2) The quality of the secondary radiation also becomes independent of the depth of penetration, because the intensities of its various spectral components bear a constant asymptotic ratio to the primary intensity. (3) Therefore, the intensity of the whole radiation is controlled by the progressive attenuation of the primary radiation, which follows an exponential law in the case of X-rays. If these circumstances applied, they would greatly simplify the study of penetration. The maximum values of intensity ratios mentioned in (1) and (2) are determined by the ratios between the attenuation coefficients [5] μ of the X-rays of various energies. The depth of penetration at which the intensity ratio between any two X-ray components attains a value within, say 10 percent of the maximum, turns out to be inversely related to the difference between the attenuation coefficients of those components.

2.2. State of Limited Equilibrium

The approach to a steady state under the conditions of hard X-ray penetration presents the following characteristic departure from the more familiar equilibrium conditions. In the Compton effect, some of the X-rays have very nearly the same energy, the same direction, and the *same attenuation coefficient* after as before the scattering. In fact, the attenuation coefficient of some among the secondary X-ray components *differs only infinitesimally* from the attenuation coefficient of the most penetrating X-rays that are present. This circumstance does not merely slow down the approach to equilibrium, it actually suppresses the features (1) and (3) in that the intensity ratio of secondary X-ray components to a monochromatic primary may *grow beyond any limit.*

At the same time one may expect that, within a finite thickness of shield, some sort of equilibrium should be attained among those components of the secondary X-ray spectrum, whose attenuation coefficient differs from that of the most penetrating component by a sufficiently large amount. In fact, the relative intensity of most secondary components and, therefore, the average "quality" of most of the secondary radiation turn out to approach a limiting value within the thickness of a finite shield, as indicated in feature (2). On the other hand, features (1) and (3) do not obtain. The intensity ratio of this secondary radiation to the primary X-rays does not approach a maximum value, but it keeps increasing; the attenuation of the primary radiation alone does *not* control the attenuation of the secondary radiation.

In a hypothetical state of true equilibrium, the flow of radiation at great depths within a shield is described as the product of an exponential function of the depth and of a function of the energy and direction of each secondary radiation component. In the sort of steady state that is actually achieved, the flow of *most* of the secondary radiation at great depths is described as a product of a *nonexponential* function of the depth and of a function of the energy and direction of each component. Calling x the depth of penetration, measured from the source, E the energy of a photon, and θ the angle that its direction of travel forms with the x axis, E_0 the energy of the primary photons, the flux of photons of various energies and directions $\Phi(x,E,\theta)$ takes then the form

$$\Phi(x, E, \theta) \sim f(x) g(E, \theta) \tag{1}$$

when x is sufficiently large and E sufficiently smaller than E_0.

At large depths most of the energy is carried by the softer secondary components. One can, therefore, use the approximate form of the flux to evaluate the quantities of practical interest. These quantities vary, as a function of depth, in proportion to $f(x)$.

[5] The term "attenuation coefficient" is used instead of "absorption coefficient" to make it clearly understood that μ includes the effect of removal of X-rays from a narrow beam by Compton effect in addition to their true absorption by photoelectric effect or pair production.

2.3. Intensity Variation at Great Depth

The over-all attenuation of the radiation does not follow an exponential law because the most penetrating secondary components never approach an equilibrium. The determination of the over-all course of attenuation, which is described by the function $f(x)$, requires, therefore, a study of the ever developing transient process of formation and destruction of the hardest secondaries. This study constitutes the main topic of the present paper.

By centering one's attention on the formation and absorption of the most penetrating secondaries, one attains a substantial simplification of the problem. The secondaries that need be considered cover only a narrow spectral range. Therefore, the variations of the scattering and absorption cross sections within this spectral range can be treated as small quantities. Furthermore, the greatest contribution to the deep penetration arises from secondaries directed in a very narrow beam perpendicular to the source plane.[6]

However, the small differences of attenuation coefficient and of direction among the most penetrating components cannot be disregarded, even in the first approximation. On the contrary, it is just *these differences that control* the relative intensity of the various components and thereby the over-all *course of attenuation* at great depths. The following argument indicates that small changes of attenuation coefficient and of direction have comparable importance. Radiation of wavelength λ, which travels in a direction forming an angle ϑ with the axis of penetration, x, travels a distance $\Delta/\cos\vartheta$ as its distance from the source plane increases from x to $x+\Delta$. Therefore, its intensity decays along x with an effective attenuation coefficient $\mu(\lambda)/\cos\vartheta$. Compton scattering of a photon, with a wavelength shift of $\delta\lambda$ Compton units, results in a change of both $\mu(\lambda)$ and $\cos\vartheta$. If a photon starts with a wavelength λ_0 in a direction with $\theta_0=0$, $\cos\theta_0=1$, after one scattering its direction changes, according to the Compton law, to $\vartheta=\arccos[1-\delta\lambda]$ and its effective absorption coefficient from $\mu(\lambda_0)/\cos\vartheta_0=\mu(\lambda_0)$ to $\mu(\lambda_0+\delta\lambda)/[1-\delta\lambda]\sim\mu(\lambda_0)+(d\mu/d\lambda)_0\delta\lambda+\mu(\lambda_0)\delta\lambda$. The corrective term $(d\mu/d\lambda)_0\delta\lambda$ arises from the change of μ, the term $\mu(\lambda_0)\delta\lambda$ from the change of $\cos\vartheta$. The ratio of the two corrective terms is $d\log\mu/d\lambda$, with λ in Compton units, which is a number of the order of 1 for most materials and wavelengths of interest.

The importance of the small deflections was not appreciated in the early stages of the investigation, which disregarded altogether the changes of direction experienced by the most penetrating radiation components, that is, relied on the "straight-ahead approximation." The results obtained in this manner, which are reported in [2, 3] and reviewed in section 5, proved useful, nevertheless, because the effect of small deflections does not change the analytical form of the intensity variation but only the value of certain numerical constants. This simple mode of action of the deflections emerges as a result of the analytical treatment of section 7 and may be visualized as follows. Photons that travel in directions oblique to the direction of maximum penetration are selected-against in the course of deep penetration, much as if they had, in effect, a larger attenuation coefficient $\mu(\lambda)$ (see above). The selection operates simultaneously in favor of photons with near-minimum μ and near-minimum, that is, near-zero, obliquity ϑ. As a result, photons of each wavelength are concentrated in a cone of directions whose aperture is the narrower the nearer to the minimum is μ. This is to say that the effect of small obliquity merely parallels and amplifies the effect of small variations of the attenuation coefficient. Indeed, the two effects result in a fixed ratio to one another (see sections 6 and 7), so that the study of the effect of variations of μ alone, in the straight-ahead approximation, gives the correct analytical form for the trend of the total intensity at great depths.

Two different analytical forms for this trend result under different circumstances, depending on the type of variation of the attenuation coefficient among the most penetrating components. As long as the energy of the primary X-rays is not too large, none of the secondary X-rays is more penetrating than the primary X-rays. The hardest secondaries are then simply those whose energy is just a little lower than the energy of the primaries. The progressive accumula-

[6] When the whole primary radiation travels in directions oblique to the source plane, there occurs a more complicated situation that will be discussed in section 2.6 and in section 8.

tion of secondary components under these conditions has been studied in [2] under the straight-ahead approximation and in [4], taking into account the deflection effects. The intensity distribution law $f(x)$ is shown there, and in section 9, to be of the form

$$f(x)=x^{K}\exp(-\mu_0 x). \tag{2}$$

Here μ_0 is the attenuation coefficient of the primary X-rays and K is a number, usually of the order of 1; K depends on the cross section for Compton scattering and on the first derivative of the attenuation coefficient with respect to energy, evaluated at the energy of the primary X-rays. (For numerical values of K and the parameters on which it depends, see appendix C, tables 1 and 2.) The exponential factor represents the usual effect of attenuation of the primaries, whereas the factor x^K represents the effect of accumulation of hard secondaries and is called the "buildup factor". Numerical values of K are given in appendix C.

At very high energies, absorption by pair production increases so rapidly that higher energy X-rays are actually less penetrating than lower energy ones. Therefore, in the case of high-energy primaries, some of the secondary X-rays are more penetrating than the primaries. The attenuation of the total radiation depends upon the formation and decay of the most penetrating secondary components, whose photon energies may be much lower than that of the primaries (in lead the hardest photons have about 3-Mev energy). This phenomenon has been treated in [3] in the straight-ahead approximation, but the effect of small deflections has not been taken into account in previous papers. Both treatments are given in sections 5, 7, and 9 and show the intensity distribution law $f(x)$ to be of the form

$$f(x)=x^{-5/6}\exp[H(\mu_m x)^{\frac{1}{3}}]\exp(-\mu_m x). \tag{3}$$

Here μ_m indicates the minimum value of the attenuation coefficient, that is, the attenuation coefficient of the most penetrating X-rays in the material under consideration. The factor $\exp(-\mu_m x)$ represents the attenuation of these X-rays. The factor $x^{-5/6}\exp[H(\mu_m x^{\frac{1}{3}}]$ is the buildup factor, corresponding to x^K in (2), and H is a constant that depends on the cross section for Compton scattering and on the values of the attenuation coefficient and of its second derivative, evaluated at the energy of the most penetrating X-rays. (For values of μ_m and H, see appendix C, table 3.)

2.4. Numerical Methods Combined With Analytical Results

The analytical results, (2) and (3), represent limiting forms of the intensity distribution, valid at extremely great depths, where the distribution depends on the generation and attenuation of an extremely narrow range of secondary spectral components. Those results must be supplemented in two directions. In the first place, one must find not only the relative X-ray intensity at different positions but the absolute intensity, particularly of lower energy components traveling in various directions, that is, the function $g(E,\vartheta)$ of eq (1). In the second place, one must be able to carry out calculations without excessively stringent limitations on the important range of secondary spectral components.

Various partially successful attempts were made to develop methods of successive approximation to take into account successively higher derivatives of the cross sections in the important spectral range. However, much greater success was eventually achieved by the semiasymptotic numerical method of Spencer [6], which fulfills both of the requirements indicated above. Additional numerical results for the X-ray distribution at moderate depths are provided by polynomial method of calculation [13].

Reliance on numerical methods to the extent of solving the transport equation numerically was dictated by the complicated dependence of the cross sections upon the X-ray energy. At the same time, since the transport equation involves three independent variables (namely distances from the source, photon energy, and direction of propagation), the reduction of the numerical burden to manageable proportion requires much guidance from qualitative analysis and as much help from analytical development as conveniently possible. Spencer has stressed the rapid gain in computational efficiency that can be derived from general information on the desired solution [6, 14]. Economy also arises from suitable substitutions of the variables of the problem (see, for example [13]).

The space variable, depth of penetration from the source, can be separated from the other variables by a Fourier-Laplace transform. This transformation has the additional advantage of replacing the depth of penetration, which varies over a wide range, with a transform variable whose important range is narrow. As will be seen in section 4, the analytical treatment that determines the intensity distribution laws (2) and (3) deals specifically with the behavior of the distribution function at a singular point of the transform variable. The numerical method holds only for values of the transform variable at a finite distance from the singular point. The analytical treatment serves as an essential complement to determine the critical behavior at infinitesimal distances from the singularity. Furthermore, the knowledge of this analytical behavior and of the resulting behavior of the corresponding inverse transform, (the function $f(x)$ of eq (1), (2), and (3)), serves as a guide for the inverse transform procedure that is also required in the numerical work.

The two remaining variables, photon energy and direction of propagation, can be separated only within the limits of application of the analytical treatment, that is, for extremely deep penetration and narrow range of photon energies (section 7). Otherwise, the interlinkage of energy and direction offers the most serious difficulty to the solution of the problem. The difficulty has been overcome in the numerical work by learning how to take into account this interlinkage without excessive complication [6]. This is done by describing the directional distribution adequately by means of a few parameters; namely, moments when the distribution is peaked, and coefficients of a Legendre expansion when it is flat. The interlinkage of energy and direction is reviewed briefly in section 6. Numerical methods have to complement the analytical treatment even when the separation of variables succeeds in order to work out the directional distribution (section 7 and appendix B).

2.5. Verification of the Qualitative Analysis

The introductory qualitative picture of the X-ray penetration, which has led to the results indicated by eq (1), (2) and (3), has been verified to a considerable extent by experimentation and by independent numerical calculations.

Two experiments dealt with a point source of Co^{60} γ-rays surrounded by a mass of water. In the first experiment [15] ionization-chamber and Geiger-counter measurements of total intensity were made at various distances from the source. The results of these measurements should be represented, in terms of the approximate eq (1), by $f(x)\int_0^\pi 2\pi \sin\vartheta d\vartheta \int_0^\infty g(E,\vartheta) r(E) dE$, where $r(E)$ indicates the response of the measuring instrument. In the absence of information on $g(E,\vartheta)$, the plot of the measurement versus the depth of penetration x should follow the trend of the function $f(x)$. The plot does, in fact, take up the trend of (2) at great distance from the source (1.5 to 2.5 m of water), with a value of K equal to that predicted by the detailed theory of [4] and of section 7.

In the second experiment [16], the spectrum of the secondary electrons in the water was measured at various distances from the source. The plot of the number of electrons of various energies ϵ, should be represented according to eq (1) by $f(x)\int_0^\pi 2\pi \sin\vartheta d\vartheta \int_0^\infty g(E,\vartheta) R(E,\epsilon) dE$, where $R(E,\epsilon)$ indicates the probability of production of an electron of energy ϵ by a photon of energy E. Qualitatively, the shape of the plot becomes independent of the penetration depth x at great depths as predicted by (1).

Equation (1) gives no information about the shape or the absolute magnitude of $g(E,\vartheta)$. However, rather detailed calculations of the complete distribution function $\Phi(x,E,\vartheta)$, for all depths of penetration covered by the experiments can be made by the independent method of polynomial expansion [13]. These calculations have successfully predicted the quantitative results of the two experiments.

