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On Methods for Obtaining Solutions of Fixed End-Point Problems in the Calculus of Variations[1]

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Two methods for constructing solutions to the problem of minimizing an integral in a certain class of arcs joining a pair of fixed points are proposed. One of these procedures is a generalization of Newton's method, while the other is a "gradient" method. Conditions for convergence to a strong relative minimum are given in both cases.

1. Introduction

In this paper we consider methods for effectively constructing solutions to the problem of minimizing an integral

$$I(y)=\int_a^b f(x,y,y')\,dx \tag{1}$$

in a certain class of arcs $y_i(x)$, $(a\leq x\leq b;\ i=1,2,\ldots,n)$ joining the two fixed points $(x=a,\ y(a)=\rho)$ $(x=b,\ y(b)=\sigma)$. As is well known, the minimum must necessarily satisfy the Euler equations

$$\frac{d}{dx}f_{y_i'}(x,y,y')=f_{y_i}(x,y,y') \qquad (i=1,2,\ldots,n).$$

Thus in any particular case the solution of these equations becomes of prime importance. The two methods to be proposed below enable us to compute arcs that satisfy the Euler equations subject to boundary conditions at two distinct points. For both methods an initial estimate of the solution is made. From this estimate a new estimate is obtained in a definite way, and so forth. These procedures have the advantage of giving estimates that satisfy the boundary conditions at each step, and their iterative nature makes them adaptable to automatic computation.

The first method of obtaining new estimates that will be discussed is a generalization of Newton's method for functions of a single real variable. This method is described in section 2 where it is actually applied to a more general class of systems of differential equations than the Euler equations, namely those systems of the form

$$\frac{d}{dx}g_i(x,y,y')=h_i(x,y,y') \qquad (i=1,2,\ldots,n).$$

To construct the convergence proof, we make the class of functions under consideration into a Banach space and regard the system of differential equations as an operator T on the space into itself. The necessary Banach space tools are developed in sections 3, 4, and 5. Sections 6, 7, and 8 are devoted to an analysis of the properties of T and of its first variation. The results obtained in these sections are used in section 9 to show that under simple conditions on the initial estimate the sequence of estimates derived from the first method will converge quadratically to an admissible function that satisfies the system of differential equations and the two-point boundary conditions. In fact, it is demonstrated in section 9 that convergence will occur if the initial estimate has nonconjugate end points (see definition 5) and is contained in some neighborhood on which the norm of T' is bounded by a suitable constant. These conditions are shown to be satisfied if a solution exists and has nonconjugate end points. In this case there will exist a neighborhood of the solution such that the sequence of estimates will converge for any initial estimate chosen from this neighborhood. In the special case of the Euler equations, considered in section 11, it is shown that the limit function not only satisfies the differential equations but under the proper conditions also affords a strong relative minimum to $I(y)$. Section 10 concerns itself with an interesting lemma about conjugate points, that proves useful in section 11.

While not having as large a range of applicability as the first method, the second of the two methods of choosing new estimates that will be discussed has the advantage of being simpler to handle at each iteration. This method is described in section 12, where it is shown to be particularly useful for the solution of the Euler equations arising from an integral whose integrand is quadratic in y and y'. In this case the integral itself, instead of the differential equations, is considered; and it is shown that under suitable hypothesis the sequence of estimates converges to an arc that affords a strong relative minimum to $I(y)$. In fact, convergence will occur if I has a lower bound on the class of admissible functions and the second variation is positive.

Neither of the methods to be described below is presented as a quick computational procedure for a desk computer. However, they are processes that might well be carried out on automatic computing machinery. Numerical examples are to be found in section 13.

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2. An Iterative Method for Solving Certain Systems of Second Order Differential Equations

In the following pages we shall be concerned with constructing solutions for systems of the form

$$\frac{d}{dx}(g_i(x,y,y'))=h_i(x,y,y') \qquad (i=1,2,\ldots,n)$$
$$y_i(a)=\rho_i, \quad y_i(b)=\sigma_i. \tag{2}$$

Here $y\equiv(y_1,y_2,\ldots,y_n)$ denotes an arc defined on the interval $a\leq x\leq b$ and ρ, σ denote arbitrary constant vectors. By formal integration of (2) it is found that an arc y that satisfies these equations must also satisfy

$$T_i(x;y)\equiv\int_a^x\left[g_i(t,y,y')-\int_a^t h_i(s,y,y')ds\right]dt$$
$$-\frac{x-a}{b-a}\int_a^b\left[g_i(t,y,y')-\int_a^t h_i(s,y,y')ds\right]dt=0 \tag{3}$$
$$y_i(a)=\rho_i, \quad y_i(b)=\sigma_i \qquad (i=1,2,\ldots,n).$$

Conversely, any arc without corners that satisfies (3) must also satisfy (2). For reasons that will be clarified below, we prefer to make use of the differential eq (2) in their integral form (3). Accordingly, let us examine some of the properties of the function $T_i(x;y)$.

That

$$T_i(a;y)=T_i(b;y)=0 \qquad (i=1,2,\ldots,n) \tag{4}$$

independent of the arc y is immediately apparent.

One recalls that a function of one or more variables is said to be of *class* C^n, $n\geq 0$, in its arguments on its domain of definition if it is continuous on that domain together with all of its (partial) derivatives up to and including those of order n. An arc $y\equiv(y_1,y_2,\ldots,y_n)$ is of class C^n if all of its components, y_i, have this property. From further inspection of the definition of $T_i(x;y)$ it is evident that if g and h are of class C^{n-1} and y is of class C^k for some k on $0\leq k\leq n$, then $T_i(x;y)$, considered as a function of x, is also of class C^k that is, $T(x;y)$ belongs to the same class of arcs as y, provided g and h have proper differentiability properties. Let a primed variable signify differentiation of that variable with respect to x. Then it will be noted that the arc $T''(x;y)$, which when set equal to zero merely presents the differential eq (2) in their usual form, does not possess the property of belonging to the same class to which y belongs. In fact, even if y is of class C^3, $T''(x;y)$ may only belong to class C'.

Before describing an iterative procedure for constructing arcs that satisfy (3), one must first introduce the functions

$$\delta g_i(x,y;\eta)\equiv g_{iy_j}(x,y,y')\eta_j+g_{iy_j'}(x,y,y')\eta_j'$$
$$\delta h_i(x,y;\eta)\equiv h_{iy_j}\eta_j+h_{iy_j'}\eta_j' \qquad (i,j=1,2,\ldots,n) \tag{5}$$

and

$$\delta T_i(x,y;\eta)\equiv\int_a^x\left[\delta g_i(t,y;\eta)-\int_a^t\delta h_i(s,y;\eta)ds\right]dt$$
$$-\frac{x-a}{b-a}\int_a^b\left[\delta g_i-\int_a^t\delta h_i ds\right]dt \qquad (i=1,2,\ldots,n). \tag{6}$$

In (5), (6), and elsewhere in this paper, unless otherwise noted, the tensor convention of summing on repeated indices is utilized. The arc η is taken to be such that it vanishes at the end points of $a\leq x\leq b$.

The function $\delta T_i(x,y;\eta)$ can be obtained from $T_i(x;y)$ as

$$\delta T_i(x,y;\eta)=\frac{d}{d\alpha}\,T_i(x;y+\alpha\eta)|_{\alpha=0}$$

and is often referred to as the "variation" of T_i. It has properties similar to those of T_i. For example, δT_i, considered as a function of x belongs to the same class as the function η, provided that g, h, and y have sufficient differentiability. This fact follows from a small bit of reflection on (6) as does the fact that

$$\delta T_i(a,y;\eta)=\delta T_i(b,y;\eta)=0 \qquad (i=1,2,\ldots,n), \tag{7}$$

regardless of the choice of y and η. It is seen that these properties do not hold for the variation of the differential equations (2) taken in their usual form.

In the iteration process to be discussed here, it is assumed that the i^{th} estimate $y^{(i)}$ of an arc satisfying (3) has been obtained. An "improved estimate" $y^{(i+1)}$ is then found as follows. Determine a "variation" η by solving the system

$$T_j(x;y^{(i)})+\delta T_j(x,y^{(i)};\eta)=0 \qquad (j=1,2,\ldots,n)$$
$$\eta_j(a)=\eta_j(b)=0. \tag{8}$$

The function $y^{(i+1)}$ is then chosen to be

$$y^{(i+1)}=y^{(i)}+\eta^{(i)}, \tag{9}$$

$y^{(i)}$ is replaced by $y^{(i+1)}$, and the steps just described are repeated.

This method may be justified heuristically by observing that upon expanding $T_i(x;y+\eta)$ by Taylor's theorem one gets

$$T_i(x;y+\eta)=T_i(x;y)+\delta T_i(x,y;\eta)+\ldots$$

Thus it is reasonable to assume that a good approximation to a solution of

$$T_i(x;y+\eta)=0 \qquad (i=1,2,\ldots,n)$$

for a known function y might be obtained by solving

$$T_i(x;y)+\delta T_i(x,y;\eta)=0 \qquad (i=1,2,\ldots,n) \quad (10)$$

for η. The analogy between this procedure and Newton's method for a function of a single real variable is clear. Consequently, it will henceforth be referred to as Newton's method.

