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Effective Circuit Bandwidth for Noise with a Power-Law Spectrum

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The effective bandwith of tuned circuits for noise with a power-law spectrum is derived and discussed. Comparison is made with the usual case of "flat" noise.

The noise bandwidth of tuned (RLC) circuits for "white," or "flat," noise is well known.[1] However, corresponding results for noise with other types of spectra have apparently not been discussed very much. In connection with some noise-measurement work currently going on at the National Bureau of Standards, the writer worked out the effective noise bandwidth in the case of an f^α spectrum (f being the frequency). The ratio between the effective bandwidth for the noise and the half-power bandwidth is, in some cases, appreciably different from the well-known factor 1.57 that applies to flat noise; in other cases, the correction is quite small. These results are offered for publication in the belief that this knowledge will be of value to others engaged in noise work. Spectra of this form are encountered in transistor noise, resistor current noise, flicker effect noise, etc.

We consider a random noise current whose spectral intensity (often called power spectrum) follows the law f^α, driving a circuit consisting of the parallel[2] elements R, L, C.

The impedance Z of such a stage is

$$Z=\frac{Rd}{d+j\left(\frac{f}{f_0}-\frac{f_0}{f}\right)}, \tag{1}$$

where $f_0=(2\pi LC)^{-1/2}$ is the resonance frequency, and

$$d=\frac{1}{2\pi f_0 RC}=\frac{1}{Q}=\frac{2\pi f_0 L}{R}=\frac{1}{R}\sqrt{\frac{L}{C}}$$

is the dissipation factor or reciprocal of Q. It is easily shown that df_0 is the frequency difference between the half-power points of the circuit; that is, it is the bandwidth b.

The mean-square output voltage of the circuit when driven by the noise is, with an obvious normalization,

$$\langle V^2\rangle=\int_0^\infty f^\alpha|Z|^2\,df \tag{2}$$

$$=bdf_0^\alpha R^2 I, \tag{2a}$$

[1] See e. g., Valley and Wallman, Vacuum tube amplifiers, p. 169 (McGraw-Hill Book Co., Inc., New York, N. Y., 1948).

[2] The results may be easily applied to the series RLC case.

where

$$I=\int_0^\infty \frac{x^\alpha\,dx}{d^2+\left(x-\frac{1}{x}\right)^2}. \tag{3}$$

The main task before us then is the evaluation of I. This evaluation is facilitated by the change of variable $y=x^2$, which gives

$$I=\tfrac{1}{2}\int_0^\infty \frac{y^{\frac{1+\alpha}{2}}\,dy}{y^2+2y\cos\lambda+1},$$

where

$$\cos\lambda\equiv\frac{d^2}{2}-1.$$

This integral may now easily be evaluated by contour integration in the complex plane. Omitting the details, the result is

$$I=\frac{\pi}{2\sin\left(\frac{1+\alpha}{2}\right)\pi}\,\frac{\sin\left(\frac{1+\alpha}{2}\right)\lambda}{\sin\lambda}, \tag{4}$$

if $-3<\alpha<1$ and $-\pi\angle\lambda<\pi$.

The restrictions on α and λ must be carefully adhered to. If values of α outside the specified range are used, the integral will diverge. Thus the device of assuming that the spectral law is constant for all frequencies, which appears to be justified in many cases when the integral will converge, cannot be used when $\alpha\geq 1$ or ≤ -3.

Now we may define the effective bandwidth as

$$B=\frac{\langle V^2\rangle}{R^2 f_0^{\alpha}} \tag{5}$$

so that

$$B=bdI. \tag{6}$$

For flat noise we insert $\alpha=0$ in (4), obtaining $I=\pi/2d$, so that

$$B_0=\frac{\pi}{2}\,b, \tag{7}$$

which is the well-known result for flat noise. The subscript refers to the case $\alpha=0$.

Using (7), (4), and (6), we may now write

$$\frac{B}{B_0}=\frac{d}{\sin\frac{\pi}{2}(1+\alpha)}\;\frac{\sin\frac{\lambda}{2}(1+\alpha)}{\sin\lambda}. \tag{8}$$

B/B_0 is the correction factor for the effective noise bandwidth resulting from the deviation of α from 0.

The correction factor B/B_0 is an even function of $\alpha+1$, that is for example, it is the same for $\alpha=0$ as it is for $\alpha=-2$. (Therefore, we have the interesting result that the effective bandwidth for $\alpha=-2$ is the same as it is for flat noise.)

A plot of B/B_0 versus α for fixed d would take the form of a U-shaped curve with the bottom of the U occuring at $\alpha=-1$: The curve is symmetrical about $\alpha=-1$. B/B_0 is less than 1 in the range $-2<\alpha<0$, is equal to 1 for $\alpha=-2$ and $\alpha=0$, and is greater than 1 outside this range; it increases toward infinity as α approaches 1 from the left or -3 from the right.

A family of curves for B/B_0 may be plotted without difficulty. However, a quick idea of the magnitude of the correction may be obtained from the following formulas to which (8) reduces in special cases:

$$\left.\begin{aligned}
&\alpha=0,\ 2;\ \frac{B}{B_0}=1\\
&\alpha=\frac{1}{2}, -2\frac{1}{2};\ \frac{B}{B_0}=\sqrt{1+\frac{d}{2}}+\frac{d}{2}\frac{1}{\sqrt{1+\frac{d}{2}}}\\
&\alpha=-\frac{1}{2}, -1\frac{1}{2};\ \frac{B}{B_0}=\frac{1}{\sqrt{1+\frac{d}{2}}}\\
&\alpha=-1;\ \frac{B}{B_0}=\frac{1-\frac{1}{\pi}\sin^{-1}d\sqrt{1-\frac{d^2}{4}}}{\sqrt{1-\frac{d^2}{4}}}
\end{aligned}\right\} \tag{9}$$

For example, for $d=0.1$, $\alpha=-1$, we find immediately $B/B_0\approx 1-d/\pi\approx 0.97$.

Thus for reasonably high values of $Q(=1/d)$, the corrections in the range $-2<\alpha<0$ are rather small. Outside this range the corrections may be large.

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