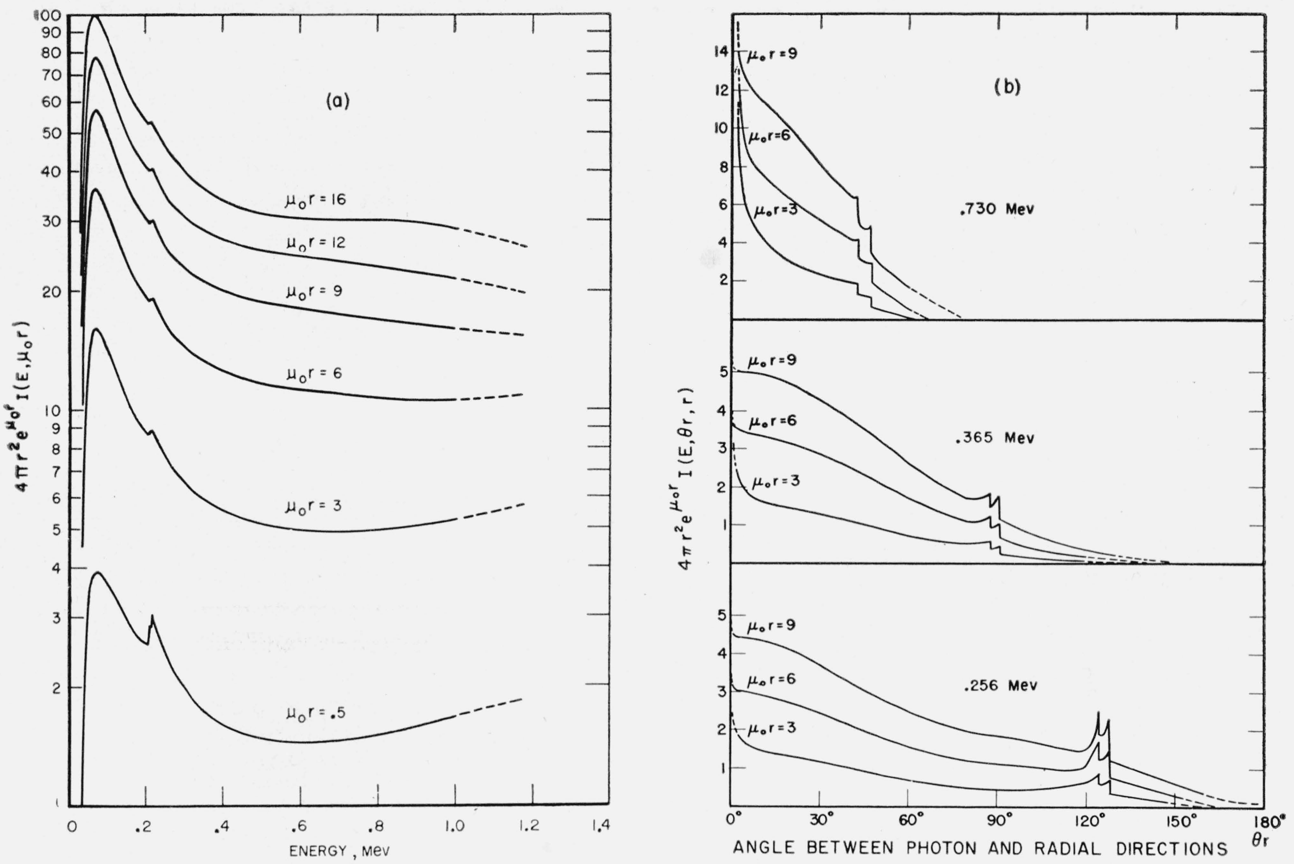

FIGURE 1. (a) *Differential spectra $I(E,\mu_0 r)$ of the X-ray intensity at various distances r from a* Co–60 *point isotropic source in water.*

The higher energy portions of the spectra have been omitted for clarity. The parallelism of the curves on the left side of this semilogarithmic plot indicates the trend towards equilibrium.

(b) *Directional distribution $I(E,\vartheta_r,r)$ of the X-ray intensity for photons of several energies E at several distances r from a point isotropic source of* Co–60 *in water.*

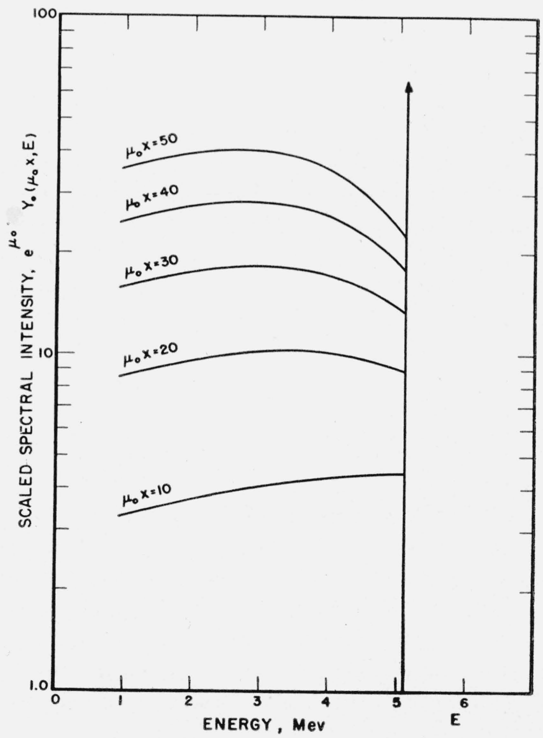

FIGURE 2. *Differential spectra $Y_0(\mu_0 x,E)$ at various distances x from a plane monodirectional source of* 5.11 *Mev photons in* Fe.

Notice the trend to parallelism on the left side as in figure 1, a.

In addition, the results of these calculations for moderately great depths provide a semiquantitative check of the predictions of the theory of penetration to very great depths. These predictions find good verification. In particular, the trend toward an equilibrium distribution for the lower-energy secondary X-rays is clearly displayed both with respect to the spectral [13] and to the directional [17] distributions (see fig. 1). The emergence of an equilibrium state at lower energies, while the higher energy components persist in a state of nonequilibrium is most clearly displayed in the results of semiasymptotic calculations [6] (see fig. 2).

2.6. Influence of the Source Direction on the Penetration Law

Previous theoretical work on the penetration of radiation from plane sources has dealt only with isotropic sources and with sources concentrated in a direction perpendicular to the source plane. In general, one may want to deal with source radiation aimed in directions that form a fixed skew angle with the source plane. However, this new problem raises questions that are largely unsolved and that will be indicated briefly here and discussed somewhat further in section 8.

At the outset one must distinguish between the conditions where the energy of the primary X-rays lies below and above the energy of minimum attenuation (corresponding to eq (2) and (3), respectively). In the first event the penetration law is controlled by secondary X-rays of energies *just below* the source energy. The overwhelming majority of these secondary X-rays travel in directions *very nearly as oblique* as those of the primary X-rays. Therefore, the obliquity of the source has a very major influence on the intensity distribution at very great depths.

In the second case, when the source energy exceeds the energy of the most penetrating secondaries by a substantial amount, the obliquity of these secondaries is less immediately related to the obliquity of the primaries. It will be shown in section 8 that the penetration is controlled by the accumulation of secondaries that have been degraded by steps to the energy of minimum attenuation and simultaneously deflected to directions nearly perpendicular to the source plane. The absolute intensity of these secondaries depends, of course, on the obliquity of the source but their rate of accumulation and, hence, the law of deep penetration are independent thereof.

Therefore, the main difficulties arising from source obliquity occur only when the energy of the primaries lies below the energy of minimum attenuation. If ϑ_0 indicates the angle formed by the source direction with the x direction, perpendicular to the source plane, the "effective attenuation coefficient" of the primary X-rays of wave length λ_0 is $\mu(\lambda_0)/\cos\vartheta_0$ (see section 2.3). After one Compton scattering with a wavelength shift $\delta\lambda$, the obliquity ϑ is given by $\cos\vartheta=\cos\vartheta_0[1-\delta\lambda]+\sin\vartheta_0[2\delta\lambda-\delta\lambda^2]^{1/2}\cos\varphi$, where φ may have any value. Minimum effective attenuation after one scattering is obtained when $\varphi=0$, but this minimum value $\mu(\lambda_0+\delta\lambda)/<\cos\vartheta>_{\max}$ may be larger or smaller than the initial value $\mu(\lambda_0)/\cos\vartheta_0$, depending on the values of $d\mu/d\lambda$ and of ϑ_0. The most favorable reduction of the effective attenuation coefficient is brought about by a succession of scatterings with infinitesimal $\delta\lambda$'s, all directed so as to reduce ϑ. An infinite sequence of such scatterings would eventually bring about a deflection all the way to $\vartheta=0$ without any finite increase of μ, but the probability of such an event is, of course, infinitesimal of a high order. Nevertheless, an unlikely process of this kind must determine the penetration at extremely great depths. That is the penetration will be eventually controlled by the scanty accumulation of secondary X-rays with energies very near the primaries and direction nearly perpendicular to the source plane, that is, with effective attenuation coefficients near μ_0. The course of accumulation of secondary X-rays that are effectively more penetrating than the primaries because of reduced obliquity has not yet been studied. Knowledge of the limiting trend at extremely great depths may not be very important because it may not even become recognizable except under unrealistic conditions. Nevertheless, lack of understanding of this process may undermine the effectiveness of the semiasymptotic and polynomial methods of calculation, whose application is now being planned.

If the source emits X-rays of various obliquities, some of which travel initially in the direction perpendicular to the source plane, it will be just the X-rays in this direction and their secondaries that achieve great penetration. Deeper and deeper penetrations arise from narrower and narrower bundles of primary X-rays. Therefore, a source with a broad distribution of directions contributes less and less toward deep penetration, as compared to a source aimed exactly in the direction perpendicular to the source ($\vartheta_0=0$). This argument will be confirmed by the analytical treatment of section 8. In particular an isotropic source, or any source with practically uniform distribution in directions nearly perpenducilar to the source plane, yields an intensity distribution law in accordance with eq (2) but with a value of K lower by 1 than the value for a source concentrated at $\vartheta_0=0$ [6].

3. Transport Equation

The following symbols will be used:

$\boldsymbol{r}=(x,y,z)$ = coordinates of a point of a medium.

λ = wavelength of a photon in Compton units (h/mc).

$\boldsymbol{u}=(u_x,u_y,u_z)=(\vartheta,\varphi)$ = unit vector indicating a photon direction.

$\mu(\lambda)$ = total (narrow-beam) attenuation coefficient of the medium for photons of wavelength λ.

μ_{Th} = Thomson scattering coefficient (probability per unit path) of the medium = low energy limit of the integral Klein-Nishina coefficient.

$k(\lambda',\lambda)=(3/8)\mu_{Th}[\lambda'/\lambda+\lambda'^3/\lambda^3+(\lambda'/\lambda)^2(\lambda-\lambda'-2)(\lambda-\lambda')]$ = Klein-Nishina differential scattering coefficient of the medium for Compton scattering with a wavelength change from λ' to λ, per unit λ, for $\lambda'\leq\lambda\leq\lambda'+2$.

$Y(\boldsymbol{r},\boldsymbol{u},\lambda)$ = number of photons per unit volume, per unit solid angle and per unit λ.

$\delta(\xi)$ = Dirac's delta function.

λ_0 = wavelength of the source photons (if monochromatic).

$\mu_0=\mu(\lambda_0)$.

λ_s = wavelength of the photons whose attenuation coefficient has the lowest value in the range of integration from λ_0 to λ.

$\mu_s=\mu(\lambda_s)$.

λ_m = wavelength of the photons whose attenuation coefficient is the absolute minimum in the medium under consideration.

$\mu_m=\mu(\lambda_m)$.

The degradation, penetration, and diffusion of X-rays is governed by a transport equation. This equation represents the rate of change of the density of photons with a specified direction and wavelength, from one point to the next in the direction of propagation, as the sum of three terms namely: (a) the attenuation of the photon density as a result of absorption and scattering, (b) the addition of photons that take up the specified direction and wavelength as a result of Compton scattering, (c) the addition of photons with the specified direction and wavelength from the source. The equation is:

$$\begin{aligned}\boldsymbol{u}\cdot\boldsymbol{grad}\,Y(\boldsymbol{r},\boldsymbol{u},\lambda)&=-\mu(\lambda)Y(\boldsymbol{r},\boldsymbol{u},\lambda)\\&+\int_{\lambda-2}^{\lambda}d\lambda' k(\lambda',\lambda)\int_{4\pi}d\boldsymbol{u}'\,(2\pi)^{-1}\delta(1-\boldsymbol{u}\cdot\boldsymbol{u}'-\lambda+\lambda')\,Y(\boldsymbol{r},\boldsymbol{u}',\lambda')+S(\boldsymbol{r},\boldsymbol{u},\lambda),\end{aligned}\tag{4}$$

where $S(\boldsymbol{r},\boldsymbol{u},\lambda)$ represents the source, that is, the density of photons produced at $\boldsymbol{r}$ with the direction $\boldsymbol{u}$ and the wavelength λ. The δ-function represents the Compton law which requires λ to equal $\lambda'+(1-\boldsymbol{u}\cdot\boldsymbol{u}')$.

Following the plan indicated in the introduction, we assume that the source distribution has a plane symmetry, that is, that it does not depend on y and z, nor on the component φ of $\boldsymbol{u}$, (but only on $\vartheta=\arccos u_x$). As a result, Y will be similarly independent of y,z and φ. We also assume that the source is concentrated on the plane $x=0$ and at a wavelength $\lambda=\lambda_0$, but this constitutes no real restriction owing to the linearity of the equation. Accordingly, we have $S(\boldsymbol{r},\boldsymbol{u},\lambda)=\delta(x)\delta(\lambda-\lambda_0)f(u_x)$ and the equation reduces to

$$u_x\partial Y(x,u_x,\lambda)/\partial x=-\mu(\lambda)Y(x,u_x,\lambda)$$
$$+\int_{\lambda-2}^{\lambda}d\lambda' k(\lambda',\lambda)\int_{4\pi}d\boldsymbol{u}'(2\pi)^{-1}\delta(1-\boldsymbol{u}\cdot\boldsymbol{u}'-\lambda+\lambda')Y(x,u_x',\lambda')+\delta(x)\delta(\lambda-\lambda_0)f(u_x). \quad (5)$$

Notice that the kernel $k(\lambda',\lambda)$ of this equation may be multiplied by any ratio $g(\lambda')/g(\lambda)$, provided Y is suitably renormalized.

4. Fourier-Laplace Transform

The transform method involves a separation of variables. It represents the X-ray distribution as a superposition of components, each of which has a certain angular and spectral distribution *uniform* over all the space. The intensity of each component varies exponentially or sinusoidally from point to point according to $\exp(-px)$, where p may be complex.[7]

The transform and its inverse are:

$$y(p,u_x,\lambda)=\int_{-\infty}^{\infty}\exp(px)Y(x,u_x,\lambda)dx, \quad (6)$$

$$Y(x,u_x,\lambda)=(2\pi i)^{-1}\int_{-i\infty}^{+i\infty}\exp(-px)y(p,u_x,\lambda)dp. \quad (7)$$

The transform method of analysis consists of two steps: (a) A study of the directional and spectral distribution $y(p,u_x,\lambda)$ for various values of p, (b) an evaluation of the inverse transform integral (7).