As one sees by setting $g_i(x,y,y')\equiv y_i'$ $(i=1,2,\ldots,n)$, the differential equations (2) contain the important class of equations

$$y_i''=h_i(x,y,y') \qquad (i=1,2,\ldots,n) \quad (11)$$

as a special case. By setting

$$\begin{aligned} g_i(x,y,y')&\equiv f_{y_i'}(x,y,y') \qquad (i=1,2,\ldots,n)\\ h_i(x,y,y')&\equiv f_{y_i}(x,y,y') \qquad (i=1,2,\ldots,n)\end{aligned} \quad (12)$$

(2) can also be made to assume the form

$$\begin{aligned} \frac{d}{dx}f_{y_i'}&=f_{y_i} \qquad (i=1,2,\ldots,n)\\ y_i(a)=\rho_i,& \qquad y_i(b)=\sigma_i.\end{aligned} \quad (13)$$

These are the Euler equations associated with the integral (1), and their solution is the chief application of Newton's method which we wish to make in this paper. The functions T_i and δT_i associated with (13) are, respectively,

$$T_i(x;y)=\int_a^x\left[f_{y_i'}-\int_a^t f_{y_i}ds\right]dt-\frac{x-a}{b-a}\int_a^b\left[f_{y_i'}-\int_a^t f_{y_i}ds\right]dt$$

and

$$\delta T_i(x,y;\eta)=\int_a^x\left[\omega_{\eta_i'}-\int_a^t \omega_{\eta_i}ds\right]dt-\frac{x-a}{b-a}\int_a^b\left[\omega_{\eta_i'}-\int_a^t \omega_{\eta_i}ds\right]dt,$$

where $\omega_{\eta_i'}$ and ω_{η_i} are derivatives of the quadratic form

$$2\omega(x,y;\eta)=f_{y_iy_j}(x,y,y')\eta_i\eta_j+2f_{y_iy_j'}\eta_i\eta_j'+f_{y_i'y_j'}\eta_i'\eta_j' \qquad (i,j=1,2,\ldots,n).$$

A deeper insight into Newton's method as applied to (13) may be acquired by applying Taylor's theorem to the integrand $f(x;y+\eta,y'+\eta')$. The expansion results in

$$\begin{aligned} f(x,y+\eta,y'+\eta')&=f(x,y,y')+f_{y_i}(x,y,y')\eta_i+f_{y_i'}(x,y,y')\eta_i'\\ &\quad+\tfrac{1}{2}f_{y_iy_j}\eta_i\eta_j+f_{y_iy_j'}\eta_i\eta_j'+\tfrac{1}{2}f_{y_i'y_j'}\eta_i'\eta_j'+\ldots\\ &=f(x,y,y')+f_{y_i}(x,y,y')\eta_i+f_{y_i'}\eta_i'+\omega(x,y,\eta)+\ldots.\end{aligned}$$

Let us drop the higher order terms and consider the problem of minimizing

$$J(y+\eta)=\int_a^b[f(x,y,y')+f_{y_i}\eta_i+f_{y_i'}\eta_i'+\omega(x,y;\eta)]dx,$$

where y is a known function and η is variable. The Euler equations with respect to η of this integral are

$$\frac{d}{dx}f_{y_i}-f_{y_i'}+\frac{d}{dx}\omega_{\eta_i'}-\omega_{\eta_i}=0 \qquad (i=1,2,\ldots,n).$$

This last system of differential equations is formally equivalent to (10). Thus Newton's method consists of repeatedly replacing $f(x,y,y')$ with an integrand that is quadratic and minimizing the new integral so obtained. Consequently, if the original integrand is quadratic, Newton's method should yield a solution in a single step independently of the choice of the initial estimate. That this actually occurs is readily verifiable.

Finally, we note that although the Euler equations often have an equivalent formulation as a system of type (11), Newton's method may not necessarily be the same for both forms of the problem. This is due to the fact that in general δT is not invariant under a transformation of T to an equivalent form. For example, the two differential equations

$$\frac{d}{dx}(yy')=1+2y'^2,$$

and

$$y''=\frac{1+y'^2}{y}$$

are equivalent for $y>0$. In the first case

$$T(x,y)=\tfrac{1}{2}[y^2(x)-y^2(a)]-\int_a^x\left[\int_a^t(1+2y'^2)ds\right]dt-\frac{x-a}{b-a}\left\{\tfrac{1}{2}[y^2(b)-y^2(a)]-\int_a^b\left[\int_a^t(1+2y'^2)ds\right]dt\right\},$$

and

$$\delta T(x,y;\eta)=y(x)\eta(x)-4\int_a^x\left[\int_a^t y'\eta'ds\right]dt+\frac{x-a}{b-a}\int_a^b\left[\int_a^t y'\eta'ds\right]dt.$$

In the second case

$$T(x;y)=y(x)-y(a)-\int_a^x\left[\int_a^t\left(\frac{1+y'^2}{y}\right)ds\right]dt-\frac{x-a}{b-a}\left\{y(b)-y(a)-\int_a^b\left[\int_a^t\left(\frac{1+y'^2}{y}\right)ds\right]dt\right\},$$

and

$$\delta T(x,y;\eta)=\eta(x)-\int_a^x\left[\int_a^t\left(\frac{2y'\eta'}{y}-\frac{1+y'^2}{y^2}\cdot\eta\right)ds\right]dt$$

$$-\frac{x-a}{b-a}\int_a^b\left[\int_a^t\left(\frac{2y'}{y}\cdot\eta'-\frac{1+y'^2}{y^2}\cdot\eta\right)ds\right]dt.$$

It is seen that the two variations are not the same. Since the convergence theory to be developed below will be general enough to include either (11) or (13), one will be free to use whichever system of differential equations seems to be simpler in any given problem.

3. Preliminaries

There are several definitions and known results of general analysis to which there will be frequent occasion to refer. It will be convenient to have these collected in one section. Accordingly, the next few pages are devoted to this end. The reader who wishes more detail about what is to follow in this section is advised to consult a paper of Graves and Hildebrandt [1].[2] The treatment he will find there is more general and couched in somewhat different terminology.

Functions defined on a linear subspace U of a Banach space Y into itself are dealt with here. These functions may depend, in addition, upon a parameter chosen from Y or a region of Y. When the arguments are indicated, the parameter will always appear first and be denoted by a lower case y perhaps bearing a superscript. It will be set off from the independent variables, which will be denoted by Greek letters η, ζ, ξ etc., by a semicolon.

Definition 1. $F(y;\eta)$ is said to be *continuous in y at $y^{(0)}$ uniformly relative to η* if given $\epsilon>0$ there exists $\delta>0$ such that

$$||F(y;\eta)-F(y^{(0)};\eta)||<\epsilon||\eta|| \tag{14}$$

for all y in $(y^{(0)})_\delta$ and every η in U. Here and elsewhere $(y^{(0)})_\delta$ and similar notations denote the δ-neighborhood of $y^{(0)}$.

The extension of definition 1 to the case of a function $F(y;\eta,\zeta,\ldots,\xi)$ of several independent variables is clear. The only essential change is made in replacing eq (14) by

$$||F(y;\eta,\zeta,\ldots,\xi)-F(y^{(0)};\eta,\zeta,\ldots,\xi)||$$
$$<\epsilon||\eta||\cdot||\zeta||\ldots||\xi||.$$

All of the definitions that follow may be generalized to functions of several variables in a similar way.

Definition 2. Let Y_0 be a region of Y. $F(y;\eta)$ is said to be *uniformly continuous in y on Y_0 uniformly with respect to η* if for every $\epsilon>0$ there exists a $\delta>0$ such that

$$||F(y^{(1)};\eta)-F(y^{(2)};\eta)||<\epsilon||\eta||$$

for all $y^{(1)}$, $y^{(2)}$ in Y_0 that satisfy $||y^{(1)}-y^{(2)}||\leq\delta$ and all η in U.

Definition 3. $F(y^{(0)};\eta)$ is said to be *bounded at $y^{(0)}$ uniformly with respect to η* if there exists a constant $M\geq0$ such that

$$||F(y^{(0)};\eta)||\leq M||\eta||$$

for all $\eta\epsilon U$. If some $M\geq0$ holds for all y belonging to some neighborhood $(y^{(0)})_\alpha$, F is said to be *uniformly bounded on $(y^{(0)})_\alpha$ uniformly with respect to η.*

For the sake of brevity the words "uniformly with respect to η", that appear in the above definitions and that should appear in the concluding definition (4) of this section, will henceforth be omitted. The reader, however, should supply them for himself in all of the necessary places.

The following lemma is an easy consequence of the definitions.

