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Pairs of Normal Matrices With Property L

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A short proof is given, under weaker assumptions, of the following theorem first proved by N. Wiegmann: If the eigenvalues α_i, β_i of two normal $n\times n$ matrices A, B may be numbered in such a way that the eigenvalues $\gamma_i(z)$ of C $(z)=a+zb$ are given by $\gamma_i(z)\ =\gamma_i+z\beta_i$ (for $i=1,\ .\ .\ .\ ,n$ and all complex values of z), then $AB=BA$.

Two $n\times n$ matrices A, B are said [2] to have property L if their eigenvalues α_k, $\beta_k (k=1,\ .\ .\ .\ ,n)$ may be ordered in such a way that the eigenvalues of $\alpha A+\beta B$ are given by $\alpha\alpha_k+\beta\beta_k$ for all complex numbers α, β. Though every pair of commuting matrices has property L, the converse is not true.[2] However, if A and B have property L and are normal, then $AB=BA$ according to a theorem of Wiegmann.[3] We wish to show that the assumptions of Wiegmann's theorem may be weakened considerably. We begin by proving the following lemma which has some interest in itself.

LEMMA: *Let A and B be arbitrary $n\times n$ matrices. Then the set Z of all points z in the complex plane for which $A+zB$ is normal, is either the whole plane, a straight line, a circle, or it contains, at most, two points. If B is normal, neither the circle case nor the two-point case may enter. If Z is the entire plane then $AB=BA$.*

PROOF: $A+zB$ is normal if, and only if,

$$(A+zB)(A^*+\bar{z}B^*)-(A^*+\bar{z}B^*)(A+zB)=0. \quad (1)$$

If $z=x+iy$, (1) is equivalent to a set of $2n^2$ real linear equations in x,y, and (x^2+y^2). Each of these $2n^2$ equations defines either the whole plane, a circle, a straight line, or else, at most, one point. The intersection of these $2n^2$ point sets obviously is one of the six types described in the lemma. That all these types may indeed occur is shown by the following matrices $A+zB$:

$$(1+z),\quad \begin{pmatrix}1 & z\\ z & 0\end{pmatrix},\quad \begin{pmatrix}0 & z\\ 1 & 0\end{pmatrix},$$

$$\begin{bmatrix}0 & z & 0\\ 0 & 0 & z\\ z & 0 & z-1\end{bmatrix},\quad \begin{pmatrix}0 & z\\ 0 & 0\end{pmatrix},\quad \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

In case B is normal, the terms of (1) containing $z\bar{z}$ cancel, hence Z is determined by linear equations in x and y only. So Z is neither a circle nor a pair of points.

In case Z is the entire plane (1) holds for every z, hence the coefficient of z vanishes, that is, $BA^*-A^*B=0$. Since B commutes with A^*, it also commutes with every polynomial $f(A^*)$, hence, with A itself. (If A is normal, then $A=f(A^*)$ for some polynomial f. This well-known fact follows by transforming A into diagonal form by means of a unitary transformation.) The lemma being proved, we turn to the generalization of Wiegmann's theorem.

THEOREM: *If the eigenvalues α_k, β_k of two normal matrices $A=(a_{ik})$, $B=(b_{ik})$ may be ordered in such a way that the eigenvalues $\gamma_k(z)$ of $A+zB$ satisfy the inequalities*

$$\sum_{k=1}^{n}|\gamma_k(z)|^2\geqq\sum_{k=1}^{n}|\alpha_k+z\beta_k|^2 \quad (2)$$

for some values $z=z_\lambda$ $(\lambda=1,\ .\ .\ .\ ,\ l)$ which are the vertices of a polygon containing 0 in its interior, then $AB=BA$. (This obviously contains Wiegmann's theorem since if A and B have property L then (2) holds even with equality sign for every z.)

PROOF: According to a theorem of Schur,[4] we have

$$\sum_{i,k}|a_{ik}+zb_{ik}|^2\geqq\sum_{k}|\gamma_k(z)|^2 \quad (3)$$

for every z, where the equality holds if, and only if, $A+zB$ is normal. From (3) and (2) we have for $z=z_\lambda$

$$\sum_{i,k}|a_{ik}+zb_{ik}|^2\geqq\sum_{k}|\alpha_k+z\beta_k|^2 \quad (4)$$

which reduces to

$$\sum_{i,k}(a_{ik}\bar{z}\bar{b}_{ik}+\bar{a}_{ik}zb_{ik})-\sum_{k}(\alpha_k\bar{z}\bar{\beta}_k+\bar{\alpha}_k z\beta_k)\geqq 0 \quad (5)$$

since $\sum|a_{ik}|^2=\sum|\alpha_k|^2$ and $\sum|b_{ik}|^2=\sum|\beta_k|^2$ in view of the fact that A and B are normal. Since the left-hand side of (5) is linear and homogeneous in x and y, (5) defines either a half plane containing 0 on its boundary or the left-hand side of (5) vanishes identically. The former case cannot occur, for the half plane contains $z_1,\ldots,z_l$, hence it contains 0 in its interior. So in (5), and hence in (4), equality holds for every z. Combining this with (2) and (3) we

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[2] T. S. Motzkin and O. Taussky, Pairs of matrices with property L, Trans. Am. Math. Soc. **73**, 108–114 (1952).

[3] N. Wiegmann, Pairs of normal matrices with property L, Proc. Am. Math. Soc. **4**, 35–36 (1953).

[4] I. Schur, Über die charakterischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen, Math. Ann. **66**, 488–510 (1909); Theorem II. The theorem of Schur has recently been used in a related problem by H. Schneider, Theorems on normal matrices, Quart. J. Math. Oxford [2] **3**, 241–249 (1952).

conclude that in (3) equality holds for $z=z_\lambda$, hence $A+z_\lambda B$ is normal. Since $z_1,\ldots,z_l$ are not collinear and B is normal, the lemma asserts that $A+zB$ is normal for every z, hence $AB=BA$.

REMARK: The theorem is best possible in the sense that there exist noncommuting normal matrices A, B such that inequality (2) holds for every z of a closed half plane containing 0 on its boundary (and for a fixed suitable ordering of the eigenvalues α_k,β_k). An example is given by

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

with the ordering $\alpha_k=1,\ -1,\ i,\ -i$; $\beta_k=i,\ -i,\ 0,\ 0$. Here we have $\gamma_k(z)=1,\ -1,\ (-1-z^2)^{\frac{1}{2}},\ -(-1-z^2)^{\frac{1}{2}}$, hence $\sum|\gamma_k(z)|^2\geqq\sum|\alpha_k+z\beta_k|^2$ for every z with $y\geqq 0$ (though $\gamma_k(z)=\alpha_k+z\beta_k$, $k=1,\ldots,4$ is valid for $z=0$ only).

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