The evaluation of the integral may proceed by the path of steepest descent. Whether this method is eventually followed or not, an analysis of the "topography" of the transform $y(p,u_x,\lambda)$, that is, of the distribution of its absolute values over the complex plane p proves very useful. This analysis indicates what region of the plane yields the largest contribution to the integral when the steepest descent method is followed. The determination of $y(p,u_x,\lambda)$ in this region, which is often very limited, proves anyhow to be very efficient for evaluating the inverse transform.

The equation that governs the directional and spectral distribution $y(p,u_x,\lambda)$, for each value of p, derives from (5) by means of the transformation (6), that is, by multiplication by $\exp(px)$ followed by integration. This equation is:

$$[\mu(\lambda)-pu_x]y(p,u_x,\lambda)=\int_0^{\lambda}d\lambda' k(\lambda',\lambda)\int_{4\pi}d\boldsymbol{u}'(2\pi)^{-1}\delta(1-\boldsymbol{u}\cdot\boldsymbol{u}-\lambda+\lambda')y(p,u_x',\lambda')+\delta(\lambda-\lambda_0)f(u_x). \quad (8)$$

The source term in this equation does not depend on p because the source term in (5) is concentrated on the plane $x=0$.

Because the wavelength of a photon increases as a result of Compton scattering, the distribution of photons of each wavelength λ depends on the distribution of photons of shorter wavelengths λ' but not on that of longer wavelength photons. If one knows the photon distribution (in spectrum and direction) at all wavelengths $\lambda'<\lambda$, one can evaluate the integral in (8) and thereby find the distribution $y(p,u_x,\lambda)$ on the left side of the equation. Thus eq (8) may be solved in principle stepwise, proceeding from the source wavelength λ_0 to longer and longer wavelengths. This procedure, which reflects the course of the physical process of energy degradation, points to an important effect. If the intensity of X-rays of a certain wavelength builds up to a high value, Compton scattering of these X-rays *builds up*, in turn, the intensity of *all longer wavelength* radiation.

[7] Notice that p has here a sign opposite to that which is most common in the literature. The convention adopted here proves convenient in problems of penetration and straggling.

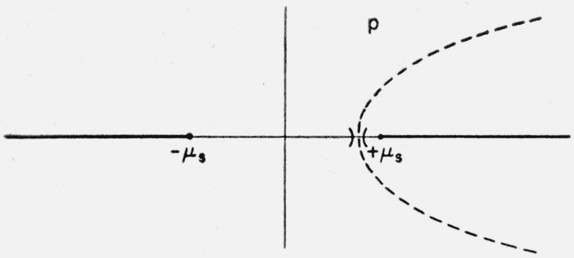

FIGURE 3. *Elements of a relief map diagram of the transform distribution y on the plane of the complex variable p.*

———, Lines over which $y(p)$ is singular; – – –, trend of the steepest descent line of $y(p)$ exp $(-px)$ for positive p;) (, approximate position of the saddle-point.

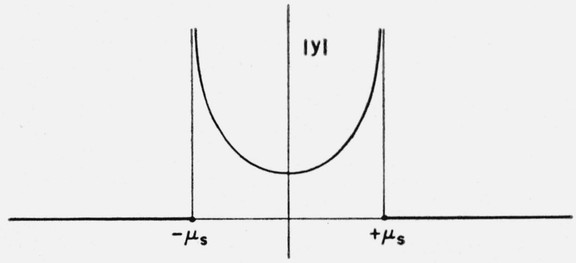

FIGURE 4. *Diagram of a plot of $|y(p)|$ along the real axis of figure 3.*

Equation (8) shows that $y(p,u_x, \lambda)$ becomes infinitely large when $\mu(\lambda)-pu_x$ vanishes. Furthermore, y grows rapidly for those combinations of p,u_x and λ for which $\mu(\lambda)-pu_x$ is particularly small. These combinations of variables are critical in various respects. We just mentioned that once $y(p,u_x, \lambda)$ has become very large for certain values of u_x and λ, it remains correspondingly large for all longer wavelengths. We also know that the distribution of large values of $y(p,u_x, \lambda)$ determines the path of steepest descent of the integral (7). Finally, the process of numerical integration becomes increasingly difficult as the rate of rise of $y(p,u_x, \lambda)$ increases, so that one may want to utilize, at least locally, analytical methods of solution.

The coefficient $\mu(\lambda)-pu_x$ of (8) vanishes only for positive real values of p larger than $\mu(\lambda)$ or for negative real values smaller than $-\mu(\lambda)$. Therefore, eq (8) becomes singular only for values of the complex variable p which are confined to the sections of the real axis marked heavily in figure 3. The lines terminate at $p=\pm\mu_s$, where μ_s indicates the smallest value of $\mu(\lambda')$ in the range of wavelengths from λ_0 to λ. There appears to be no other point in the plane p where $y(p,u_x, \lambda)$ diverges.

Figure 4 shows a schematic plot of $|y|$ against real values of p. The plot rises to infinity at $p=\pm\mu_s$, where the equation has become singular for $u_x=1$ and for the wavelength $\lambda_s\leq\lambda$ at which μ takes its smallest value μ_s.

The integrand of (7) equals $y(p,u_x, \lambda)$ times $\exp(-px)$. The factor $\exp(-px)$, regarded as a function of p, slopes down to the right for positive values of x and to the left for negative values. The slope increases in proportion to the distance from the source, $|x|$. Therefore the integrand has a minimum, for real p, where $\partial \log y(p,u_x\lambda)/\partial p=x$, that is, at a point that approaches $p=\mu_s$ when x is large and positive, or $p=-\mu_s$ when x is large and negative.

This lowest point of $\exp(-px)y(p,u_x\lambda)$ against *real* values of p constitutes the one and only saddle point along the steepest descent path when $\exp(-px)y(p,u_x, \lambda)$ is integrated with respect to the *complex* variable p (see fig. 3).

The minimum and the saddle are sharp when $|x|$ is large. Under these conditions, a knowledge of $y(p,u_x, \lambda)$ over a narrow range of *real* values of p suffices to evaluate the integral (7) and thus to calculate the total distribution of X-rays far from the source. The main object of this paper is just to study the trend of y for real values of p near μ_s (or $-\mu_s$). Section 9 deals with the actual evaluation of the integral (7).

For smaller values of the depth of penetration x, the saddle point lies nearer to $p=0$. The direct calculation of $Y(x,u_x,\lambda)$ at moderate depths also provides convenient expansions of y in powers of p, or of related variables like $p/(\mu_s-p)$ and $p^2/(\mu_s^2-p^2)$ [13]. This information may serve to interpolate y between the proximity of $p=0$ and of $p=\pm\mu_s$.

5. Special Analytical Solutions—Straight-ahead Approximation

In order to study the trend of y for real values of p near μ_s, we seek approximate analytical solutions of the transform equation (8) valid for values of $\lambda\sim\lambda_s$ and of $u_x\sim 1$. This search will be conducted first in the straight-ahead approximation even though this approximation is inadequate.

The straight-ahead approximation consists of setting $u_x=1$ on the left side of (5) or (8). This procedure is equivalent to the replacement of the depth of penetration x in (5) with the path length traveled by the photons, irrespective of the obliquity of penetration [18]. The variables u_x and $\boldsymbol{u}$ that still remain in the eq (8) can then be eliminated by integrating over $\boldsymbol{u}$. Setting

$$\int_{4\pi} d\boldsymbol{u} y(p,u_x,\lambda)=\Phi(p,\lambda), \tag{9}$$

we obtain the straight-ahead approximation equation

$$[\mu(\lambda)-p]\Phi(p,\lambda)=\int_{\lambda_0}^{\lambda} k(\lambda',\lambda)\Phi(p,\lambda')d\lambda'+\delta(\lambda-\lambda_0)\mathrm{I}, \tag{10}$$

where $\mathrm{I}=\int_{4\pi} f(u_x)d\boldsymbol{u}$.

Because we are interested in an approximate solution valid over a narrow range of variation of λ, we disregard the variations of the differential scattering coefficient k over an interval $\lambda'\sim\lambda$ and take

$$k(\lambda',\lambda)\sim k(\lambda,\lambda)=(3/4)\mu_{Th}=C, \tag{11}$$

in agreement with the definition of k (see section 3).

Before entering this approximation into (10), we must distinguish two different situations, as in section 2.3. (a) If the primary radiation is the most penetrating one, that is, if $\mu_s=\mu(\lambda_0)=\mu_0$, the range of approximate integration over λ is right next to the source wavelength λ_0. (b) If the most penetrating radiation is that of wavelength λ_m, for which μ has its absolute minimum value $\mu_m=\mu(\lambda_m)$, that is, if $\lambda_0<\lambda_m$, the range of approximate integration cannot run all the way from λ_0 to λ_m. In this event, we do not attempt to solve the inhomogeneous equation (10), but only the corresponding homogeneous equation, in the range of $\lambda\sim\lambda_m$.

Therefore, we seek the solution of

$$[\mu(\lambda)-p]\Phi(\lambda,p)=C\int_{\lambda_0}^{\lambda}\Phi(\lambda',p)d\lambda'+\mathrm{I}\,\delta(\lambda-\lambda_0)$$

for

$$\lambda\sim\lambda_0>\lambda_m, \tag{12a}$$

and of

$$[\mu(\lambda)-p]\Phi(\lambda,p)=C\int^{\lambda}\Phi(\lambda',p)d\lambda'$$

for

$$\lambda\sim\lambda_m>\lambda_0. \tag{12b}$$

These equations reduce to differential form, if a derivative is taken with respect to λ, and their solutions are

$$\Phi(\lambda,p)=\mathrm{I}\,(\mu_0-p)^{-1}C\,[\mu(\lambda)-p]^{-1}\exp\left\{C\int_{\lambda_0}^{\lambda}d\lambda'/[\mu(\lambda')-p]\right\} \tag{13a}$$

$$\Phi(\lambda,p)=C\,[\mu(\lambda)-p]^{-1}\exp\left\{C\int^{\lambda}d\lambda'/[\mu(\lambda')-p]\right\}. \tag{13b}$$

These solutions are derived as special cases of a more general procedure in [18 section 11b].

We are interested in the behavior of these solutions when p approaches the singular point μ_s, that is, μ_0 in (13a) and μ_m in (13b).

In case (a) the first derivative of $\mu(\lambda')$ at the singular point μ_0, $(d\mu/d\lambda')_0=\dot{\mu}_0$, is different from zero. Therefore, the integral in the exponent of (13a) has a logarithmic singularity. In case (b) the first derivative of $\mu(\lambda')$ vanishes at $\mu(\lambda')=\mu_m$, $\lambda'=\lambda_m$. Therefore the integral in the exponent of (13b) over an interval of $\lambda'\sim\lambda_m$ is of the arctangent type and its singularity has the form $(\mu_m-p)^{-1/2}$. If $\mu(\lambda')$ is regarded as linear in case (a) and parabolic in case (b), the solutions (12) take the characteristic singular forms [3]:

$$\Phi(\lambda,p)\sim \mathrm{I}(\mu_0-p)^{-1}C[\mu(\lambda)-p]^{-1}\exp\left\{\frac{C}{\dot{\mu}_0}\log\frac{\mu(\lambda)-p}{\mu_0-p}\right\}=\mathrm{I}C[\mu(\lambda)-p]^{C/\dot{\mu}_0-1}[\mu_0-p]^{-C/\dot{\mu}_0-1} \quad (14a)$$

$$\Phi(\lambda,p)\sim C\,[\mu(\lambda)-p]^{-1}\exp\{2\pi C/[2\ddot{\mu}_m(\mu_m-p)]^{1/2}\}, \quad (14b)$$

where $\ddot{\mu}_m=(d^2\mu/d\lambda^2)_{\lambda m}$.

Even though the solution of the transform equation (10) for values of $\lambda>>\lambda_s$ cannot be obtained in this simple manner, its dependence on p, as p approaches μ_s, is described by (14) for the following reason. As soon as λ is sufficiently larger than λ_s so that $\mu(\lambda)-p>>\mu_s-p$, p may be replaced in the equation by μ_m. This implies that the further variations of $\Phi(\lambda,p)$, as λ keeps increasing, are independent of p, that is, that a state of limited equilibrium, within the meaning of section 2.2, has been achieved. As p approaches μ_s, λ need depart from λ_s less and less, before the condition $\mu(\lambda)-p>>\mu_s-p$ is adequately fulfilled. Therefore the approximations made in the derivation become increasingly good.

The inversion of Laplace transforms with the singularities (14a) and (14b) will be considered in section 9. However, we anticipate here that the inverse transforms behave, respectively, as (2) and (3) for large depths of penetration x.

For the sake of completeness, we mention yet another situation, which combines the features of (a) and (b), namely, the case where $\lambda_0<\lambda<\lambda_m,\lambda_s=\lambda$. Here the pertinent form of the equation is (12b) and its solution (13b). However, the derivative $(d\mu/d\lambda)_s$ at $\lambda=\lambda_s$ is different from zero. Hence, the integral in the exponent of (13b) has a logarithmic singularity and we find, instead of (14b),

$$\phi(\lambda,p)\sim C\,[\mu(\lambda)-p]^{C/\dot{\mu}_s-1}. \quad (14c)$$

Various attempts were made in the past to develop methods of successive approximations, starting from the approximate solutions (13) and (14), to solve the straight-ahead equation (9) for all values of λ. Direct numerical integration eventually proved superior [6], even for values of p rather near μ_s, despite the fact that the rapid rise of y along λ, for λ near λ_s, requires small steps of numerical integration.

6. Possible Treatments of the Obliquity of Penetration

As pointed out in section 2.3, the deep penetration of X-rays is influenced to a substantial extent by the small obliquity of the most penetrating components. In order to evaluate this effect, one must, in principle, work out the directional distribution of the X-rays of each wavelength, that is, solve the complete two-variable equation (8). In practice, one may want to learn about the distribution-in-angle only the least amount necessary to evaluate the effect on the trend of penetration.

In order to analyze this situation, one may expand the distribution-in-angle of $y(p,u_x,\lambda)$ into Legendre polynominals $P_l(u_x)$,

$$y(p,u_x,\lambda)=\textstyle\sum_l(2l+1)(4\pi)^{-1}y_l(p,\lambda)P_l(u_x). \quad (15)$$

with

$$y_l(p,\lambda)=\int_{4\pi}d\boldsymbol{u}P_l(u_x)y(p,u_x,\lambda). \quad (16)$$

Equation (6) multiplied by each of the P_l's and integrated, reduces to the system of equations

$$\mu(\lambda)y_0(p,\lambda)-py_1(p,\lambda)=\int_{\lambda_0}^{\lambda}k(\lambda',\lambda)y_0(p,\lambda')d\lambda'+\delta(\lambda-\lambda_0)\int_{4\pi}f(u_x)d\boldsymbol{u}$$

. . . . (17)

$$\mu(\lambda)y_l(p,\lambda)-p[(l+1)y_{l+1}(p,\lambda)+ly_{l-1}(p,\lambda)]/(2l+1)$$

$$=\int_{\lambda_0}^{\lambda}k(\lambda',\lambda)P_l(1-\lambda+\lambda')y_l(p,\lambda')d\lambda'+\delta(\lambda-\lambda_0)\int_{4\pi}P_l(u_x)f(u_x)d\boldsymbol{u}.$$

. . . .