Lemma 1. *If $F(y;\eta)$ to U is continuous in y at $y^{(0)}$ and bounded at $y^{(0)}$, there exists a neighborhood of $y^{(0)}$ such that F is uniformly bounded on this neighborhood.*

The conclusion follows from the inequality

$$||F(y;\eta)||\leq||F(y^{(0)};\eta)||+||F(y;\eta)-F(y^{(0)};\eta)||.$$

Definition 4. $F(y;\eta)$ to U is said to be of *class C' in y uniformly on Y_0* in case there exists a function $dF(y;\eta,\zeta)$ on Y_0UU to U having the following properties:

(1) dF is uniformly continuous in y on Y_0.
(2) dF is linear in ζ and uniformly bounded on Y_0.
(3) the function $R(y^{(1)};y^{(2)};\eta)$ defined by

$$F(y^{(2)};\eta)-F(y^{(1)};\eta)-dF(y^{(1)};\eta,y^{(2)}-y^{(1)})$$
$$=R(y^{(1)};y^{(2)};\eta)||y^{(2)}-y^{(1)}||,\quad y^{(2)}\neq y^{(1)},$$
$$R(y^{(1)};y^{(1)};\eta)=\theta \tag{15}$$

is such that given $\epsilon>0$ there exists $\delta>0$ such that

$$||R(y^{(1)};y^{(2)};\eta)||<\epsilon||\eta||$$

for all $y^{(1)},y^{(2)}$ in Y_0 satisfying $||y^{(2)}-y^{(1)}||\leq\delta$ and all η in U.

Here and elsewhere θ refers to the zero of the space Y.

Lemma 2. *If F is of class C' in y uniformly on Y_0, there exist nonnegative constants α and M such that for every $y^{(1)}$ and $y^{(2)}$ of Y_0 for which $||y^{(1)}-y^{(2)}||\leq\alpha$ it is true that*

$$||F(y^{(1)};\eta)-F(y^{(2)};\eta)||\leq M||y^{(1)}-y^{(2)}||\cdot||\eta||$$

for all η in U.

The conclusion of this lemma follows from an application of (2) and (3) of definition 4 to eq (15).

Finally, we state a lemma that corresponds to lemma 16.3 of Graves [1].

Lemma 3. *Let $F(y;\eta)$ to U be bounded at y and linear in η for each y of the region Y_0. In addition let $y^{(0)}$ be a point of Y_0 and let F have a bounded inverse*

[2] Figures in brackets indicate the literature references at the end of this paper.

$F^{-1}(y^{(0)};\eta)$ *with respect to* η, *and be of class* C' *in* y *uniformly on* Y_0. *Then there exists a constant* α *such that for each* y *in* $(y^{(0)})_\alpha$, $F^{-1}(y;\eta)$ *exists and is of class* C' *in* y *uniformly on* $(y^{(0)})_\alpha$.

4. A Convergence Theorem

In section 2 an iterative method for constructing solutions of the Euler equations was described. However, no mention was made of conditions of convergence. In this section we will describe and give conditions for the convergence of a similar procedure that can be applied to a large class of functions. Later the process of section 2 will be considered as a special case of that to be given here, and it will be shown that under simple assumptions the special process satisfies the hypothesis of the convergence theorem of this section.

Let $S(y)$ be a continuous function not identically zero defined on a region Y_0 of a Banach space Y to a subspace U. The problem of interest is that of obtaining solutions of the equation

$$S(y)=\theta. \tag{16}$$

Toward this end one introduces a linear operator $K(y;\eta)$ on U to U, which depends in addition on some parameter taken from Y. It is assumed that $K(y;\eta)$ is nonsingular in η. Let the ith estimate $y^{(i)}$ of the solution to (16) be known. The $i+1$st estimate is constructed as follows. A function $\eta^{(i)}$ is determined by solving

$$S(y^{(i)})+K(y^{(i)};\eta)=\theta \tag{17}$$

and the function $y^{(i+1)}$ is chosen to be

$$y^{(i+1)}=y^{(i)}+\eta^{(i)}. \tag{18}$$

The process is then repeated.

In order to simplify the statement of the convergence theorem, several conditions will now be introduced on the function $S(y)$ and on $K^{-1}(y;\eta)$, the inverse with respect to η of $K(y;\eta)$.

(I). *The arc* $y^{(0)}$ *of* Y_0, *the constant* $\gamma>0$, *and the operator* $K^{-1}(y;\eta)$ *are such that there exists a positive constant* M *for which*

$$||K^{-1}(y^{(1)};\eta)-K^{-1}(y^{(2)};\eta)||\leq M||y^{(1)}-y^{(2)}||\cdot||\eta|| \tag{19}$$

for all η *in* U *and all* $y^{(1)}$, $y^{(2)}$ *in* $(y^{(0)})_\gamma$. *Furthermore, there exists* $B\geq 0$ *such that*

$$||K^{-1}(y;\eta)||\leq B||\eta|| \tag{20}$$

for all y *in* $(y^{(0)})_\gamma$ *and all* η *in* U.

From (19) it is clear that K^{-1} is continuous in y at $y^{(0)}$. Therefore by lemma 1 one sees that in order for (20) to hold on some neighborhood of $y^{(0)}$, which is also a subneighborhood of $(y^{(0)})_\gamma$, it is sufficient to assume that $K^{-1}(y;\eta)$ is bounded at $y^{(0)}$. The remaining part of (I) is, of course, undisturbed on a subneighborhood of $(y^{(0)})_\gamma$.

In the next two conditions certain non-negative constants H, L, and N appear. These are taken to be such that they satisfy

$$H=B\cdot L+M\cdot N<1. \tag{21}$$

(II), *For all* $y^{(1)}$, $y^{(2)}$ *in* $(y^{(0)})_\gamma$ *the inequality*

$$||K(y^{(1)};y^{(1)}-y^{(2)})-S(y^{(1)})+S(y^{(2)})||\leq L||y^{(1)}-y^{(2)}|| \tag{22}$$

is true.

This condition asserts that K is an approximate differential of S on the neighborhood $(y^{(0)})_\gamma$.

(III). *There exists a constant* C *such that* $0<C<\gamma$ *and*

$$||S(y^{(0)})||\leq(1-H)C/B, \tag{23}$$

while

$$||S(y)||\leq N \tag{24}$$

for all other y *in* $(y^{(0)})_\gamma$.

THEOREM 1. *Let the initial estimate* $y^{(0)}$ *and the positive constant* γ *be such that (I), (II), (III) are valid. Then the sequence of estimates determined by (18) converges to a unique element* $\bar{y}$ *of* $(y^{(0)})_\gamma$. *The arc* $\bar{y}$ *is the unique solution of eq(16) in* $(y^{(0)}_\gamma$.

First we prove that

$$||y^{(i)}-y^{(i-1)}||\leq H^{i-1}(1-H)C,\ y^{(i)} in (y^{(0)})_\gamma; \qquad i\geq 1. \tag{25}$$

In order to construct the proof by induction consider the inequality

$$||y^{(1)}-y^{(0)}||=||y^{(0)}+K^{-1}(y^{(0)};-S(y^{(0)}))-y^{(0)}||$$
$$\leq B||S(y^{(0)})||\leq(1-H)C<\gamma.$$

This last follows from (23) and establishes (25) for $i=1$. Assume that (25) is true for all $i\leq m-1$. By this assumption in conjunction with (19), (22), and (24) one may write

$$\begin{aligned}
||y^{(m)}-y^{(m-1)}||&=||y^{(m-1)}+K^{-1}(y^{(m-1)};-S(y^{(m-1)}))\\
&\quad -y^{(m-2)}-K^{-1}(y^{(m-2)};-S(y^{(m-2)}))||\\
&\leq||y^{(m-1)}-y^{(m-2)}-K^{-1}(y^{(m-1)};S(y^{(m-1)}))\\
&\quad +K^{-1}(y^{(m-1)};S(y^{(m-2)}))||+||K^{-1}(y^{(m-2)};\\
&\quad S(y^{(m-2)}))-K^{-1}(y^{(m-1)};S(y^{(m-2)}))||\\
&\leq||K^{-1}(y^{(m-1)};K(y^{(m-1)};\\
&\quad y^{(m-1)}-y^{(m-2)})-S(y^{(m-1)})+S(y^{(m-2)}))||\\
&\quad +M||y^{(m-1)}-y^{(m-2)}||\cdot||S(y^{(m-2)})||\\
&\leq B||K(y^{(m-1)};y^{(m-1)}-y^{(m-2)})-S(y^{(m-1)})+S(y^{(m-2)})||\\
&\quad +M||y^{(m-1)}-y^{(m-2)}||.||S(y^{(m-2)})||\\
&\leq(B\cdot L+M\cdot N)||y^{(m-1)}-y^{(m-2)}||\\
&\leq H\cdot H^{m-2}(1-H)C.
\end{aligned}$$

This establishes the first assertion of (25) for $i=m$. The rest of the proof of (25) follows from the inequality

$$||y^{(m)}-y^{(0)}||=||\sum_{i=1}^{m}(y^{(i)}-y^{(i-1)})||\leq\sum_{i=1}^{m}||y^{(i)}-y^{(i-1)}||$$

$$\leq\sum_{i=1}^{m}H^{i-1}(1-H)C=(1-H^{m})C<\gamma. \tag{26}$$

If a sequence $\{y^{(i)}\}$ in a Banach space is such that the series $\Sigma||y^{(i)}-y^{(i-1)}||$ is convergent, it has a unique limit. Therefore, by (25) there exists a unique element $\overline{y}$ of Y that satisfies

$$\lim_{i\to\infty}||y^{(i)}-\overline{y}||=0. \tag{27}$$

It follows from (26) and (27), that $\overline{y}$ must belong to $(y^{(0)})_\gamma$.

