Pairs of Normal Matrices With Property L

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A short proof is given, under weaker assumptions, of the following theorem first proved by N. Wiegmann: If the eigenvalues $\alpha_i$, $\beta_k$ of two normal $n \times n$ matrices $A$, $B$ may be numbered in such a way that the eigenvalues $\gamma_i(z)$ of $C(z) = A + zB$ are given by $\gamma_i(z) = \gamma_i + z\beta_k$, for $i = 1, \ldots, n$ and all complex values of $z$, then $AB = BA$.

Two $n \times n$ matrices $A$, $B$ are said to have property $L$ if their eigenvalues $\alpha_i$, $\beta_k$($k = 1, \ldots, n$) may be ordered in such a way that the eigenvalues of $A + zB$ are normal, for all complex numbers $\alpha_i$, $\beta_k$. Though every pair of commuting matrices has property $L$, the converse is not true. However, if $A$ and $B$ have property $L$ and are normal, then $AB = BA$ according to a theorem of Wiegmann. We wish to show that the assumptions of Wiegmann's theorem may be weakened considerably. We begin by proving the following lemma which has some interest in itself.

**Lemma:** Let $A$ and $B$ be arbitrary $n \times n$ matrices. Then the set $Z$ of all points $z$ in the complex plane for which $A + zB$ is normal, is either the whole plane, a straight line, a circle, or it contains, at most, two points. If $B$ is normal, neither the circle case nor the two-point case may occur. If $Z$ is the entire plane then $AB = BA$.

**Proof:** $A + zB$ is normal if, and only if,

$$(A + zB)(A^* + zB^*) - (A^* + zB^*)(A + zB) = 0,$$  

(1)

If $z = x + iy$, (1) is equivalent to a set of $2n^2$ real linear equations in $x,y$, and $(x^2 + y^2)$. Each of these $2n^2$ equations defines either the whole plane, a circle, a straight line, or else, at most, one point.

The intersection of these $2n^2$ points sets obviously is one of the six types described in the lemma. That all these types may indeed occur is shown by the following matrices $A + zB$:

$$
\begin{pmatrix}
1 & z \\
z & 1
\end{pmatrix}, \begin{pmatrix}
z & 0 \\
0 & z
\end{pmatrix}, \begin{pmatrix}
0 & z \\
z & 0
\end{pmatrix}, \\
\begin{pmatrix}
z & 0 \\
0 & z
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
$$

In case $B$ is normal, the terms of (1) containing $zz^*$ cancel, hence $Z$ is determined by linear equations in $x$ and $y$ only. So $Z$ is neither a circle nor a pair of points.

In case $Z$ is the entire plane (1) holds for every $z$, hence the coefficient of $z$ vanishes, that is, $BA^* - A^*B = 0$. Since $B$ commutes with $A^*$, it also commutes with every polynomial $f(A^*)$, hence, with $A$ itself. (If $A$ is normal, then $A = f(A^*)$ for some polynomial $f$. This well-known fact follows by transforming $A$ into diagonal form by means of a unitary transformation.) The lemma being proved, we turn to the generalization of Wiegmann's theorem.

**Theorem:** If the eigenvalues $\alpha_i$, $\beta_k$ of two normal matrices $A = (a_{ik})$, $B = (b_{ik})$ may be ordered in such a way that the eigenvalues $\gamma_i(z)$ of $A + zB$ satisfy the inequalities

$$\sum_{k=1}^{n} |\gamma_k(z)|^2 \geq \sum_{k=1}^{n} |\alpha_k + z\beta_k|^2$$  

(2)

for some values $z = z_\lambda$ ($\lambda = 1, \ldots, l$) which are the vertices of a polygon containing 0 in its interior, then $AB = BA$. (This obviously contains Wiegmann's theorem since if $A$ and $B$ have property $L$ then (2) holds even with equality sign for every $z$.)

**Proof:** According to a theorem of Schur, we have

$$\sum_{i,k} |a_{ik} + zb_{ik}|^2 \geq \sum_{k} |\gamma_k(z)|^2$$  

(3)

for every $z$, where the equality holds if, and only if, $A + zB$ is normal. From (3) and (2) we have for $z = z_\lambda$

$$\sum_{i,k} |a_{ik} + zb_{ik}|^2 \geq \sum_{k} |\alpha_k + z\beta_k|^2$$  

(4)

which reduces to

$$\sum_{i,k} (a_{ik}z\bar{b}_{ik} + a_{ik}z\bar{b}_{ik}) - \sum_{k} (\alpha_kz\bar{\beta}_k + \alpha_kz\beta_k) \geq 0$$  

(5)

since $\sum |a_{ik}|^2 = \sum |\alpha_k|^2$ and $\sum |b_{ik}|^2 = \sum |\beta_k|^2$ in view of the fact that $A$ and $B$ are normal. Since the left-hand side of (5) is linear and homogeneous in $x$ and $y$, (5) defines either a half plane containing 0 on its boundary or the left-hand side of (5) vanishes identically. The former case cannot occur, for the half plane contains $z_1, \ldots, z_l$, hence it contains 0 in its interior. So in (5), and hence in (4), equality holds for every $z$. Combining this with (2) and (3) we

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conclude that in (3) equality holds for \( z=z_3 \), hence \( A+z_3B \) is normal. Since \( z_1, \ldots, z_l \) are not collinear and \( B \) is normal, the lemma asserts that \( A+zB \) is normal for every \( z \), hence \( AB=BA \).

Remark: The theorem is best possible in the sense that there exist noncommuting normal matrices \( A, B \) such that inequality (2) holds for every \( z \) of a closed half plane containing 0 on its boundary (and for a fixed suitable ordering of the eigenvalues \( \alpha_k, \beta_k \)). An example is given by

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & -i
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

with the ordering \( \alpha_k=1, -1, i, -i; \beta_k=i, -i, 0, 0 \).

Here we have \( \gamma_k(z)=1, -1, (-1-z^2)^{1/2}, -(1-z^2)^{1/2} \)

hence \( \sum |\gamma_k(z)|^2 \geq \sum |\alpha_k+z\beta_k|^2 \) for every \( z \) with \( y \geq 0 \)

(though \( \gamma_k(z)=\alpha_k+z\beta_k, k=1, \ldots, 4 \) is valid for \( z=0 \) only).

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