The first equation of this system has the same form as the straight-ahead equation (10), except for the fact that y_1 appears in the second term on the left side instead of y_0. If there were a way of guessing the ratio

$$y_1(p,\lambda)/y_0(p,\lambda)=\int_{4\pi} u_x y(p,u_x,\lambda)d\boldsymbol{u}\Big/\int_{4\pi} y(p,u_x,\lambda)d\boldsymbol{u}=\bar{u}_x(p,\lambda), \tag{18}$$

y_1 could be replaced with $\bar{u}_x(p,\lambda)y_0(p,\lambda)$ in the first eq. (17), which could then be solved as a single-variable-straight-ahead equation. Thus a knowledge of $\bar{u}_x(p,\lambda)$, or of an equivalent parameter, embodies all the information about the directional distribution that is immediately relevant to the penetration problem.

Instead of $\bar{u}_x$ one may want to use the equivalent parameter [6]

$$g_1(p,\lambda)=1-\bar{u}_x(p,\lambda)=\int_{4\pi}(1-u_x)y(p,u_x,\lambda)d\boldsymbol{u}\Big/\int_{4\pi} y(p,u_x)d\boldsymbol{u}, \tag{19}$$

which represents the first moment of the angular distribution. If we set $y_1=(1-g_1)y_0$ in (17), the first equation becomes

$$[\mu(\lambda)-p+pg_1(p,\lambda)]y_0(p,\lambda)=\int_{\lambda_0}^{\lambda} k(\lambda',\lambda)y_0(p,\lambda')d\lambda'+\delta(\lambda-\lambda_0)\int_{4\pi} f(u_x)d\boldsymbol{u}. \tag{20}$$

Thus, $g_1(p,\lambda)$ is seen to measure the departure of the eq (20) from the straight-ahead eq (10).

In the critical range of wavelengths, $\lambda\sim\lambda_s$, the solution of (20) may be expressed in a manner similar, for example, to (13b), that is,

$$\begin{aligned} y_0(p,\lambda)&\sim C[\mu(\lambda)-p+pg_1(p,\lambda)]^{-1}\exp\left\{C\int^{\lambda} d\lambda'/[\mu(\lambda')-p+pg_1(p,\lambda')]\right\} \\ &=\frac{C[\mu(\lambda)-p]^{-1}}{1+pg_1(\lambda,p)/[\mu(\lambda)-p]}\exp\left\{\int^{\lambda}\frac{C}{1+pg_1(p,\lambda')/[\mu(\lambda')-p]}\,\frac{d\lambda'}{\mu(\lambda')-p}\right\} \end{aligned} \tag{21}$$

The last expression differs from (13b) in that C is now divided by $1+pg_1(\lambda,p)/[\mu(\lambda)-p]$. We shall see in the next section that this factor tends to a constant limit as p approaches the singular point μ_s.

If the ratio $\bar{u}_x=y_1/y_0$ is assumed to be known, y_0 can be calculated, and thereby y_1 becomes also known. The second equation (17) yields then y_2 directly, the third yields y_3, and so on, that is, the whole system of equations unravels automatically. It was pointed out by Wick [1], and also by Waller [19], that the set of values thus obtained for the y_l, diverges, in general, unless the correct value of $\bar{u}_x$ was chosen at the start. Thus the solution of the system (17) takes the character of an eigenvalue problem. (This property is common to all systems of which the first n equations contain $n+1$ unknowns).

A simple trial and error procedure on the choice of $\bar{u}_x=y_1/y_0$ in (17) is impractical in the critical range of λ where the directional distribution is sharply peaked forward and the successive coefficients y_l, y_{l+1} . . . are nearly equal, so that the trend of the sequences y_l evolves exceedingly slowly. In other words, the angular distribution cannot be characterized in this range by just a few Legendre coefficients. Therefore Wick [1] replaced the sequence of the y_l's with a continuous function and the expression $[(l+1)y_{l+1}+ly_{l-1}]/(2l+1)-y_l$ in (17) with the second derivative of the function. In this manner the problem of determining the value of $\bar{u}_x=y_1/y_0$ was reduced to a problem of differential equations. An approximate determination of the eigenvalue by variational methods offers little difficulty and was carried out by Wick for a constant μ (λ) and for a linear $\mu(\lambda)$ [1].

Whereas variational methods are well suited for deriving rapidly an approximate eigenvalue, they tend to yield poorer approximations to the eigenfunctions and generally do not appear convenient as a basis for higher approximations. In fact, Wick effectively characterizes the angular distribution of radiation by a single adjustable parameter, in the first approximation. A substantially better approximation is required to take into account more general variations of $\mu(\lambda)$ as well as variations of the kernel $k(\lambda', \lambda)$. Extensions of Wick's procedure in this direction have been developed but they appear cumbersome.

When the directional distribution is sharply peaked, as it is in the critical range, $\lambda \sim \lambda_s$, it can be characterized conveniently by its set of moments, which in this case tend to converge far more quickly than the set of Legendre coeffcients y_l. The moments, $m_0 = y_0$, $m_1 = g_1 y_0 = y_1 - y_0$, . . . $m_n = \int_{4\pi} (1-u_x)^n y(p, u_x, \lambda) d\boldsymbol{u}$ are linear combinations of the Legendre coefficients. Determinations of g_1 by trial and error procedures based on calculations of the m_n's should be more practical than procedures based on the y_l's.

Spencer eventually developed an efficient iteration procedure that permits to determine g_1 by trial and error, utilizing only a very few—as few as two—of the equations for the moments [6]. This method has superseded all previously attempted approaches.

One point, which emerged in the course of earlier attempts, might be worth mentioning because of its possible more general interest. The testing of trial values of $g_1(p,\lambda)$ becomes particularly easy when the obliquity of penetration is less effective than the variations of $\mu(\lambda)$ in shaping the X-ray spectrum, that is, according to section 2.3, when $d \log \mu / d\lambda < 1$. In this event, an incorrect initial assumption regarding g_1, not only causes the sequence of moments m_1, m_2 . . . m_n . . . eventually to diverge, but quickly to assume a grossly erratic trend. This critical instability appears to be a general property of systems that may be called "strongly nonself-adjoint." It is discussed in appendix B.

7. Effects of Small Deflections on Special Analytical Solutions

As mentioned in section 6, Wick did the basic work on the effect of small deflections on penetration [1], for two typical situations, namely, constant $\mu(\lambda)$ and linear $\mu(\lambda)$. Either of these assumptions about μ, together with the assumption (11) of a constant kernel $k(\lambda',\lambda)$, makes it possible to separate the direction variable u_x from the energy variable λ in the transform equation (8).

Specifically, the homogeneous part of the equation thus simplified has solutions that are products of a function of λ and of a function of $(1-u_x)p/[\mu(\lambda)-p]$. This to say that the directional distribution maintains a constant *shape* as λ varies, but it contracts in width, as $\mu(\lambda)-p$ grows smaller, in proportion to $[\mu(\lambda)-p]/p$. As a result, the factor $1+pg_1(\lambda,p)/[\mu(\lambda)-p]$ in (21) remains *independent of* λ *and* p. Therefore, the dependence of the photon number on λ and p, irrespective of direction, which is given by (21), has *the same analytical form as the straight-ahead* approximation (13b), except that C is replaced with the constant $C/\{1+pg_1/(\mu-p)\}$. There is a whole set of solutions of the homogeneous equation, with different values of $pg_1/(\mu-p)$.

Wick superposes solutions of this set so as to construct a solution of the inhomogeneous equation with the $\delta(\lambda-\lambda_0)$ source. The component solution with the smallest value of $pg_1/(\mu-p)$ predominates over the other components as p approaches μ_s and as λ increases through the critical range $\lambda \sim \lambda_s$. Therefore, the singularity of the Laplace transform is characterized by the variation of $\mu(\lambda)-p$ in (21) (or in (13b)) as p approaches μ_s and by the smallest value of $pg_1/(\mu-p)$.

The application of this procedure to the X-ray problem is straightforward when the primary radiation is the most penetrating one ($\lambda_0 > \lambda_m$, $\mu_s = \mu_0$) [4], since in the very proximity of λ_0 the variations of $\mu(\lambda)$ may well be regarded as linear, $\mu(\lambda) \sim \mu_0 + \dot{\mu}_0(\lambda - \lambda_0)$, and those of $k(\lambda',\lambda)$ may be disregarded.

The applicability is less clear when the primary radiation is not the most penetrating one ($\lambda_0 < \lambda_m$), that is, when $\mu(\lambda)$ goes through a minimum in the critical range. However, one may surmise that the problem at the minimum ($\lambda \sim \lambda_m$) can be treated by assuming the same value of $pg_1/(\mu-p)$ *as though* $\mu(\lambda)$ were constant. This assumption, which has not been discussed previously, implies that the singularity of the transform is given by the straight-ahead formula (14b) with C replaced with the value of $C/\{1+pg_1/(\mu-p)\}$ corresponding to a constant μ.

We present here first a qualitative argument and then a more detailed treatment which confirm the correctness of the surmise.

The trend of the solution of the transform equation in the critical range depends on the variations of the attenuation coefficient $\mu(\lambda)$ and on the effect of small deflections. We want to show that the deflection effects predominate over the variations of $\mu(\lambda)$, in the limit as p approahces μ_m, so that the deflection effects may be treated as though $\mu(\lambda)$ had the constant value μ_m.

Obliquity of penetration effectively impedes the build-up of photons if the value of $\mu - pu_x$ in (7) becomes substantially larger than the corresponding "straight-ahead" value $\mu - p$. Therefore, the main build-up is confined to photons for which $1 - u_x \lesssim (\mu - p)/p$. Compton scattering of photons within this range of directions corresponds to wavelength shifts $\delta\lambda \lesssim 1 - u_x \lesssim (\mu - p)/p$.

On the other hand, variations of the absorption coefficient that affect the asymptotic behavior must be of the order $\Delta\mu \sim \mu_m - p$. These variations may be described, in the neighborhood in the minimum, as $\Delta\mu \sim \frac{1}{2}\ddot{\mu}_m \Delta\lambda^2$ (where $\ddot{\mu}_m = (d^2\mu/d\lambda^2)_m$). Therefore, the wavelength change $\Delta\lambda$ required to bring about a substantial change of μ is of the order of magnitude $[2(\mu_m - p)/\ddot{\mu}_m]^{1/2}$.

Hence, in the limit for $\mu_m - p \to 0$, this shift $\Delta\lambda$ becomes proportionately much larger than the shift $\delta\lambda$ required to bring about substantial effects of obliquity.

Proceeding beyond this qualitative argument, a complete treatment of the problem would consist of solving the transform equation (8) by a procedure of successive approximations, whose first step should yield the Wick-type results. Efforts in this direction have yielded rather cumbersome developments. In practice such a complete treatment has been made unnecessary by the development of the semiasymptotic numerical method [6] (see section 2.4), which requires, as an analytical complement, only a knowledge of the singularity at $p = \mu_s$, $u_x = 1$. One possible approach to the remaining problem would be to seek a special solution of the eq (8) for the case when the primary radiation is *not* the most penetrating one, that is, when $\lambda_0 < \lambda_m$, $\mu_s = \mu_m$, since Wick has already solved the problem for $\lambda_0 > \lambda_m$, $\mu_s = \mu_0$. Whereas Wick assumed a linear variation of μ, near $\lambda = \lambda_0$, we should assume here a parabolic variation near $\lambda = \lambda_m$, that is, $\mu(\lambda) \sim \mu_m + \frac{1}{2}\ddot{\mu}_m(\lambda - \lambda_m)^2$. Instead of following this approach, we shall take a somewhat more general one, namely, to seek special solutions of the equation valid "in the asymptotic limit"

$$[\mu(\lambda) - p]/p \to 0, \tag{22}$$

that is, for $\lambda \sim \lambda_s$, $u_x \sim 1$. This will be done without utilizing any initial approximation about $\mu(\lambda)$ or $k(\lambda', \lambda)$ but the approximations considered above will result automatically from the consistent application of the limit process.

The first step of our treatment consists in choosing instead of u_x a new direction variable that measures the effect of obliquity on penetration. Equation 21 shows that this effect is proportional to the mean value of $p(1 - u_x)/[\mu(\lambda) - p]$. Accordingly, we replace the unit space vector $\boldsymbol{u} = (\vartheta, \varphi)$ with a homolographic projection variable that is represented in plane polar coordinates by the vector

$$\boldsymbol{v} = ([2p(1 - u_x)/(\mu - p)]^{1/2}, \varphi) \sim ([p/(\mu - p)]^{1/2}\vartheta, \varphi). \tag{23}$$

(The approximate equality corresponds to the small angle approximation $\vartheta << 1$, which we need not utilize at this point.) The inverse transformation of (23) is

$$u_x = 1 - \tfrac{1}{2}v^2[\mu(\lambda) - p]/p, \qquad \boldsymbol{u}(\boldsymbol{v}, \lambda) = (\arccos u_x, \varphi). \tag{24}$$

It is also convenient, though not essential, to renormalize the dependent variable $y(p, u_x, \lambda)$ by multiplication with $\mu(\lambda) - p$, that is, to take

$$w(v, \lambda) = [\mu(\lambda) - p]y(p, u_x, \lambda). \tag{25}$$

The dependence of w on p will not be indicated explicitly.