Upon passing to the limit in the expression

$$y^{(m)}=y^{(m-1)}-K^{-1}(y^{(m-1)};S(y^{(m-1)})),$$

and recalling the fact that K^{-1} and S are continuous, one obtains

$$K^{-1}(\overline{y};S(\overline{y}))=\theta.$$

This implies $S(\overline{y})=\theta$.

To see that $\overline{y}$ is the unique solution of (16) in $(y^{(0)})_\gamma$ let y^* be another element of this neighborhood satisfying (16). Then by (20) and (22)

$$||\overline{y}-y^*||=||\overline{y}-y^*-K^{-1}(\overline{y};S(\overline{y}))+K^{-1}(\overline{y};S(y^*))||$$

$$=||K^{-1}(\overline{y};K(\overline{y};\overline{y}-y^*)-S(\overline{y})+S(y^*))||$$

$$\leq B||K(\overline{y};\overline{y}-y^*)-S(\overline{y})+S(y^*)||$$

$$\leq BL||\overline{y}-y^*||.$$

Since (21) implies that $BL<1, \overline{y}\equiv y^*$.

It is possible to modify the method of obtaining iterated estimates by always holding y in $K(y;\eta)$ fixed at the initial estimate $y^{(0)}$. This is equivalent to originally introducing an operator $K(\eta)$, which depends only on functions from U. In this case, of course, the convergence theorem may be simplified. Condition (I) clearly no longer applies, and condition (III) may be modified so that (24) is eliminated. The resulting theorem may be stated in the following way.

Theorem 2. *Let $K(\eta)$ be a nonsingular linear operator. Let $K^{-1}(\eta)$ be bounded and let the positive constant L be such that $||K^{-1}||L<1$. Then if the initial estimate $y^{(0)}$ and the positive constant α are such that*

(1) *for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\alpha$*

$$||K(y^{(1)}-y^{(2)})-S(y^{(1)})+S(y^{(2)})||\leq L||y^{(1)}-y^{(2)}||$$

(2) *for some C, $0<C<\alpha$*

$$||S(y^{(0)})||<\frac{(1-||K^{-1}||\cdot L)C}{||K^{-1}||}$$

the sequence of estimates $\{y^{(i)}\}$ will converge to a unique element $\overline{y}$ of $(y^{(0)})_\alpha$. The arc $\overline{y}$ will satisfy eq (16) uniquely in $(y^{(0)})_\alpha$.

Thus when K is independent of y, only an initial approximation to the solution at one point and an approximate differential of S are required to insure convergence. Consequently, this last theorem is seen to be just the classical implicit function theorem in general analysis. One notes that in this case the continuity of S is not required.

This section will be completed by establishing conditions under which the sequence of estimates will converge quadratically.

(IV). *Let $y^{(0)}$ be an element of Y and let γ and L' be constants such that*

$$||K(y^{(1)};y^{(1)}-y^{(2)})-S(y^{(1)})+S(y^{(2)})||$$
$$\leq L'||y^{(1)}-y^{(2)}||^2 \tag{28}$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\gamma$.

We may now state the following theorem.

Theorem 3. *Let the initial estimate $y^{(0)}$ and the positive constant γ be such that (I), (III), and (IV) are valid. Then the sequence of estimates determined by (18) converges quadratically to a unique element $\overline{y}$ of some neighborhood $(y^{(0)})_\alpha$, $\alpha\leq\gamma$. The arc $\overline{y}$ is the unique solution of eq (16) in $(y^{(0)})_\alpha$.*

To see that (IV) implies (II) on $(y^{(0)})_\alpha$, $\alpha\leq\gamma$, merely take α small enough so that $L'||y^{(1)}-y^{(2)}||\leq L$ for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\alpha$. As (I) and (III) are undisturbed on the smaller neighborhood, all conclusions but that of quadratic convergence follow at once from theorem 1. The quadratic nature of the convergence can be established by the following calculations. From (17) and (18) it follows that

$$y^{(i)}-y^{(i-1)}=K^{-1}(y^{(i-1)};-S(y^{(i-1)})). \tag{29}$$

Therefore,

$$K(y^{(i-1)};y^{(i)}-y^{(i-1)})=-S(y^{(i-1)}).$$

Thus by (28)

$$||S(y^{(i)})||\leq L'||y^{(i)}-y^{(i-1)}||^2.$$

Making use of this last result in (29) one obtains

$$||y^{(i)}-y^{(i-1)}||\leq BL'||y^{(i-1)}-y^{(i-2)}||^2.$$

This establishes theorem 3.

5. Some Properties of $\delta T(x,y;\eta)$

In this section some properties of the function δT, which was defined in section 2, are developed with an eye toward applying the considerations of section 4 to Newton's method. First, however, the space Y to which the arcs y belong must be more precisely defined. The space Y is chosen to be the collection $(y_i(x))$ of all n-tuples of real valued functions that are of class C' on the interval $a \leq x \leq b$. Addition of elements in Y and multiplication of elements of Y by real scalars are defined in the usual manner. Under these definitions of addition and multiplication Y forms a linear vector space over the real numbers.

For any y in Y let

$$|y(x)| = \left(\sum_{i=1}^{n} |y_i(x)|^2\right)^{\frac{1}{2}}.$$

Then a topology may be introduced into Y by defining a norm

$$||y|| = \sup_{a \leq x \leq b} |y(x)| \vee \sup_{a \leq x \leq b} |y'(x)|. \tag{30}$$

The fact that (30) satisfies the properties of a norm is easily verified as is the fact that $||y^{(1)} - y^{(2)}||$ has all of the properties of a metric. It is also not difficult to establish that any Cauchy sequence of elements of Y must converge in the sense of the norm to an element of Y. In fact, convergence in Y implies uniform convergence in the ordinary sense of the sequence of first derivatives. Thus in the topology specified by (30) Y is a Banach space.

The subclass of arcs in Y that have the property

$$y(a) = y(b) = 0 \tag{31}$$

will be denoted by the symbol U. It is clear that U is a linear subspace of Y.

In order to eliminate repetition when discussing certain properties of the norms of elements of Y, it is convenient to have the following simple lemma available for reference.

Lemma 4. *Let y, η, and $Q(x,y;\eta)$ belong to Y. Let there correspond to each component Q_i of Q a non-negative number $C^{(i)}$ such that*

$$\sup_x |Q_i(x,y;\eta)| \leq C^{(i)} ||\eta||.$$

Also let there correspond to each component Q_i' of Q', the first derivative with respect to x of Q, a non-negative number $C^{(i)\prime}$ such that

$$\sup_x |Q_i'(x,y;\eta)| \leq C^{(i)\prime} ||\eta||.$$

Then there exists $C \geq 0$ such that

$$||Q(x,y;\eta)|| \leq C ||\eta||.$$

By (30) either

$$||Q(x,y;\eta)|| = \sup_x \left(\sum_{i=1}^{n} |Q_i|^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{n} \sup_x |Q_i| \leq \sum_{i=1}^{n} C^{(i)} ||\eta||,$$

or

$$||Q(x,y;\eta)|| = \sup_x \left(\sum_{i=1}^{n} |Q_i'|^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{n} \sup_x |Q_i'| \leq \sum_{i=1}^{n} C^{(i)\prime} ||\eta||.$$

Obviously C can be chosen to be the greater of the numbers $\sum_{i=1}^{n} C^{(i)}$ and $\sum_{i=1}^{n} C^{(i)\prime}$.

In addition to defining the space of functions under consideration, it is desirable at this point to state some assumptions that will hold throughout the remainder of the paper. Unless otherwise noted, it is assumed that the functions $g(x,y,y')$ and $h(x,y,y')$ are of class C^2 in a region R of Euclidean $2n+1-$space. Any arc of Y, all of whose elements $(x,y(x),y'(x))$ are in R, will be termed *admissible*. At least one admissible arc is required to exist. This implies the existence of a subregion Y_0, all of whose elements are admissible. It is also assumed that the determinant $|g_{iy_j'}(x,y,y')|$ never vanishes in R. In the case of the Euler equations, that is, when g and h are given by (12), the above assumptions imply that the derivatives f_{y_i} and $f_{y_i'}$ of the integrand $f(x,y,y')$ must be of class C_2 in R.

In eq (6), which defines the function $\delta T_i(x,y;\eta)$, let the parameter y range over Y_0 and the variable η range over U. The following result is then valid.

Lemma 5. *For each y belonging to Y_0 the transformation $\delta T(x,y;\eta) \equiv (\delta T_1, \ldots, \delta T_n)$ is on Y_0U to U and is linear in η and bounded at y.*

The fact that δT is an element of U follows from the considerations of section 2. The linearity in η is a consequence of the corresponding property of eq (5) and of the operation of integration.

To simplify the proof of the boundedness we introduce a lemma that will be found useful in what follows.

Lemma 6. *Let $q(x;y)$ be given by*

$$q(x;y) \equiv a_j(x) y_j(x) + b_j(x) y_j'(x),$$

where $a_j(x)$, $b_j(x)$ $(j=1, 2, \ldots, n)$ are continuous on $a \leq x \leq b$ and y is in Y. Then there exists a constant $C \geq 0$ such that for each y in Y

$$\sup_x |q(x;y)| \leq C ||y|| \tag{32}$$

and

$$\sup_x \left| \int_a^x q(t;y) dt \leq (b-a) C ||y||. \tag{33}$$

On recalling that $\sup_x |y_i| \leq ||y||$ and $\sup_x |y_i'| \leq ||y||$ for all $(i=1, 2, \ldots, n)$ one quickly obtains the inequality

$$\sup_x |q(x;y)| \leq (\sum_j \sup_x |a_j|) ||y|| + (\sum_j \sup_x |b_j|) ||y|| \equiv C ||y||.$$

The second statement follows from the first by means of the inequality

$$\sup_x\left|\int_a^x q(t;y)\,dt\right|\leq\int_a^b|q(t;y)|dt\leq\int_a^b\sup_t|q(t;y)|dt.$$

Upon noting that δg_i and δh_i are of the same type as $q(x;y)$ and applying lemma (6) one obtains the inequality

$$\begin{aligned}|\delta T_i|&\leq\sup_x|\delta T_i|\\
&=\sup_x\left|\int_a^x\left[\delta g_i-\int_a^t\delta h_i ds\right]dt-\right.\\
&\qquad\left.\frac{x-a}{b-a}\int_a^b\left[\delta g_i-\int_a^t\delta h_i ds\right]dt\right|\\
&\leq 2\int_a^b\left|\delta g_i-\int_a^t\delta h_i ds\right|dt\\
&\leq 2\int_a^b\left[\sup_x|\delta g_i|+\sup_t\left|\int_a^t\delta h_i ds\right|\right]dt\\
&\leq C^{(i)}||\eta||,\quad C^{(i)}\geq 0.\end{aligned}$$

A similar result follows for $|\delta T_i'|$. Thus the conclusion follows from lemma 4.

6. An Existence Lemma

In section 7 the existence of an arc η in U that satisfies

$$\begin{aligned}\delta T_i(x,y;\eta)\equiv\int_a^x\left[\delta g_i-\int_a^t\delta h_i ds\right]dt-&\\
\frac{x-a}{b-a}\int_a^b\left[\delta g_i-\int_a^t\delta h_i ds\right]dt&\\
=l_i(x),\quad(i=1,2,\ldots,n)&\end{aligned}\tag{34}$$

for an arbitrary function $l(x)$ in U will be discussed. To facilitate this discussion it is convenient to have available the following lemma.

LEMMA 7. *For each admissible arc y and every n-tuple of functions $m_i(x)$ continuous on $a\leq x\leq b$ the system*

$$\delta g_i(x,y;\eta)-\int_a^x\delta h_i(t,y;\eta)dt=m_i(x)\qquad(i=1,2,\ldots,n)\tag{35}$$

has a unique solution $\eta(x)$ which is of class C' and satisfies $\eta(a)=\theta$.

An alternate form for (35) to which known methods of proof used in constructing standard existence theorems can be directly applied will be introduced. Let

$$\zeta_i\equiv\delta g_i=g_{iy_j}\eta_j+g_{iy_j'}\eta_j'\qquad(i=1,2,\ldots,n).\tag{36}$$

Because by hypothesis the determinant $|g_{iy_j'}|$ is never zero, eq (36) can be solved for η_i'. The solution is

$$\eta_i'=\pi_i(x,\eta,\zeta)\qquad(i=1,2,\ldots,n)\tag{37}$$

where π_i is a linear and homogeneous function of η_i and ζ_i. By use of (37) δh_i may be expressed as

$$\delta h_i=h_{iy_j}\eta_j+h_{iy_i'}\pi_j(x,\eta,\zeta),$$

a linear and homogeneous function in η_i and ζ_i. Therefore, any pair η, η' satisfying (35) are transformed by (36) into a pair η, ζ satisfying

$$\zeta_i-\int_a^x[h_{iy_j}\eta_j+h_{iy_j'}\pi_j(t,\eta,\zeta)]dt=m_i(x)\quad(i=1,2,\ldots,n)$$

$$\eta_i-\int_a^x\pi_i(t,\eta,\zeta)dt=0\qquad(i=1,2,\ldots,n).\tag{38}$$

The last n equations of (38) are obtained by integrating (37) and recalling that $\eta_i(a)=0$.

Conversely, assume η,ζ satisfy (38) the second condition of that set implies (37). Inverting this last system of equations, one obtains (36), and by substitution in the first condition of (38) eq (35) result. Thus instead of seeking a solution of (35) which is of class C' one may equivalently seek a continuous solution of (38).

The system (38) has the form

$$\gamma_i(x)=\int_a^x A_{ij}(t)\gamma_j(t)dt+\overline{m}_i(x)\qquad(i=1,2,\ldots,2n)\tag{39}$$

where A_{ij}, $\overline{m}_i$ are all continuous on $a\leq x\leq$ b. From well known methods of proof which have been used by Bliss [2, p. 274–278] and others for systems of differential equations which can be obtained by formally differentiating (39) it can be shown that (39) has a unique continuous solution satisfying $\gamma_i(a)=\overline{m}_i(a)$. Since

$$\gamma_{n+1}(a)\equiv\eta_i(a)=0\qquad(i=1,2,\ldots,n)$$

the lemma is proved.

7. Existence of an Inverse of $\delta T(x,y,\eta)$

In this section the existence of a linear bounded inverse in η for the function $\delta T(x,y;\eta)$ is discussed. An idea that plays a leading role in this discussion is that of conjugate points.

DEFINITION 5. A point P_3 is said to be *conjugate* to the point P_1 on an arc joining P_1 and P_2 if there exists a solution of

$$\frac{d}{dx}\delta g_i(x,y;\eta)-\delta h_i(x,y;\eta)=0\qquad(i=1,2,\ldots,n)\tag{40}$$

along this arc, which vanishes at P_1 and P_3 but is not identically zero between P_1 and P_3.

A consequence of definition 5 that will be apropos to our present purpose is stated in the following lemma.

LEMMA 8. *Let $\eta_i^{(j)}$ $(i,j=1,2,\ldots,n)$ be solutions of*

$$L_i(\eta)\equiv\delta g_i-\int_a^x \delta h_i dt=\delta_{ij},\quad \eta_i^{(j)}(a)=0 \qquad (i=1,2,\ldots,n). \tag{41}$$

Then $x=b$ is conjugate to $x=a$ if, and only if, the determinant $|\eta_i^{(j)}(b)|$ is zero.

By lemma 7 the $\eta_i^{(j)}(x)$ described in the hypothesis exist and are of class C'. In addition to satisfying (41), the $\eta_i^{(j)}$ are n linearly independent solutions of (40). Thus the n-parameter family of solutions of (40) which vanish at $x=a$ is given by

$$\eta_i(x,C_1,C_2,\ldots,C_n)\equiv\eta_i^{(j)}(x)C_j \qquad (i,j=1,2,\ldots,n). \tag{42}$$

Setting $\eta_i(b)=0$ in (42), one obtains the system of linear algebraic equations

$$\eta_i^{(j)}(b)C_j=0 \qquad (i=1,2,\ldots,n). \tag{43}$$

In order for $x=b$ to be conjugate to $x=a$, the system (43) must have a nontrivial solution. This is possible if, and only if, $|\eta_i^{(j)}(b)|=0$.