The transform equation (8) takes now the form

$$(1+\tfrac{1}{2}v^2)w(v,\lambda)=(2\pi p)^{-1}\int_0^\lambda d\lambda' k(\lambda',\lambda)\int d\boldsymbol{v}'\delta(1-\boldsymbol{u}\cdot\boldsymbol{u}-\lambda+\lambda')w(v',\lambda')+\mathrm{I}\,\delta(\lambda-\lambda_0)f(u_x)$$

$$=(2\pi p)^{-1}\int d\boldsymbol{v}' k(\lambda-\Lambda[\lambda,\boldsymbol{v},\boldsymbol{v}'],\lambda)w(v',\lambda-\Lambda[\lambda,\boldsymbol{v},\boldsymbol{v}'])+\mathrm{I}\,\delta(\lambda-\lambda_0)f(1-\tfrac{1}{2}v^2(\mu_0-p)/p), \tag{26}$$

where Λ stands for $1-\boldsymbol{u}\cdot\boldsymbol{u}'$. Since $\boldsymbol{u}$ is given by (24) as a function of $\boldsymbol{v}$ and λ, and similarly $\boldsymbol{u}'$ as a function of $\boldsymbol{v}'$ and $\lambda'=\lambda-\Lambda$, $\Lambda(\lambda,\boldsymbol{v},\boldsymbol{v}')$ is defined as the root of the equation

$$\Lambda=1-\boldsymbol{u}(\boldsymbol{v},\lambda)\cdot\boldsymbol{u}'(\boldsymbol{v}',\lambda-\Lambda). \tag{27}$$

We are now interested in the solutions of (26) in the critical region $\lambda\sim\lambda_s$, $\mu(\lambda)\sim\mu_s$ and in the asymptotic limit $[\mu(\lambda)-p]/p\sim 0$. Following Wick, we seek here solutions of the homogeneous equation corresponding to (26), that is, of

$$(1+\tfrac{1}{2}v^2)w(v,\lambda)=(2\pi p)^{-1}\int d\boldsymbol{v}' k(\lambda-\Lambda,\lambda)w(v',\lambda-\Lambda), \tag{28}$$

and leave it for the next section to superpose solutions of (28) in such a manner as to fulfil the inhomogeneous equation (26) at the source where $\lambda=\lambda_0$.

Our knowledge of the straight-ahead solution and the discussion of section 6 on the effect of small deflection suggest that we split off from $w(v,\lambda)$ a factor $\exp\left\{\overline{C}\int^\lambda d\lambda'/[\mu(\lambda')-p]\right\}$, where $\overline{C}$ remains to be determined, with the intention of showing that this factor contains the whole singularity of w in the asymptotic limit. Thus we set

$$w(v,\lambda)=\chi(v,\lambda)\exp\left\{\overline{C}\int^\lambda d\lambda'/[\mu(\lambda')-p]\right\} \tag{29}$$

$$y(p,u_x,\lambda)=[\mu(\lambda)-p]^{-1}\chi(v,\lambda)\exp\left\{\overline{C}\int^\lambda d\lambda'/[\mu(\lambda')-p]\right\}. \tag{29'}$$

Equation (28) becomes

$$(1+\tfrac{1}{2}v^2)\chi(v,\lambda)=(2\pi p)^{-1}\int d\boldsymbol{v}' k(\lambda-\Lambda,\lambda)\exp\left\{-\overline{C}\int_{\lambda-\Lambda}^\lambda d\lambda'/[\mu(\lambda')-p]\right\}\chi(v',\lambda), \tag{30}$$

and we shall show that in the asymptotic limit Λ vanishes, so that $k(\lambda-\Lambda,\lambda)$ equals C, and the exponential becomes independent of λ. As a result χ becomes, in this limit, a nonsingular function of v only, so that the exponential in (29) does indeed represent the singular behavior of w. The values of $\overline{C}$ will be fixed by the condition that $\chi(v)$ converge for large v.

To carry out this program, we begin by studying the equation (27) which determines Λ. If we substitute $\boldsymbol{u}$ and $\boldsymbol{u}'$ from (24), (27) becomes

$$\Lambda=1-\left[1-\frac{1}{2}\frac{\mu(\lambda)-p}{p}v^2\right]\left[1-\frac{1}{2}\frac{\mu(\lambda-\Lambda)-p}{p}v'^2\right]$$
$$-\left\{\frac{1}{2}\frac{\mu(\lambda)-p}{p}\left[1-\frac{1}{4}\frac{\mu(\lambda)-p}{p}v^2\right]\frac{1}{2}\frac{\mu(\lambda-\Lambda)-p}{p}\left[1-\frac{1}{4}\frac{\mu(\lambda-\Lambda)-p}{p}v'^2\right]\right\}^{1/2}2\,\boldsymbol{v}\cdot\boldsymbol{v}'. \tag{31}$$

Terms of higher order in $[\mu(\lambda)-p]/p$ will be disregarded henceforth in the asymptotic limit. Such terms should be treated by a power expansion in case one intended to proceed to higher approximations. This procedure has the same effect upon (31) as the "small angle approximation" indicated in (23). Equation (31) reduces then to

$$\Lambda=\frac{1}{2}\frac{\mu(\lambda)-p}{p}\left\{\left[v^2+\frac{\mu(\lambda-\Lambda)-p}{\mu(\lambda)-p}v'^2\right]-\left[\frac{\mu(\lambda-\Lambda)-p}{\mu(\lambda)-p}\right]^{1/2}2\,\boldsymbol{v}\cdot\boldsymbol{v}'\right\}. \tag{32}$$

Since this equation indicates that Λ tends to vanish in proportion to $[\mu(\lambda)-p]/p$, in the asymptotic limit, we replace Λ with the variable

$$\xi=\Lambda p/[\mu(\lambda)-p], \tag{33}$$

which we surmise to remain finite in the asymptotic limit. The dependence of $\mu(\lambda-\Lambda)$ on Λ will also be indicated through a parameter which tends to a finite limit, namely

$$\eta=[\mu(\lambda)-\mu(\lambda-\Lambda)]/p\Lambda=[\mu(\lambda)-\mu(\lambda-\xi[\mu(\lambda)-p]/p)]/\xi[\mu(\lambda)-p] \tag{34}$$

$$\mu(\lambda-\Lambda)=\mu(\lambda)-\eta p\Lambda. \tag{35}$$

With these substitutions, (32) reduces to the quadratic form

$$(2+v'^2\eta)^2\xi^2-2[2(v^2+v'^2)+(v'^4+v^2v'^2-2\boldsymbol{v}\cdot\boldsymbol{v}'^2)\eta]\xi+[(v^2+v'^2)^2-4\boldsymbol{v}\cdot\boldsymbol{v}'^2]=0. \tag{36}$$

This equation depends on ξ not only explicitly but also through η. However, if the right side of (34) is expanded into powers of $[\mu(\lambda)-p]/p$ and only the first term is retained, one finds that

$$\eta=(d\mu/d\lambda)/p=\dot{\mu}(\lambda)/p \tag{37}$$

independently of ξ. Therefore, ξ is given by (36) as an algebraic expression $\xi(\boldsymbol{v}\cdot\boldsymbol{v}',\dot{\mu}/p)$ which is finite and independent of $[\mu(\lambda)-p]/p$ in the asymptotic limit. Owing to (33), Λ vanishes then in the asymptotic limit, as we had surmised.

Therefore, in this limit, $k(\lambda-\Lambda,\lambda)=k(\lambda,\lambda)=C$, the exponent of (30) becomes

$$\overline{C}\int_{\lambda-\Lambda}^{\lambda}d\lambda'/[\mu(\lambda')-p]=\overline{C}\int_{\lambda-\xi[\mu(\lambda)-p]/p}^{\lambda}d\lambda'/[\mu(\lambda')-p]=(\overline{C}/\dot{\mu})\ln[1-\xi\dot{\mu}/p]. \tag{38}$$

and the equation (30) reduces to

$$(1+\tfrac{1}{2}v^2)\chi(v,\lambda)=(C/2\pi p)\int d\boldsymbol{v}'\,[1-\xi(\boldsymbol{v},\boldsymbol{v}',\dot{\mu}/p)\dot{\mu}/p]^{\overline{C}/\dot{\mu}}\chi(v',\lambda). \tag{39}$$

The kernel of this integral equation depends on λ through $\dot{\mu}$. Within the narrow critical range of photon energies, $\lambda\sim\lambda_s$, in which we seek an analytical solution, $\mu(\lambda)$ can be expressed as a power series

$$\dot{\mu}(\lambda)=\dot{\mu}_s+\ddot{\mu}_s(\lambda-\lambda_s)+1/2\dddot{\mu}_s(\lambda-\lambda_s)^2+\ldots \tag{40}$$

Accordingly, one may seek the solution of (39) by expanding both the kernel $[1-\xi(\boldsymbol{v},\boldsymbol{v}',\dot{\mu}/p)\dot{\mu}/p]\overline{C}/\dot{\mu}$ and the unknown function $\chi(v,\lambda)$ into powers of $(\lambda-\lambda_s)$. Since the critical range around λ_s becomes infinitely narrow in the asymptotic limit, the singularity of $y(p,u_x,\lambda)$ (at $p=\mu_s,u_x=1$) depends only on the zero-order term of the expansion into $(\lambda-\lambda_s)^n$. Therefore, for the purpose of studying this singularity, we may set

$$\chi(v,\lambda)\sim\chi(v,\lambda_s)=\chi_0(v), \tag{41}$$

where $\chi_0(v)$ fulfils the eq (39) at $\lambda=\lambda_s$, and with $p=\mu_s$, that is,

$$(1+\tfrac{1}{2}v^2)\chi_0(v)=(C/2\pi\mu_s)\int d\boldsymbol{v}'[1-\xi(\boldsymbol{v},\boldsymbol{v}',\dot{\mu}_s/\mu_s)\dot{\mu}_s/\mu_s]^{C/\dot{\mu}_s}\chi_0(v'). \tag{42}$$

In this manner we have reduced the problem to the solution of an equation in the single variable v, that is, we have achieved the same separation of the energy and direction variables (λ and v) as was achieved by Wick. The only difference lies in having achieved the separation by a limiting procedure rather than by initial assumptions regarding the variation of $\mu(\lambda)$ and the constancy of $k(\lambda,\lambda)$. Notice that (42) holds whether λ_s coincides with the primary wavelength λ_0 (that is, for $\lambda_0>\lambda_m$), with the longest wavelength λ (that is for $\lambda_0<\lambda<\lambda_m$) or with the wavelength of minimum absorption λ_m(that is for $\lambda_0<\lambda_m<\lambda$). In the last event $\mu_s=\dot{\mu}_m=0$ and, according to (36) and (37), the kernel of (42) reduces to

$$\{[1-\xi(\boldsymbol{v},\boldsymbol{v}',\dot{\mu}_s/\mu_s)\dot{\mu}_s/\mu_s]^{C/\ddot{\mu}_s}\}_{\dot{\mu}_s=0}=\exp[-(\overline{C}/\mu_m)\xi(\boldsymbol{v},\boldsymbol{v}',0)]=\exp\{-\tfrac{1}{2}(\overline{C}/\mu_m)|\boldsymbol{v}-\boldsymbol{v}'|^2\} \tag{43}$$

Equation (42) is equivalent to the Wick eigenvalue equations and has finite solutions $\chi_0(v)$ only for special values of $\overline{C}$. As in the standard eigenvalue problems with a single independent variable, the successive eigenfunctions $\chi_0^{(0)}(v),\chi_0^{(1)}(v),\ldots\chi_0^{(n)}(v)\ldots$ have $0,1\ldots n\ldots$ nodes. The corresponding eigenvalues $\overline{C}^{(0)}, \overline{C}^{(1)},\ldots\overline{C}^{(n)}\ldots$ depend monotonically on n. The eigenvalue $\overline{C}^{(0)}$ is the largest one, algebraically. Each of the eigenvalues $\overline{C}^{(n)}$ characterizes a type of singularity of $y(p,u_x,\lambda)$. As mentioned before, these singularities coincide with those of the straight-ahead solutions (14) in section 5 except for the replacement of the constant C with $\overline{C}$. The solution of the inhomogenous equation (8) or (26) will be constructed, in the next section, as a superposition of components, each of which is a solution of the homogeneous equation, with different values of $\overline{C}$. We anticipate here that the component with the largest value of $\overline{C}$ in (29′) is clearly the one with the sharpest singularity and therefore the one which predominates and which alone matters in the asymptotic limit.

The equivalence of (42) with the Wick equations is immediately apparent when $\lambda_s=\lambda_m$, $\mu_s=\mu_m$, $\dot{\mu}_s=0$, that is, when the kernel of (42) takes the form (43), so that

$$\left[1+\frac{1}{2}v^2\right]\chi_0(v)=[C/2\pi\mu_m]\int d\boldsymbol{v}'\exp\left[-\frac{1}{2}(\overline{C}/\mu_m)|\boldsymbol{v}-\boldsymbol{v}'|^2\right]\chi(v'). \tag{44}$$

This equation coincides with (W53) [1], as shown by substituting

$$\boldsymbol{v}=\frac{1}{2}(M/k)^{1/2}\boldsymbol{s},\quad \overline{C}/\mu_m=2k/M,\quad C/\mu_m=(M+1)^2/2M^2. \tag{45}$$

When $\dot{\mu}_s\neq0$ no better procedure is seen, to show the eigenvalence with the Wick equations, than to backtrack from (42) to an equivalent homogeneous equation in the variables λ and u_x and to handle this problem by Laplace and Fourier transforms in the manner of Wick. One may then identify $\chi_0(v)$ with a transform of solutions of Wick equations according to the formulae

$$\begin{aligned}\chi_0(v)&=(\dot{\mu}_s/2\pi i\mu_s)\int_{-i\infty}^{i\infty}dt\,\mathrm{expt}\,t^{1-\overline{C}/\ddot{\mu}_s}f([2t\dot{\mu}_s/\mu_s]^{1/2}v)\\&=(\dot{\mu}_s/8\pi^2 i)\int_{-i\infty}^{i\infty}dt\int d\boldsymbol{q}\exp(t-it^{1/2}\boldsymbol{q}\cdot\boldsymbol{v})\,t^{1-\overline{C}/\dot{\mu}_s}\phi([\mu_s/2\dot{\mu}_s]^{1/2}\boldsymbol{q})\\&=(\dot{\mu}_s/8\pi^2 i)\int_{-i\infty}^{i\infty}dt\int d\boldsymbol{q}\exp(t-it^{1/2}\boldsymbol{q}\cdot\boldsymbol{v}+q^2/u)t^{1-\overline{C}/\dot{\mu}_s}U([\mu_s/2\dot{\mu}_s]^{1/2}\boldsymbol{q}).\end{aligned} \tag{46}$$

The functions f, Φ and U obey the equations

$$\left[\overline{C}-\dot{\mu}_s-\frac{1}{2}\dot{\mu}_s u(d/du)+\frac{1}{4}\mu_s u^2\right]f(u)=(C/4\pi)\int d\boldsymbol{u}'\exp\left(\frac{1}{4}|\boldsymbol{u}-\boldsymbol{u}'|^2\right)f(u'), \tag{47}$$

$$\left\{\frac{1}{4}\mu_s[(d/d\sigma)^2+\sigma^{-1}(d/d\sigma)]-\frac{1}{2}\dot{\mu}_s\sigma(d/d\sigma)+C\exp(-\sigma^2)\right\}\phi(\sigma)=\overline{C}\phi(\sigma), \tag{48}$$

$$\left\{\frac{1}{4}\mu_s[(d/d\sigma)^2+\sigma^{-1}(d/d\sigma)]+\frac{1}{2}\dot{\mu}_s+C\exp(-\sigma^2)-\frac{1}{4}\dot{\mu}_s\sigma^2/\mu_s\right\}U(\sigma)=\overline{C}U(\sigma). \tag{49}$$

All of these equations obtain when $\mu_s=0$ as well as when $\mu_s\neq 0$, and for $\mu_s<0$ (that is, for $\lambda_0<\lambda_s<\lambda_m$) as well as for $\mu_s>0$. In particular, (47) coincides with (44), for $\mu_s=0$, if $\boldsymbol{u}$ is replaced with $(2\overline{C}/\mu_m)^{1/2}\boldsymbol{v}$.

The self-adjoint differential equation (49) has the form of the Schroedinger equation for the motion of a particle oscillating radially in a cylindrically symmetrical potential. It is identical with (W105) [1] as shown by performing the substitutions

$$C/\mu_s=(M+1)^2/2M^2,\quad \dot{\mu}_s/\mu_s=2\gamma/M,\quad \overline{C}/\dot{\mu}_s=2-\rho. \tag{50}$$

It is also identical with eq (V4) [4] as shown by replacing σ^2 with $\sigma^2/2$ and $\overline{C}$ with $-Q/\dot{\mu}_s$.

Wick has estimated the largest eigenvalue of his Schroedinger-type equations by a variational method. When $\dot{\mu}_s=0$ he finds

$$\overline{C}\sim(C^{1/2}-\tfrac{1}{2}\mu_m{}^{1/2})^2=C[1-\tfrac{1}{2}(\mu_m/C)^{1/2}]^2. \tag{51}$$

This eigenvalue reduces to C as μ_m/C approaches zero. The ratio C/μ_m represents the differential cross section for Compton scattering, for $\lambda'=\lambda$, expressed in units of the minimum value of the total cross section. When $\dot{\mu}_s\neq 0$ the largest approximate eigenvalue is

$$\overline{C}\sim\mu_s[(C/\mu_s)r(r+1)^{-1}+\tfrac{1}{2}\dot{\mu}_s/\mu_s-\tfrac{1}{4}(\dot{\mu}_s/\mu_s)^2r^{-1}], \tag{52}$$

where the parameter r indicates the solution of the algebraic equation

$$4(C/\mu_s)(r+1)^{-2}+(\dot{\mu}_s/\mu_s)^2r^{-2}=1. \tag{53}$$

When the logarithmic derivative $|\dot{\mu}_s/\mu_s|$ is large, one may solve (49) by a perturbation method starting from the solutions of the quantum mechanical harmonic oscillator problem and treating $\epsilon=\mu_s/\dot{\mu}_s$ as a small quantity. The result, given in (V6) [4], is

$$\overline{C}=C-\tfrac{1}{2}\dot{\mu}(\alpha-1)+C\epsilon^2/\alpha(\alpha+\epsilon)+C^2\epsilon^4(2\alpha+\epsilon)^2/\dot{\mu}_s\alpha^3(\alpha+\epsilon)^4+\ldots, \tag{54}$$

with

$$\alpha=[1+4(C/\dot{\mu}_s)\epsilon]^2. \tag{55}$$

In practice $\mu_s/\dot{\mu}_s$ is seldom very small, and the range of application of the simple perturbation method is limited. The investigation of eigenvalues proceeds more effectively by the simple variational method of Wick or by the methods that were developed by Spencer [6] for the numerical solution of the semiasymptotic problem. Appendix C contains a description of these latter methods and tables of eigenvalues.

8. Effect of Obliquity of the Primary Radiation

We resume here the discussion of section 2.6 regarding the influence of the source direction on the penetration law.

In the last section, we have studied the singularity of special solutions of the transform equation at $p=\mu_s$. These special solutions obey a homogeneous transform equation shorn of any source term. It was understood that the solution $y(p,u_x,\lambda)$ of the complete, inhomogeneous, transport equation (8), in the critical range $\lambda\sim\lambda_s$ and in the asymptotic limit, should be constructed as a superposition of special solutions (29′)

$$\begin{aligned} y(p,u_x,\lambda)&=\Sigma_n a_n[\mu(\lambda)-p]^{-1}\chi^{(n)}(v,\lambda)\ \exp\left\{\overline{C}^{(n)}\int^{\lambda}d\lambda'/[\mu(\lambda')-p]\right\}\\ &\sim\Sigma_n a_n[\mu(\lambda)-p]^{-1}\chi_0^{(n)}(v)\ \exp\left\{\overline{C}^{(n)}\int^{\lambda}d\lambda'/[\mu(\lambda')-p]\right\}. \end{aligned} \tag{56}$$

Here $\chi_0^{(n)}(v)$ and $\overline{C}^{(n)}$ represent, respectively, the n-th eigenfunction and eigenvalue of (42). The function $\chi^{(n)}(v,\lambda)$ in the middle part of (56) represents the complete expansion $\chi^{(n)}=\Sigma_r\chi_r^{(n)}(v)(\lambda-\lambda_s)^r$ which begins with $\chi_0^{(n)}$ and can be carried to higher order of approximation if necessary.

As p approaches μ_s, the factors $\exp\left\{\overline{C}^{(n)}\int^{\lambda}d\lambda'/[\mu(\lambda')-p]\right\}$ become infinitely large, provided $\overline{C}^{(n)}>0$. Wick pointed out that only one of the eigenvalues $\overline{C}^{(n)}$, namely $\overline{C}^{(0)}$, may happen to be positive. At any rate the term $n=0$ of the sum (56), with the algebraically largest eigenvalue $\overline{C}^{(0)}$, will increase fastest as p approaches μ_s and will predominate over all others in the asymptotic limit. Therefore, the singularity of $y(p,u_x,\lambda)$ depends primarily on the behavior of $\exp\left\{\overline{C}^{(0)}\int^{\lambda}d\lambda'/[\mu(\lambda')-p]\right\}$, as anticipated in section 7. The behavior of this factor is described by (14) in section 5.

In addition, one must consider the factor $[\mu(\lambda)-p]^{-1}$, if $\mu(\lambda)\sim\mu_s$ and especially the dependence, if any, of the coefficient a_0 on p. The coefficients a_n of the superposition (56) are determined by the inhomogenity of the transform eqn (8), that is by the characteristics of the radiation source. Therefore, these characteristics may influence the singularity of $y(p,u_x,\lambda)$, and hence the penetration law at great depths, through the dependence of a_0 on p.

It has been pointed out in section 2.6 that the obliquity of the primary radiation has much less influence on the penetration for high source energies, $\lambda_0<<\lambda_m$ than for $\lambda_0>\lambda_m$. This important difference shows up clearly in the process of determining the coefficients a_n of the expansion (56).

If the source energy is considerably larger than the energy of maximum penetration, one will probably have to resort to numerical integration of the transform eq (8) from the source wavelength λ_0 at least up to some wavelength λ_f somewhat lower than λ_m. If $\lambda_0<<\lambda_m$, and one is interested in the distribution of photons of energy higher than the energy of maximum penetration, then λ_s equals the wavelength of these photons and one must choose $\lambda_f<\lambda_s$. Whether $\lambda_s<\lambda_m$ or $\lambda_s=\lambda_m$, we define λ_f so that $\lambda_0<\lambda_f\lesssim\lambda_s\leq\lambda_m$. A superposition of analytical solutions valid in the critical range $\lambda\sim\lambda_s$ should then be fitted to the numerical solution at λ_f. Under asymptotic conditions, $(\mu_s-p)/p<<1$, the fitting-wavelength λ_f can be so chosen that $\mu(\lambda_f)-\mu_s>>\mu_s-p$, that is so that $\mu(\lambda)-p$ can be safely replaced with $\mu(\lambda)-\mu_s$ for $\lambda\leq\lambda_f$. When this is so, the photon distribution at λ_f varies no longer as p approaches μ_s still further. Similarly, the analytical solutions $\chi^{(n)}(v,\lambda)\ \exp\left\{\overline{C}^{(n)}\int^{\lambda}d\lambda'/[\mu(\lambda')-p]\right\}$, extended from $\lambda\sim\lambda_s$ to λ_f, no longer depend on p in the proximity of λ_f, where p may be effectively replaced with μ_s.

Under these conditions, the fitting of the superposition of analytical solutions to the actual photon distribution at λ_f, that is, the determination of the coefficients a_n of (56), gives a result independent of p. Any change of obliquity of the source radiation causes a change of the directional distribution of the photons at λ_f and, therefore, a change of the coefficients a_n. The photon distribution through the critical range will vary with p in a manner which depends on the intensity ratio of the component solutions $a_n\chi_n(v,\lambda)\exp\{\ .\ .\ .\ \}$ of (56) and, therefore, on the obliquity of the source radiation.

However, the intensity of the lower energy radiation depends only on the $n=0$ term of (56), once p is sufficiently near to μ_m, and therefore it will vary with p only through $\exp\left\{\overline{C}^{(0)}\int^{\lambda}d\lambda'/[\mu(\lambda')-p]\right\}$, independently of the ratios among the a_n's. That is, the source obliquity influences the intensity of the lower energy radiation through the same factor a_0 independently of depth of penetration, provided this depth is adequate to insure fully asymptotic conditions.

A quite different situation prevails when $\lambda_0>\lambda_m$, since the coefficients of the analytical solutions in (56) must be fitted to the directional distribution of the source radiation right at $\lambda_0=\lambda_s$, in the middle of the critical range. Here, the dependence of the a_n's on p has been worked out by Wick for an isotropic source [1], and in reference [4] for a monodirectional source.

General formulae equivalent to a determination of the a_n's will be given here below, but they are of very limited usefulness because of their complication and, especially, because they imply a knowledge of the directional eigenfunctions $\chi_0^{(n)}(v)$. A practical solution of the effect of source obliquity under these conditions will probably require much additional work.

An analytical solution for $\lambda_0 > \lambda_m$, equivalent to (56) and fitted to the directional distribution of the X-ray source, can be obtained performing a Laplace transformation of the energy variable λ, according to Wick [1]. This is a solution of the transport equation (8) schematized in the following manner:

(1) $\mu(\lambda)$ is taken as $\mu_0 + \dot{\mu}_0(\lambda - \lambda_0)$,

(2) $k(\lambda', \lambda)$ is replaced with C,

(3) $\boldsymbol{u} = (\vartheta, \varphi)$ is replaced with a "small-angle approximation" vector $\boldsymbol{s} = (s, \varphi)$, with $0 \leq s \leq \infty$, such that u_x may be replaced with $1 - \frac{1}{2}s^2$, and $\boldsymbol{u}\cdot\boldsymbol{u}'$ with $1 - \frac{1}{2}|\boldsymbol{s} - \boldsymbol{s}'|^2$. For small angles, one sets $s \sim \vartheta$. The solution of the equation is then written as

$$y(p,s,\lambda) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\eta\, e^{\eta(\lambda-\lambda_0)}\frac{1}{4\pi^2}\int d\boldsymbol{k}\, e^{i\sqrt{2\eta}\boldsymbol{k}\cdot\boldsymbol{s} + \dot{\mu}_0 k^2/2p}$$

$$\times \Sigma_n U_n(k)\frac{\eta}{\mu_0 - p}\int_0^\infty dl\, e^{-\eta l}\left[1 + \frac{\dot{\mu}_0 l}{\mu_0 - p}\right]^{\overline{C}^{(n)}/\dot{\mu}_0 - 1} D_n\left(\eta\left[1 + \frac{\dot{\mu}_0 l}{\mu_0 - p}\right]\right). \tag{57}$$

Here U_n and $\overline{C}^{(n)}$ are the eigenfunctions and eigenvalues of (49) and

$$D_n(\alpha) = \frac{\alpha}{4\pi^2}\int d\boldsymbol{s}\, f\left(1 - \frac{1}{2}s^2\right)\int d\boldsymbol{k}\, e^{-i\sqrt{2\alpha}\boldsymbol{k}\cdot\boldsymbol{s} - \dot{\mu}_0 k^2/2p}\, U_n(k) \tag{58}$$

is, in essence, a coefficient of the expansion of the directional distribution of the source—indicated by $f(u_x)$ in (8)— in terms of the eigenfunctions of (47). Some simplification is attained by considering only the X-ray flux integrated over all directions, since in this case (57) yields

$$\int d\boldsymbol{s}\, y(p,s,\lambda) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\eta\, e^{\eta(\lambda-\lambda_0)}\,\Sigma_n\frac{U_n(0)}{2(\mu_0 - p)}\cdot\int dl\, e^{-\eta l}\left[1 + \frac{\dot{\mu}_0 l}{\mu_0 = p}\right]^{\overline{C}^{(n)}/\dot{\mu}_0 - 1} D_n\left(\eta\left[1 + \frac{\dot{\mu}_0 l}{\mu_0 - p}\right]\right). \tag{59}$$

Much further simplification obtains for two special types of source, namely, (a) monodirectional perpendicular to the source plane, $f(1 - \frac{1}{2}s^2) = \delta(\boldsymbol{s})$, and (b) isotropic, or practically isotropic, $f(1 - \frac{1}{2}s^2) \sim \overline{f} = \text{const.}$, since (58) reduces, respectively, to

$$D_n(\alpha) = \frac{\alpha}{2\pi}\int_0^\infty k\,dk\, e^{-\dot{\mu}_0 k^2/2p}\, U_n(k), \tag{60a}$$

and

$$D_n(\alpha) = \tfrac{1}{2}U_n(0)\overline{f}. \tag{60b}$$

Whether D_n is constant or proportional to its argument, the two integrals in (59) may be treated as a Laplace transform followed by its inverse, so that (59) reduces to

$$\int d\boldsymbol{s}\, y(p,s,\lambda) = \Sigma_n\frac{U_n(0)}{4\pi}\frac{d}{d\lambda}\frac{1}{\mu_0 - p}\left[1 + \frac{\dot{\mu}_0(\lambda - \lambda_0)}{\mu_0 - p}\right]^{\overline{C}^{(n)}/\dot{\mu}_0 - 1}\int_0^\infty k\,dk\, e^{-\dot{\mu}_0 k^2/2p}\, U_n(k), \tag{61a}$$

and

$$\int d\boldsymbol{s}\, y(p,s,\lambda) = \Sigma_n\frac{1}{4}[U_n(0)]^2\overline{f}\,\frac{1}{\mu_0 - p}\left[1 + \frac{\dot{\mu}_0(\lambda - \lambda_0)}{\mu_0 - p}\right]^{\overline{C}^{(n)}/\dot{\mu}_0 - 1}, \tag{61b}$$

respectively, that is, to algebraic functions of $\mu_0 - p$. (The p in the exponential of (61a) may be safely replaced with μ_0).