LEMMA 9. *For each admissible y in Y whose end points are not conjugate $\delta T(x,y;\eta)$ is a 1—1 mapping U onto itself.*

By lemma 5, $\delta T(x,y;\eta)$ is an element of U for each η in U and any admissible function y. It remains to be demonstrated that the system (34) has a unique solution η in U for each $l(x)$ in U provided that the end points of y are not conjugate. By lemma 7, there exists a unique $\bar{\eta}(x)$ of class C' which satisfies

$$\delta g_i-\int_a^x \delta h_i\, dt=l_i'(x),\quad \bar{\eta}_i(a)=0 \qquad (i=1,2,\ldots,n).$$

Let $\eta_i^{(j)}$ be the functions mentioned in lemma 8. Then by lemma 8 there always exists a unique set $\{C_i\}$ $(i=1,2,\ldots,n)$ such that

$$\eta_i(b)+\bar{\eta}_i^{(j)}(b)C_j=0 \qquad (i=1,2,\ldots,n).$$

Consequently, by (41),

$$\eta_i(x)\equiv\bar{\eta}_i(x)+\eta_i^{(j)}(x)C_j \qquad (i=1,2,\ldots,n) \tag{44}$$

is a solution of

$$\delta g_i-\int_a^x \delta h_i dt=l_i'+C_i \qquad (i=1,2,\ldots,n) \tag{45}$$

which vanishes at $x=a$ and $x=b$. That $\eta(x)$ is of class C' is apparent. That $\eta(x)$ is unique follows from the condition of nonconjugate end points. For if $\eta^{(1)}$ and $\eta^{(2)}$ both satisfy (45) and the boundary conditions, their difference must satisfy (40) and vanish at both end points of $a\leq x\leq b$, that is, their difference must be identically zero.

By integration of (45) it follows that $\eta(x)$ must satisfy

$$\int_a^b\left[\delta g_i-\int_a^t \delta h_i ds\right]dt-C_i(x-a)=l_i(x) \qquad (i=1,2,\ldots,n). \tag{46}$$

Letting $x=b$ in (46) one sees that

$$\int_a^x\left[\delta g_i-\int_a^t \delta h_i ds\right]dt-C_i(b-a)=l_i(b)=0 \qquad (i=1,2,\ldots,n).$$

Therefore,

$$C_i=\frac{1}{b-a}\int_a^b\left[\delta g_i-\int_a^t \delta h_i ds\right]dt \qquad (i=1,2,\ldots,n).$$

Consequently, $\eta(x)$ satisfies (34).

We conclude with a lemma that states the result we have been pointing at in this and the preceding section.

LEMMA 10. *For each admissible y in Y whose end points are not conjugate the inverse of $\delta T(x,y;\eta)$ exists and is linear and bounded at y.*

By lemma 5, and lemma 9, $\delta T(x,y;\eta)$ is a bounded linear transformation mapping a Banach space in a 1—1 manner onto itself. As is shown in Hille [3, p. 28–29], a transformation having the above properties must possess a bounded inverse.

The inverse of a linear transformation is linear.

8. Differentiability Properties of $\delta T(x,y;\eta)$

In this section two lemmas that will be useful in establishing the fact that δT is one of the class of operators that was denoted by $K(y;\eta)$ in section 4 are proved. The first lemma will help to show that δT satisfies condition (I) of section 4, whereas the second relates to conditions (II) and (IV) of the same section. First, however, we present a lemma that will considerably simplify the statement of the proofs of our two principal ones.

LEMMA 11. *Let $p(x,y,y',\eta,\xi)$ be of the form*

$$p(x,y,y',\eta,\xi)=a_{ij}(x,y,y')\eta_i\xi_j+b_{ij}(x,y,y')\eta_i\xi_i'+c_{ij}(x,y,y')\eta_i'\xi_j' \quad (i,j=1,\ldots,n), \tag{47}$$

where a_{ij}, b_{ij}, c_{ij}, are all continuous functions on the same region R on which g and h are defined, y is an admissible element of Y, and η, ξ are elements of U. Let $y^{(0)}$ be any admissible element of Y and $\delta>0$ be such that the closure of $(y^{(0)})_\delta$ contains only admissible arcs. Then given $\epsilon>0$ it follows that there exists $d>0$ such that

$$|p(x,y^{(1)},y^{(1)\prime},\eta,\xi)-p(x,y^{(2)},y^{(2)\prime},\eta,\xi)|<\epsilon||\eta||\cdot||\xi|| \tag{48}$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$, which satisfy $||y^{(1)}-y^{(2)}||<d$ and all η,ξ in U. Furthermore, there exists a constant $C\geq 0$ such that

$$\sup_x |p(x,y,y',\eta,\xi)|\leq C||\eta||\cdot||\xi||, \tag{49}$$

and

$$\sup_x \left|\int_a^x p(x,y,y',\eta,\xi)dx\right|\leq(b-a)C||\eta||\cdot||\xi|| \tag{50}$$

for y in the closure of $(y^{(0)})_\delta$, η, ξ in U.

Statements (49) and (50) follow from a simple extension of lemma 6.

In order to establish the remaining part of the lemma, let

$$\varphi(x,y^{(1)},y^{(2)},\eta,\xi)=p(x,y^{(1)},y^{(1)'},\eta,\xi)-p(x,y^{(2)},y^{(2)'},\eta,\xi).$$

Then φ is a continuous function on the closed and bounded set $S{:}a\leq x\leq b$, $y^{(1)}$, $y^{(2)}$ in the closure of $(y^{(0)})_\delta$, $\|\eta\|=\|\xi\|=1$. That it is sufficient to merely consider η, ξ on the unit sphere follows from the bilinearity of p. It is clear that $\varphi=0$ on the subset of S for which $y^{(1)}=y^{(2)}$. Thus $|\varphi|<\epsilon$ on a neighborhood of this set, that is, on $a\leq x\leq b$, $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$, $\|y^{(1)}-y^{(2)}\|<d$, $\|\eta\|=\|\xi\|=1$ for d sufficiently small. This proves lemma 11.

LEMMA 12. *For each admissible $y^{(0)}$ in Y there exists a constant $\delta>0$ such that $\delta T(x,y;\eta)$ is of class C' in y uniformly on $(y^{(0)})_\delta$.*

Let $\delta^2 g_i(x,y;\eta,\xi)$ and $\delta^2 h_i(x,y;\eta,\xi)$ be defined by

$$\delta^2 g_i\equiv g_{iy_jy_k}(x,y,y')\eta_j\xi_k+g_{iy_jy'_k}(x,y,y')\eta_j\xi'_k+g_{iy'_jy_k}\eta'_j\xi_k+g_{iy'_jy'_k}\eta'_j\xi'_k, \tag{51}$$

and

$$\delta^2 h_i\equiv h_{iy_jy_k}(x,y,y')\eta_j\xi_k+h_{iy_jy'_k}\eta_j\xi'_k+h_{iy'_jy_k}\eta'_j\xi_k+h_{iy'_jy'_k}\eta'_j\xi'_k \tag{52}$$

where y is some admissible element of Y and η,ξ are arcs in U. Then the function $d(\delta T)$ mentioned in definition 4 can be defined as

$$d(\delta T_i)\equiv\delta^2 T_i(x,y;\eta,\xi)=\int_a^x\left[\delta^2 g_i-\int_a^t \delta^2 h_i ds\right]dt-\frac{x-a}{b-a}\int_a^b\left[\delta^2 g_i-\int_a^t\delta^2 h_i ds\right]dt \qquad (i=1,2,\ldots,n).$$

By reasoning similar to that used in considering δT it follows that $\delta^2 T$ is on Y_0UU to U. That $\delta^2 T$ is bilinear in η and ξ is obvious. To complete the proof of the lemma conditions (1), (2), and (3) of definition 4 must be established.

(1) *For every admissible $y^{(0)}$ in Y and every finite $\delta>0$ such that the closure of $(y^{(0)})_\delta$ contains only admissible arcs $\delta^2 T(x,y;\eta,\xi)$ is continuous in y uniformly on $(y^{(0)})_\delta$.*

Let $F(x,y,y';\eta,\xi)$ be any function defined on Y and U. Then we define

$$\Delta F(1,2)\equiv F(x,y^{(1)},y^{(1)'};\eta,\xi)-F(x,y^{(2)},y^{(2)'};\eta,\xi).$$

Keeping this convention in mind, one may write

$$\begin{aligned}&\sup_x |\delta^2 T_i(x,y^{(1)};\eta,\xi)-\delta^2 T_i(x,y^{(2)};\eta,\xi)|\\ &=\sup_x\left|\int_a^x\left[\Delta\delta^2 g_i(1,2)-\int_a^t\Delta\delta^2 h_i(1,2)ds\right]dt-\frac{x-a}{b-a}\int_a^b\left[\Delta\delta^2 g_i(1,2)-\int_a^t\Delta\delta^2 h_i(1,2)ds\right]dt\right| \qquad (53)\\ &\leq 2\int_a^b\left|\Delta\delta^2 g_i(1,2)-\int_a^t\Delta\delta^2 h_i(1,2)ds\right|dt\\ &\leq 2\int_a^b|\Delta\delta^2 g_i(1,2)|dt+2(b-a)\int_a^b|\Delta\delta^2 h_i(1,2)|dt.\end{aligned}$$

Examination of (51) and (52) reveals that both $\delta^2 g_i$ and $\delta^2 h_i$ are of the type $p(x,y,y',\eta,\xi)$ defined in lemma 11. Thus application of lemma 11 to (53) yields the conclusion

$$\sup_x|\delta^2 T_i(x,y^{(1)};\eta,\xi)-\delta^2 T_i(x,y^{(2)};\eta,\xi)|<\epsilon||\eta||\cdot||\xi||$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$ which satisfy a condition of the form $\|y^{(1)}-y^{(2)}\|\leq d$. The same result can be established when $\delta^2 T_i$ is replaced by $\delta^2 T'_i$. Therefore, (1) follows at once from lemma 4.