For other types of directional distribution, the dependence of (57) or (59) on $\mu_0 - p$ is quite complicated and even difficult to survey qualitatively, as had been anticipated.

9. Inversion of the Laplace Transform

Following the discussion of the singularity of $y(p,u_x,\lambda)$ at $p=\mu_s$ in sections 5 and 7, we shall now translate each type of singularity of the Laplace transform into a corresponding distribution-in-depth of the X-rays for large values of the depth x within the medium. The translation is done by means of the inverse transform formula (7)

$$Y(x,u_x,\lambda)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \exp(-px)\, y(p,u_x,\lambda)dp=\exp\,(-\mu_s x)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \exp\,[(\mu_s-p)x]\, y(p,u_x,\lambda)dp. \quad (62)$$

Having split off the factor $\exp(-\mu_s x)$ in the last expression, the remaining integral represents the buildup factor $B(x,u_x,\lambda)$ (to within a constant factor equal to the source intensity). Henceforth, we shall be interested in the dependence of B on x for any given obliquity u_x and wavelength λ and, therefore, we shall omit explicit reference to u_x and λ. Similarly, it will be convenient to indicate $y(p,u_x,\lambda)$ simply as $y(\mu_s-p)$, so that (62) takes the form

$$B(x)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp\, \exp\,[(\mu_s-p)x]y(\mu_s-p). \quad (63)$$

The preceding sections have called attention to two types of singularities of $y(\mu_s-p)$, namely: (a) an inverse power law, which we indicate as

$$y(\mu_s-p)\propto(\mu_s-p)^{-K-1}, \quad (64a)$$

and (b) an exponential function of the inverse square root, which we indicate as

$$y(\mu_s-p)\propto \exp\,[D\mu_s/(\mu_s-p)]^{1/2}. \quad (64b)$$

Equation (64a) describes the singularity when $\lambda_0>\lambda_m$ and the directional distribution of the source is either perpendicular to the source plane or effectively isotropic, as shown, respectively, by (61a) and (61b). In addition, the same type of singularity arises for $\lambda_0<\lambda_m$ in the calculation of the buildup factor for $\lambda<\lambda_m$, that is, when $\lambda_0<\lambda_s<\lambda_m$, in which case the straight-ahead result (14c) applies, except for the replacement of C with $\overline{C}^{(0)}$. Henceforth $\overline{C}^{(0)}$ will be indicated briefly as $\overline{C}$. The value of K in (64a) is given by

$$K=\overline{C}/|\dot{\mu}_s| \quad (65)$$

for $\lambda_s<\lambda_m$, according to (14c), and for $\lambda_s>\lambda_m$ with an isotropic source, according to (61b), whereas

$$K=\overline{C}/\dot{\mu}_s+1 \quad (66)$$

for $\lambda_s>\lambda_m$ with a monodirectional source, owing to the derivative in (61a). Equation (64b) describes the singularity when $\lambda_s=\lambda_m$ and $\lambda>\lambda_m$, which implies $\lambda_0<\lambda_m$, in which case (14b) applies, except for the replacement of C with $\overline{C}$, and

$$D=2\pi^2\overline{C}^2/\mu_m(d^2\mu/d\lambda^2)_m. \quad (67)$$

It may be added that the build-up of the radiation of maximum penetration $\lambda=\lambda_m$, for which $\lambda_s=\lambda_m$ also, involves an integration of the arctangent type of integral in (13b) over a range of $\pi/2$ of the arctangent, instead of π. Moreover, in this case, the factor $[\mu(\lambda)-p]^{-1}$ in (14b) represents $(\mu_s-p)^{-1}$. As a result, (64b) must be replaced with

$$y(\mu_s-p)\propto(\mu_s-p)^{-1}\exp\,[F\mu_s/(\mu_s-p)]^{1/2}, \quad (64c)$$

where

$$F=\pi^2\overline{C}^2/2\mu_m(d^2\mu/d\lambda^2)_m=D/4. \quad (68)$$

262279—53——4

Notice that both K and D have been defined as numerical coefficients whose value has to be calculated for each medium and for the appropriate wavelength but is always of the order of 1. To emphasize the dimensional structure of K and D, we rewrite (65) and (67) in the following manner:

$$K=\frac{\overline{C}}{C}\,k(\lambda,\lambda)\left|\left(\frac{d\lambda}{d\mu}\right)_s\right| \tag{65'}$$

$$D=2\pi^2\left(\frac{\overline{C}}{C}\right)^2\left[\frac{k(\lambda,\lambda)}{\mu_m}\right]^2\Big/\left(\frac{d^2\log\mu}{d\lambda^2}\right)_m. \tag{67'}$$

The power-law singularity (64a), entered into the integral (63) yields the build-up factor proportional to x^K, which was anticipated in (2). This result can be easily derived by replacing $(\mu_s-p)x$ with $-t$ in the integral (63):*

$$B(x)\propto\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\,\exp\,[(\mu_s-p)x](\mu_s-p(^{-K-1}=x^K\,\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dt\,\exp\,(-t)(-t)^{-K-1}=\frac{x^K}{\Gamma(K+1)} \tag{69}$$

(See appendix C, tables 1, 2 for values of K, $\dot{\mu}_s/\mu_s$, C/μ_s.)

If desirable, one can also carry out the build-up factor integration by entering into (63) not merely the singularity of y as given by (64a), but the expression of $\int d\boldsymbol{s}y(p,s,\lambda)$ as given by (61a) or (61b). We may write

$$B(x)\propto\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\,\exp\,[(\mu_0-p)x]\,(\mu_0-p)^{-1}\left[1+\frac{\dot{\mu}_0(\lambda-\lambda_0)}{\mu_0-p}\right]^K, \tag{70}$$

and proceed, for example, by expanding the binomial

$$\left[1+\frac{\dot{\mu}_0(\lambda-\lambda_0)}{\mu_0-p}\right]^K=\sum_\nu\frac{\Gamma(-K+\nu)}{\Gamma(-K)\nu!}\left[-\frac{\dot{\mu}_0(\lambda-\lambda_0)}{\mu_0-p}\right]^\nu, \tag{71}$$

and integrating term by term, which yields

$$B(x)\propto\sum_\nu\frac{\Gamma(-K+\nu)}{\Gamma(-K)\nu!^2}\,[-\dot{\mu}_0(\lambda-\lambda_0)x]^\nu={}_1F_1(-K,1,-\dot{\mu}_0[\lambda-\lambda_0]x)=L_{-K}(-\dot{\mu}_0[\lambda-\lambda_0]x). \tag{72}$$

Here ${}_1F_1$ indicates, as usual, the confluent hypergeometric function whose power expansion is represented by the Σ_ν, and L_{-K} indicates the Laguerre function, to which the ${}_1F_1$ reduces when its second parameter equals 1. The binomial expansion (71) is actually unnecessary since (70) is directly recognizable as an integral representation of ${}_1F_1$ or L_{-K}. Equation (72) was derived from a schematized treatment of the straight-ahead problem in (I4) [2].

The singularity (14b) for $\lambda_s=\lambda_m$, entered into the integral (63), yields a build-up factor represented by a transcendental function of unusual type, namely

$$\begin{aligned}B(x)\propto&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\,\exp\,\{(\mu_m-p)x+[D\mu_m/(\mu_m-p)]^{1/2}\}\\&=D\mu_m(D\mu_m x)^{-2/3}\frac{1}{2\pi i}\int_A dt\,\exp\,\{(t+t^{-1/2})(D\mu_m x)^{1/3}\}=D\mu_m(D\mu_m x)^{-2/3}G([D\mu_m x]^{1/3}).\end{aligned} \tag{73}$$

Here $t=(\mu_m-p)\;(D\mu_m)^{-2/3}x^{1/3}$, the contour A loops around the origin in the positive direction and

$$G(z)=\frac{1}{2\pi i}\int_A dt\,\exp\,[(t+t^{-1/2})z]. \tag{74}$$

*The expression of the integral as a Γ-function is found, for example, in reference [20], last formula on p. 245.

For large values of z, the function G can be evaluated by steepest descent method in the proximity of the saddle point, which lies at $t=2^{-2/3}$. The result is

$$G(z)\sim(2\pi)^{-1/2}(2^{2/3}3z)^{-1/2}\exp\,(2^{-2/3}3z). \tag{75}$$

This formula, entered into (73), yields

$$B(x)\propto(D\mu_m)(D\mu_m x)^{-5/6}2^{-5/6}(3\pi)^{-1/2}\exp\,[2^{-2/3}3(D\mu_m x)^{1/3}]. \tag{76}$$

This build-up factor is proportional to $x^{-5/6}\exp\,[H(\mu_m x)^{1/3}]$, as anticipated in (3), with

$$H=2^{-2/3}3D^{1/3}=3(\pi^2\overline{C}^2/2\mu_m\ddot{\mu}_m)^{1/3}. \tag{77}$$

This result was derived in (II 13c) [3] for the straight-ahead approximation, that is with C in the place of $\overline{C}$. (See appendix C, table 3 for values of μ_m, H, D, and $\overline{C}$).

The case $\lambda=\lambda_s=\lambda_m$ with the singularity (64c), may be treated by taking the second derivative of $B(x)$ with repsect to $D^{1/2}$, since this operation generates (64c) from (64a), and then replacing D with the appropriate value (68) of F, namely $D/4$, and H with $H/2^{2/3}=0.63\,H$.

The function $G(z)$ has been tabulated throughout the range $z>0$ by carrying out the integration (74) numerically along the steepest descent path $\mathrm{Im}(t+t^{-1/2})=0$. However no application of this tabulation has been required because Spencer [6] found it convenient to evaluate integrals of the type

$$\frac{1}{2\pi i}\int dp(\mu_0-p)^\alpha\exp\{(\mu_m-p)x+[D\mu_m/(\mu_m-p)]^{1/2}\} \tag{78}$$

by expansion into powers of $[D\mu_m/(\mu_m-p)]^{1/2}$ followed by term-by-term integration in the manner of (70) and (72). The method proved practical because the convergence of the expansion was moderately rapid despite the large value of the argument. (The number of terms in the expansion corresponds to the multiplicity of scattering [6]).

I thank L. V. Spencer for many helpful discussions and for contributing the appendix C.

10. References

[1] G. C. Wick, Phys. Rev. **75,** 738 (1949) referred to as W. See also, M. Verde and G. C. Wick, Phys. Rev. **71,** 852 (1947), R. E. Marshak, Rev. Mod. Phys. **19,** 201 (1947).
[2] H. A. Bethe, U. Fano, and P. R. Karr, Phys. Rev. **76,** 538 (1949), referred to as I.
[3] U. Fano, Phys. Rev. **76,** 739 (1949), referred to as II.
[4] U. Fano, H. Hurwitz Jr. and L. V. Spencer, Phys. Rev. **77,** 425 (1950), referred to as V.
[5] S. Chandrasekhar, Radiative transfer (Oxford University Press, Oxford, England, 1950).
[6] L. V. Spencer, Phys. Rev. **88,** 793 (1952).
[7] J. O. Hirschfelder, J. L. Magee, and M. H. Hull, Phys. Rev., **73,** 853 (1948); J. O. Hirschfelder and E. N. Adams, Phys. Rev. **73,** 863 (1948); P. Maignan, Compt. rend. **230,** 2018 (1950).
[8] G. H. Peebles and M. S. Plesset, Phys. Rev. **81,** 430 (1951).
[9] L. Cave, J. Corner and R. H. A. Liston, Proc. Roy. Soc. [A] **204,** 223 (1950); J. Corner and R. H. A. Liston, Proc. Roy. Soc. [A] **204,** 323 (1950); J. Corner, F. A. G. Day and R. E. Weir, Proc. Roy. Soc. [A] **204,** 329 (1950).
[10] L. L. Foldy, Phys. Rev. **81,** 395 (1951); L. L. Foldy and R. K. Osborn, Phys. Rev. **81,** 400 (1951).
[11] W. R. Faust and M. H. Johnson, Phys. Rev. **75,** 467 (1949); W. R. Faust, Phys. Rev. **77,** 227 (1950).
[12] H. Kahn, Nucleonics **6,** 60 (1950).
[13] L. V. Spencer and U. Fano, J. Research, NBS **46,** 446 (1951). RP2213.
[14] L. V. Spencer, Phys. Rev. **90,** 146 (1953).
[15] G. R. White, Phys. Rev. **80,** 154 (1950).
[16] E. Hayward, Phys. Rev. **86,** 493 (1952).
[17] L. V. Spencer and F. A. Stinson, Phys. Rev. **85,** 662 (1952).
[18] U. Fano, Phys. Rev., in press.
[19] I. Waller, Arkiv Mat. **34A,** No. 3, 4, 5.
[20] E. T. Whittaker and G. N. Watson, Modern Analysis, Cambridge and New York (1946).
[21] R. C. O'Rourke, Phys. Rev. **85,** 881 (1952).

11. Appendix A. Connection Between the X-Ray Distributions From a Plane Isotropic and From a Point Isotropic Source

The X-ray distribution from a plane isotropic source is represented by the solution of eq (5), with $f(u_x)=1$. The equation for a point isotropic monoenergetic source is a particular case of (4) in which one takes the source term $S(\boldsymbol{r},\boldsymbol{u},\lambda)=\delta(\boldsymbol{r})\delta(\lambda-\lambda_0)$. The solution of this equation will be indicated as $F(r,u_r,\lambda)$, where $u_r=\boldsymbol{u}\cdot\boldsymbol{r}/r$ is the radial component of $\boldsymbol{u}$.