(2) *For every admissible $y^{(0)}$ in Y there exists a $\delta>0$ such that $\delta^2 T(x,y;\eta,\xi)$ is uniformly bounded on $(y^{(0)})_\delta$.*

From an application of (49) and (50) of lemma 11 the inequality

$$\begin{aligned}\sup_x|\delta^2 T_i(x,y^{(0)};\eta,\xi)|&\leq 2\int_a^b\left|\delta g_i-\int_a^t\delta h_i ds\right|dt\\ &\leq 2\int_a^b\left[\sup_t|\delta g_i|+\sup_t\left|\int_a^t\delta h_i ds\right|\right]dt\\ &\leq C^{(i)}||\eta||\cdot||\xi||,\quad C^{(i)}\geq 0\end{aligned} \tag{54}$$

is obtained. From analogous considerations the existence of $C^{(i)'}\geq 0$ such that

$$\sup_x|\delta^2 T'_i(x,y^{(0)};\eta,\xi)|\leq C^{(i)'}||\eta||\cdot||\xi|| \tag{55}$$

can be demonstrated. Because of lemma 4, inequalities (54) and (55) imply that $\delta^2 T$ is bounded in y at $y^{(0)}$. By (1) $\delta^2 T$ is continuous in y at $y^{(0)}$. Thus by lemma 1 there exists $\delta>0$ such that $\delta^2 T$ is uniformly bounded in y on $(y^{(0)})_\delta$.

(3) *For each admissible $y^{(0)}$ in Y and each finite $\delta>0$ which is such that the closure of $(y^{(0)})_\delta$ contains only admissible arcs, the function R $(y^{(1)},y^{(2)};\eta)$ defined by replacing F by δT and dF by $\delta^2 T$ in (4) is*

such that given $\epsilon>0$ there exists $d>0$ for which

$$\|R(y^{(1)},y^{(2)};\eta)\|<\epsilon\|\eta\|$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$ satisfying $\|y^{(2)}-y^{(1)}\|\leq d$ and all η in U.

Consider

$$\delta T_i(x,y^{(1)};\eta)-\delta T_i(x,y^{(2)};\eta)=\int_a^x [\Delta g_{iy_j}(1,2)\eta_j$$

$$+\Delta g_{iy_j'}(1,2)\eta_j'-\int_a^t (\Delta h_{iy_j}(1,2)\eta_j+\Delta h_{iy_j'}(1,2)\eta_j')ds]dt$$

$$-\frac{x-a}{b-a}\int_a^b [\Delta g_{iy_j}(1,2)\eta_j+\Delta g_{iy_j'}(1,2)\eta_j'$$

$$-\int_a^t (\Delta h_{iy_j}(1,2)\eta_j+\Delta h_{iy_j'}(1,2)\eta_j')ds]dt. \quad (56)$$

Applying the law of the mean to a typical Δ-symbol in (56), one obtains

$$\Delta g_{iy_j}(1,2)=g_{iy_jy_k}(x,y^{(3)},y^{(3)\prime})(y_k^{(1)}-y_k^{(2)})$$
$$+g_{iy_jy_k'}(x,y^{(3)},y^{(3)\prime})(y_k^{(1)\prime}-y_k^{(2)\prime}),$$

where

$$y^{(3)}=y^{(2)}+\mu(y^{(1)}-y^{(2)}), \qquad 0<\mu<1.$$

Thus by applying the law of the mean to the remaining Δ-symbols and taking note of (51) and (52), we may write

$$\delta T_i(x,y^{(1)};\eta)-\delta T_i(x,y^{(2)};\eta)-\delta^2 T_i(x,y^{(1)};\eta,y^{(1)}-y^{(2)})$$

$$=\int_a^x \boldsymbol{D}(x,y;\eta,\xi)dt-\frac{x-a}{b-a}\int_a^b \boldsymbol{D}dt,$$

where

$$\boldsymbol{D}\equiv\Delta g_{iy_jy_k}(3,1)\eta_j(y_k^{(1)}-y_k^{(2)})+\Delta g_{iy_jy_k'}(3,1)\eta_j(y_k^{(1)\prime}-y_k^{(2)\prime})$$

$$+\Delta g_{iy_j'y_k}(3,1)\eta_j'(y_k^{(1)}-y_k^{(2)})+\Delta g_{iy_j'y_k'}(3,1)\eta_j'(y_k^{(1)\prime}-y_k^{(2)\prime})$$

$$-\int_a^t [\Delta h_{iy_jy_k}(3,1)\eta_j(y_k^{(1)}-y_k^{(2)})+\Delta h_{iy_jy_k'}(3,1)\eta_j\times (y_k^{(1)\prime}-y_k^{(2)\prime})$$

$$+\Delta h_{iy_j'y_k}(3,1)\eta_j'(y_k^{(1)}-y_k^{(2)})+\Delta h_{iy_j'y_k'}(3,1)\eta_j'\times (y_k^{(1)\prime}-y_k^{(2)\prime})]ds. \quad (57)$$

Of course, the 3 appearing in the arguments of the various Δ-symbols does not always refer to the same value of μ.

The function $g_{iy_jy_k}\eta_j(y_k^{(1)}-y_k^{(2)})$, that is typical of the functions to which the Δ operation is applied in (57), is of the type considered in lemma 11. Thus it readily follows from that lemma that,

$$\sup_x |\delta T_i(x,y^{(1)};\eta)-\delta T_i(x,y^{(2)};\eta)-\delta^2 T_i(x,y^{(1)};\eta,y^{(1)}-y^{(2)})|\leq 2\int_a^b \left|\boldsymbol{D}-\int_a^t \boldsymbol{D}\,ds\right|dt<\epsilon\|\eta\|\cdot\|y^{(1)}-y^{(2)}\|$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$ such that $\|y^{(1)}-y^{(2)}\|\leq d$. A similar result holds if δT_i and $\delta^2 T_i$ are replaced by their derivatives. Therefore, by lemma 4 it is true that

$$\|R(y^{(1)},y^{(2)};\eta)\|\cdot\|y^{(1)}-y^{(2)}\|$$

$$=\|\delta T(x,y^{(1)};\eta)-\delta T(x,y^{(2)};\eta)-\delta^2 T(x,y^{(2)};\eta,y^{(1)}-y^{(2)})\|$$

$$<\epsilon\|\eta\|\cdot\|y^{(1)}-y^{(2)}\| \quad (58)$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$ that are close enough together. Statement (3) follows at once from (58).

By choosing the δ's of (1) and (3) to be equal to each other and less than the δ mentioned in (2), one obtains a δ that is suitable to be used in lemma 12.

LEMMA 13. *For any admissable $y^{(0)}$ in Y there exist constants $\alpha>0$, $L'\geq 0$ such that*

$$\|\delta T(x,y^{(1)};y^{(1)}-y^{(2)})-T(x;y^{(1)})+T(x;y^{(2)})\|\leq L'\|y^{(1)}-y^{(2)}\|^2$$

for all $y^{(1)}$, $y^{(2)}$ in $y^{(0)})_\alpha$.

Consider

$$T_i(x;y^{(2)})-T_i(x;y^{(1)})$$

$$=\int_a^x\left[\Delta g_i(2,1)-\int_a^t\Delta h_i(2,1)ds\right]dt-$$

$$\frac{x-a}{b-a}\int_a^b\left[\Delta g_i(2,1)-\int_a^t\Delta h_i(2,1)ds\right]dt.$$

By the law of the mean

$$\Delta g_i(2,1)=g_{iy_j}(x,y^{(3)},y^{(3)\prime})(y_j^{(2)}-y_j^{(1)})+$$
$$g_{iy_j}{}'(x,y^{(3)},y^{(3)\prime})(y_j^{(2)\prime}-y_j^{(1)\prime})$$

where

$$y^{(3)}=y^{(1)}+\mu(y^{(2)}-y^{(1)}), \qquad 0<\mu<1.$$

As a similar result holds for Δh_i, we may write

$$\delta T_i(x,y^{(1)};y^{(1)}-y^{(2)})-T_i(x;y^{(1)})+T_i(x;y^{(2)})-$$

$$\int_a^x Z(x,y;\eta,\xi)dx=\frac{x-a}{b-a}\int_a^b Zdx,$$

where

$$Z\equiv\Delta g_{iy_j}(1,3)(y_j^{(1)}-y_j^{(2)})+\Delta g_{iy_j'}(1,3)(y_j^{(1)\prime}-y_j^{(2)\prime})-$$

$$\int_a^t[\Delta h_{iy_j}(1,3)(y_j^{(1)}-y_j^{(2)})+\Delta h_{iy_j'}(1,3)(y_j^{(1)}-y_j^{(2)\prime})]ds.$$