Following up the discussion of plane-symmetrical geometries in section 1, we notice that the plane isotropic source $\delta(x)\delta(\lambda-\lambda_0)$ may be regarded as a plane integral over a distribution of point isotropic sources located at $\boldsymbol{r}_0=(0,y_0,z_0)$

$$\delta(x)\delta(\lambda-\lambda_0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dy_0dz_0\delta(\boldsymbol{r}-\boldsymbol{r}_0)\delta(\lambda-\lambda_0). \tag{A1}$$

Owing to the linearity of the transport eq (4), the distribution $Y(x,u_x,\lambda)$ will be similarly expressed as an integral over the distribution $F(r,u_r,\lambda)$. Consider an isotropic point source at $(0,y_0,z_0)$ and a point P at $(x,0,0)$. The distance from the source to P is $r=[x^2+y_0^2+z_0^2]^{1/2}$ and u_r is equal to $(u_x x-u_y y_0-u_z z_0)/[x^2+y_0^2+z_0^2]^{1/2}$. We have

$$Y(x,u_x\lambda)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dy_0dz_0F\left([x^2+y_0^2+z_0^2]^{1/2},\frac{u_xx-u_yy_0-u_zz_0}{[x^2+y_0^2+z_0^2]^{1/2}},\lambda\right). \tag{A2}$$

If we take cylindrical coordinates (ρ,ψ,x), with $\rho^2=y_0^2+z_0^2$ and $\psi=\arccos\,[-(u_yy_0+u_zz_0)/\rho(1-u_x^2)^{1/2}]$, (A2) becomes

$$Y(x,u_x,\lambda)=\int_0^{2\pi}d\psi\int_0^{\infty}\rho d\rho F\left([x^2+\rho^2]^{1/2},\frac{u_xx+[1-u_x^2]^{1/2}\rho\cos\psi}{[x^2+\rho^2]^{1/2}},\lambda\right). \tag{A3}$$

The distribution $F(r,u_r,\lambda)$ may be represented by the Legendre polynomial expansion

$$F(r,u_r,\lambda)=\sum\nolimits_l(2l+1)(4\pi)^{-1}F_l(r,\lambda)P_l(u_r), \tag{A4}$$

and similarly

$$Y(x,u_x,\lambda)=\sum\nolimits_l(2l+1)(4\pi)^{-1}Y_l(x,\lambda)P_l(u_x). \tag{A5}$$

Since the $\int_0^{2\pi}d\,\psi P_l([u_xx+(1-u_x^2)^{1/2}\,\rho\,\cos\,\psi]/[x^2+\rho^2]^{1/2})$ equals simply $2\pi P_l(u_x)P_l(x/[x^2+\rho^2])$, (A3) reduces to

$$\begin{aligned}Y_l(x,\lambda)&=\int_0^{\infty}\rho d\rho F_l([x^2+\rho^2]^{1/2},\lambda)P_l(x/[x^2+\rho^2]^{1/2})\\&=\int_x^{\infty}RdR\,F_l(R,\lambda)P_l(x/R).\end{aligned} \tag{A6}$$

The Legendre coefficients with $l=0$ represent the total intensity distribution at each point, irrespective of direction of propagation. Since $P_0=1$, the relationship (A6) simplifies for $l=0$ to

$$Y_0(x,\lambda)=\int_x^{\infty}RdR\,F_0(R,\lambda) \tag{A7}$$

$$F_0(x,\lambda)=x^{-1}\partial Y_0(x,\lambda)/\partial x. \tag{A8}$$

12. Appendix B. Note on the Treatment of Strongly Nonself-adjoint Systems

Consider the system of equations.

$$\sum_{m=1}^{n}a_{nm}x_m+px_{n+1}=c_n. \tag{B1}$$

In the limit $p=0$ this system has a trivially simple solution since the first equation contains the variable x_1 alone and all other equations can be solved chainwise. If p is sufficiently small, the terms px_{n+1} may be regarded as small perturbations with respect to the trival case $p=0$.

Our method of proceeding is this: If one assumes a trial value of the ratio x_2/x_1, the first equation determines the absolute value of x_1 and x_2. Thereafter the n-th equation determines the $n+1$-th variable, chainwise. Owing to the smallness of p, any error in the initial estimate of x_2/x_1 which causes a wrong estimate of $\sum_{m=1}^{n}a_{nm}x_m$, causes a much larger error in the estimate of x_{n+1}, and so on. A trial and error method with successive estimates of x_2/x_1 may thus proceed quite rapidly, because the amplification of the error in successive equations soon causes the trial solution to become grossly unrealistic, as judged by some suitable criterion.

Experimental calculations were made with the homogeneous system

$$x_n=(x_{n-3}+x_{n-2}+x_{n-1}-x_{n+1})/2, \tag{B2}$$

assuming $x_n=0$ for negative n. This system may be regarded as representing a game of chance of the parchesee type, in which case x must be positive. Successive variables are linear functions of the trial value of x_1/x_0 with alternate and rapidly increasing slope. Calculation up to $n=8$ suffices to show that the solution becomes unacceptable unless $x_1/x_0\sim0.325$.

As a physical model for the considerations presented here, one may think of a chain of pendulums with asymmetrical, or "coaster", couplings, such that the motion of each pendulum influences the next pendulum on its right strongly, but the one on its left only weakly. If the right-to-left coupling were nil, then the motion of the whole set would be fully determined proceeding from the left to the right. Because the right-to-left coupling is weak, one can still determine the whole motion easily by approximation methods proceeding in the same way from left to right. Any error in the initial estimate of the motion of one pendulum implies a large error in the corresponding estimate for the pendulums further to the right.

We call this situation "strongly nonself-adjoint" because it is a typical feature of the self-adjoint problem with symmetric coupling that one must consider the whole set of pendulums simultaneously. Similarly, we would characterize the trivial case of a completely one-way coupling as "extreme nonself-adjoint".

In general terms one might state the situation as follows. Self-adjoint systems are represented by symmetrical matrices. Extremely nonself-adjoint systems are represented by matrices all of whose elements vanish on one side of the diagonal; these systems have simple solutions. Systems with matrices having small-valued elements on one side of the diagonal should generally be amenable to easy approximate treatment, especially if these elements are confined to a strip near the diagonal.

13. Appendix C.* Determination of Wick's Eigenvalue by Semiasymptotic Iteration Procedures

This method for determining eigenvalues can perhaps best be illustrated by proceeding step by step through the solution of a specific example. Thus, suppose we want to determine the eigenvalue of eq (44) for the case of Pb, that is, for $C/\mu_m=2.90$. To begin with, we backtrack a little and define a variable $u=(\overline{C}/\mu_m)^{1/2}v$. In terms of this variable, eq (44) becomes

$$\left[\overline{C}/\mu_m+\frac{1}{2}u^2\right]\chi_0(u)=(C/\mu_m)\int d\boldsymbol{u}'\,\frac{1}{2\pi}e^{-1/2|\boldsymbol{u}-\boldsymbol{u}'|^2}\chi_0(u'). \tag{C1}$$

Next, we multiply this equation by powers of $(u^2/2)$, integrate over the $\boldsymbol{u}$ space, and thus write down formally moment equations of which the first two are

$$\left.\begin{aligned}(\overline{C}/\mu_m)K_0+K_1&=(C/\mu_m)K_0\\(\overline{C}/\mu_m)K_1+K_2&=(C/\mu_m)(K_1+K_0)\end{aligned}\right\}, \tag{C2}$$

where $K_n=(1/2\pi)\int d\boldsymbol{u}(u^2/2)^n\chi_0(u)$. If we write $\xi=\overline{C}/\mu_m$ and insert numbers, these two equations become

$$\left.\begin{aligned}\xi K_0+K_1&=2.90K_0\\\xi K_1+K_2&=2.90(K_1+K_0)\end{aligned}\right\}. \tag{C3}$$

Now the right sides of these equations are the moments of a smooth, positive, monotonically decreasing (as $u\rightarrow\infty$) function, which we shall call $R(u)$. We shall assume that a reasonable representation for this function is $R(u)=be^{-\beta(u^2/2)}$. From (C1) we then have $\chi_0(u)=[\xi+u^2/2]^{-1}be^{-\beta(u^2/2)}$.
If we take moments of this quantity, we readily obtain the relation

$$K_1/K_0=g=\xi\left\{\frac{1}{f(\beta\xi)}-1\right\}, \tag{C4}$$

*By L. V. Spencer.

where $f(x)=xe^{x}[-Ei(-x)]$. We now establish an iteration procedure using eq (C3) and (C4), namely

$$\left.\begin{aligned} g^{(n+1)}&=\xi^{(n)}\left\{\frac{1}{f(\beta^{(n)}\xi^{(n)})}-1\right\} \\ \xi^{(n+1)}&=2.90-g^{(n+1)} \\ \beta^{(n+1)}&=\frac{1}{g^{(n+1)}+1} \end{aligned}\right\}, \qquad \text{(C5)}$$

where n refers to n'th approximation. The middle equation in (C5) is simply a rewrite of the first of eq (C3), while the last equation in (C5) is obtained by fitting moments of the assumed $R(u)$ to the right sides of eq (C3).

The function $f(x)$ can be easily calculated from standard tables. For interpolation between tabulated values we use the relation $f(x+\delta)\approx f(x)[1+\delta(1+1/x)]-\delta$.

To begin the calculation we make a guess that $g^{(0)}=1$. This yields $\xi^{(0)}=1.90$, $\beta^{(0)}=0.5$, and $\beta^{0}\xi^{0}=0.95$. Continuing, $f(0.95)=0.5863$. Thus

$$\begin{array}{lllll} g^{(1)}=1.3407 & g^{(2)}=1.4556 & g^{(3)}=1.4864 & g^{(4)}=1.4931 & g^{(5)}=1.4951 \\ \xi^{(1)}=1.5593 & \xi^{(2)}=1.4444 & \xi^{(3)}=1.4136 & \xi^{(4)}=1.4069 & \xi^{(5)}=1.405 \\ \beta^{(1)}=0.42722 & \beta^{(2)}=0.40723 & \beta^{(3)}=0.40219 & \beta^{(4)}=0.40111 & \\ f(\beta^{(1)}\xi^{(1)})=0.5172 & f(\beta^{(2)}\xi^{(2)})=0.49284 & f(\beta^{(3)}\xi^{(3)})=0.48632 & f(\beta^{(4)}\xi^{(4)})=0.48481 & \end{array}$$

The difference between successive ξ's is converging by a factor $\sim$3–4 at each step. Consequently, $\xi^{(\infty)}$ should be, within a small fraction of a percent, equal to 1.404. This yields $\overline{C}=0.484C$. This compares with Wick's variational value of $\overline{C}=0.499C$. (See eq (51).)

This type of calculation can be extended easily by taking into account more of the moment equations (C2). A calculation making use of four moment equations yields an approximate eigenvalue in agreement with that of the variational method to better than 1 percent.

The iteration procedure for the case of a monotonically increasing attenuation coefficient can be set up in the same way. Our original intention was to calculate the eigenvalues of tables 2 and 3 by this method using four moment equations and starting from a variational method value. It turned out, however, that the semiasymptotic calculations agreed to within about a percent or better with the variational calculations in all parts of the tables 2 and 3 and thus we decided to present the numerical values of the variational calculation results obtained from eq (51) and (52).

The agreement between variational and semiasymptotic calculations argues strongly for an accuracy of 1 percent or so in the eigenvalue determinations. However, in practical siuations, K and H depend also upon first and second derivatives, respectively, of μ. These are difficult to determine accurately. There may thus be quite sizeable errors in the tabulated values of H and D (table 3) and of $\dot{\mu}_s/\mu_s$ (table 1).

TABLE 1. *Values of $\dot{\mu}_s/\mu_s$ and C/μ_s for various materials and energies*

Source energy	H_2O		Al		Fe	
	$\dot{\mu}_s/\mu_s$	C/μ_s	$\dot{\mu}_s/\mu_s$	C/μ_s	$\dot{\mu}_s/\mu_s$	C/μ_s
Mev						
10	7.5	7.54	4.1	6.21	------	------
8	6.6	6.87	4.0	5.91	0.48	4.64
6	5.5	5.96	3.8	5.43	1.2	4.56
4	4.0	4.91	3.3	4.68	2.0	4.25
3	3.1	4.22	2.7	4.10	2.1	3.89
2	2.1	3.38	2.0	3.33	1.8	3.30
1	.96	2.37	.96	2.37	.89	2.36
.8	.73	2.12	.73	2.12	.72	2.11
.6	.52	1.86	.52	1.86	.54	1.84
.4	.32	1.58	.32	1.57	.37	1.52
.3	.22	1.41	.22	1.40	.32	1.32
.2	.13	1.23	.155	1.21	.31	1.01
.15	.086	1.12	.145	1.09	.33	0.767
Source energy	**Sn**		**Pb**		**U**	
10	------	------	------	------	------	------
8	------	------	------	------	------	------
6	------	------	------	------	------	------
4	0.40	3.61	------	------	------	------
3	.86	3.49	0.33	2.88	0.22	2.72
2	1.4	3.13	1.7	2.65	2.0	2.48
1	1.1	2.22	1.7	1.75	1.9	1.53
.8	.92	1.95	1.6	1.43	1.8	1.21
.6	.83	1.63	1.5	1.033	1.6	.852
.4	.77	1.17	1.35	.579	1.4	.449
.3	.74	.798	1.2	.333	1.2	.260
.2	.72	.417	.95	.133	.97	.100
.15	.70	.223	.77	.0647	.82	.0485

TABLE 2. *Values of the exponent $K=\overline{C}/\dot{\mu}_s$ of the build-up factor for various combinations of the parameters $\dot{\mu}_s/\mu_s$ and C/μ_s*

[Values of $\dot{\mu}_s/\mu_s$ and C/μ_s for various materials and energies are given in table 1]

$\dot{\mu}_s/\mu_s$	$C/\mu_s=0.02$	0.05	0.1	0.25	0.5	1	2	4	6	8
0.02	0.0200	0.0516	0.1095	0.374	2.63	13.00	42.3	113.0	189.5	272
.05	.01939	.0494	.1052	.332	1.328	5.49	17.21	45.5	76.5	108.9
.1	.01846	.0473	.0988	.293	.871	2.98	8.84	23.0	38.5	54.7
.25	.01617	.0411	.0844	.231	.547	1.439	3.81	9.48	15.69	22.2
.5	.01341	.0338	.0687	.1798	.388	.887	2.11	4.96	8.16	11.32
1	.01003	.0252	.0506	.1289	.265	.559	1.211	2.67	4.24	5.87
2	----------	------	.0335	.0841	.1696	.345	.709	1.477	2.28	3.11
4	----------	------	------	------	------	------	------	.821	1.173	1.674
6	----------	------	------	------	------	------	------	.577	.870	1.165
8	----------	------	------	------	------	------	------	.447	.671	.897

TABLE 3.—*Values of μ_m, $H=3[\pi^2\overline{C}^2/2\mu_m(d^2\mu/d\lambda^2)_n]^{1/3}$, $D=4H^3/27$ and $\overline{C}$ for various materials*

Material	μ_m	H	D	$\overline{C}$ (cm²/g per Compton wavelength unit)
	cm²/g			
H_2O	0.0167	2.0	1.3	0.118
Al	.0216	2.1	1.4	.0942
Fe	.0300	2.8	3.4	.0826
Sn	.0351	2.6	2.7	.0689
W	.0391	2.5	2.4	.0619
Pb	.0410	2.3	1.8	.0594
U	.0425	2.1	1.4	.0565

WASHINGTON, March 17, 1953.

U. S. GOVERNMENT PRINTING OFFICE: 1953