Of course, the 3 appearing in the arguments of Δg_i and Δh_i does not always refer to the same value of μ. Application of the law of the mean to a typical term of Z yields

$$\Delta g_{iy_j'}(1,3)=g_{iy_j'y_k}(x,y^{(4)},y^{(4)\prime})(y_j^{(1)\prime}-y_j^{(2)\prime})(y_k^{(1)}-y_k^{(2)})\mu$$
$$+g_{iy_j'y_k'}(x,y^{(4)},y^{(4)\prime})(y_j^{(1)\prime}-y_j^{(2)\prime})(y_k^{(1)\prime}-y_k^{(2)\prime})\mu,$$

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where

$$y^{(4)}=y^{(1)}+\mu^*(y^{(1)}-y^{(2)}), \qquad 0<\mu^*<1.$$

It will be noted that $\Delta g_{iy_j'}(1,3)$ is of the same form as the functions considered in lemma 11. Let α be chosen small enough so that for any $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\alpha$, $y^{(4)}$ lies in some fixed neighborhood $(y^{(0)})_\alpha$ whose closure contains only admissible functions. Then because of our previous hypothesis on the class of h_i and g_i, we may apply lemma 11 from which the existence of $C^{(i)}\geq 0$ such that

$$\sup_x |\delta T_i(x,y^{(1)};y^{(1)}-y^{(2)})-T_i(x,y^{(1)})+T_i(x;y^{(2)})| \leq C^{(i)}||y^{(1)}-y^{(2)}||^2$$

follows. As similar considerations are valid when $\delta T_i, T_i$ are replaced by $\delta T_i', T_i'$, respectively, the conclusion of lemma 13 follows from lemma 4.

9. Convergence of the Sequence of Estimates Derived by Newton's Method

The main theorem on the convergence of the sequence of estimates derived from Newton's method is the following.

THEOREM 4. *Let $y^{(0)}$ be an admissible element of Y having nonconjugate end points. Then if there exists a sufficiently small constant $D>0$ such that $||T(x,y^{(0)})||<D$, the sequence of estimates derived from Newton's method will converge quadratically to a unique arc $\bar{y}$ of class C^3 which lies within a predetermined neighborhood of $y^{(0)}$ and satisfies $T(x;\bar{y})=\theta, \bar{y}(a)=y^{(0)}(a), \bar{y}(b)=y^{(0)}(b)$.*

By lemma 12, there exists a constant $\alpha_1>0$ such that $\delta T(x,y;\eta)$ is of class C' in y uniformly on $(y^{(0)})_{\alpha_1}$. By lemma 10, $\delta T^{-1}(x,y^{(0)};\eta)$ exists, and is linear and bounded at $y^{(0)}$. By lemma 5, δT is on $(y^{(0)})_{\alpha_1}U$ to U and is bounded in y and linear in η. Therefore, by lemma 3, there exists α_2 such that δT^{-1} exists on $(y^{(0)})_{\alpha_2}$ and is of class C' in y uniformly on $(y^{(0)})_{\alpha_2}$. Thus by lemma 2, there exist nonnegative constants α_3 and M such that for $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_{\alpha_2}$ and $||y^{(1)}-y^{(2)}||\leq_{\alpha_3}$ it follows that

$$||\delta T^{-1}(x,y^{(1)};\eta)-\delta T^{-1}(x,y^{(2)};\eta)||\leq M||y^{(1)}-y^{(2)}||\cdot||\eta||.$$

Consequently, if we choose $\alpha_4\equiv\alpha_3/2$, a part of condition (I) of section 4 is satisfied on $(y^{(0)})_{\alpha_4}$.

By an argument similar to that applied in section 4, it can be shown that there exist nonnegative constants α_5, B such that

$$||\delta T^{-1}(x,y;\eta)||\leq B||\eta|| \tag{59}$$

for all y in $(y^{(0)})_{\alpha_5}$ and all η in U. If α_6 is taken to be less than or equal inf (α_4,α_5), and such that $(y^{(0)})_{\alpha_6}$ contains only admissible arcs, the part of condition (I) previously proved and inequality (59) are still valid on $(y^{(0)})_{\alpha_6}$, that is, condition (I) holds on $(y^{(0)})_{\alpha_6}$.

The existence of nonnegative constants H, L, N, which along with the nonnegative constants B, M already determined in this section satisfy relation (21) is clear. By lemma 13 there also exist α_7 and L' such that

$$||\delta T(x,y^{(1)};y^{(1)}-y^{(2)})-T(x;y^{(1)})+T(x;y^{(2)})||\leq L'||y^{(1)}-y^{(2)}||^2$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_{\alpha_7}$. That is, condition (IV) of section 4 is valid on $(y^{(0)})_{\alpha_7}$. Thus we may choose $\alpha_8\leq\alpha_7$ such that conditions (I) and (IV) both hold on $(y^{(0)})_{\alpha_8}$. By continuity there exists α_9 such that $||T(x;y)||\leq D$ on $(y^{(0)})_{\alpha_9}$. Let C be such that $O<C<$inf (α_8,α_9). Then if the constant D is less than or equal to the smaller of N and $(1-H)C/B$, condition (III) is satisfied on $(y^{(0)})_{\alpha_9}$. Consequently, if the positive constant γ in theorem 3 is defined to be the inf (α_8,α_9), we may conclude by that theorem that Newton's method converges quadratically to a unique element $\bar{y}$ of $(y^{(0)})_\alpha$, $\alpha\leq\gamma$, which is such that (3) is satisfied. That

$$\bar{y}(a)=y^{(0)}(a), \qquad \bar{y}(b)=y^{(0)}(b)$$

follows at once from the fact that all the variations $\eta^{(i)}$ vanish at $x=a$ and $x=b$.

The arc $\bar{y}$ is of class C' by virtue of being in Y. To see that $\bar{y}$ is actually of class C^3 consider the equations in (x,v)

$$G_i(x,v)\equiv g_i(x,\bar{y}(x),v)-\int_a^x h_i(t,\bar{y}(t),\bar{y}'(t))dt-\frac{1}{b-a}\int_a^b\Big[g_i(x,\bar{y}(x),\bar{y}'(x))-\int_a^t h_i(s,\bar{y}(s),\bar{y}'(s))ds\Big]dt=0 \quad (i=1,2,\ldots,n). \tag{60}$$

By what has just been proved the point $(x,\bar{y}'(x))$ is a particular solution of (60) for each x on $a\leq x\leq b$. Since for each such x, $(x,\bar{y}(x),\bar{y}'(x))$ is an interior point of the region R in which g and h are of class C^2, $G(x,v)$ is of the same class in a neighborhood of each point $(x,\bar{y}'(x))$. By hypothesis the Jacobian

$$|G_{iv_j}|=|g_{iy_j'}|\neq 0$$

in the region R. Thus the implicit function theorem (see Bliss [2, p. 269]) is applicable. This theorem tells us that $\bar{y}'(x)$ is of the same class as $G(x,v)$, that is, $\bar{y}(x)$ is of class C^3.

The following theorem gives further conditions under which the conclusions of theorem 4 will be valid.

THEOREM 5. *Let y^* be an admissible arc with nonconjugate end points such that $T(x;y^*)=\theta$. Then there exists a constant $\alpha>0$ such that for any initial estimate $y^{(0)}$ taken from $(y^*)_\alpha$ the conclusions of theorem 4 are valid.*

By reasoning similar to that employed in the proof of theorem 4 there exists $\beta > 0$ such that conditions (I) and (II) and (IV) are satisfied on $(y^*)_\beta$. By continuity β may be made to have the additional property that $\|T(y)\| \leq N$ for all y in $(y^*)_\beta$. Let α be such that $0 < \alpha < \beta/2$. Then for any $y^{(0)}$ belonging to $(y^*)_\alpha$ it follows that $(y^{(0)})_{\beta/2}$ is contained in $(y^*)_\beta$. Thus (I), (II), (IV), and the inequality $\|T(y)\| \leq N$ hold on $(y^{(0)})_{\beta/2}$. Let C be such that $0 < C < \beta/2$. Then by continuity α may be taken so small that for all $y^{(0)}$ in $(y^*)_\alpha$ $\|T(y^{(0)})\| \leq (1-H)C/B$. This says condition (III) also holds on the $(\beta/2)$-neighborhood of any $y^{(0)}$ taken from $(y^*)_\alpha$. We now invoke theorem 3 and the remainder of the proof follows as in theorem 4.

If one wishes, he may hold $y^{(0)}$ fixed in $\delta T(x,y;\eta)$ throughout the entire process. Theorem 2 is applicable in this simplified case, and with its help all of the previous results can be shown to be true for the simplified Newton's method.

10. A Lemma Concerning Conjugate Points

It is easily seen that the equation

$$\delta T(x,y;\eta) = \int_a^x \left[\delta g_i(t,y;\eta) - \int_a^t \delta h_i(s,y;\eta)ds \right] dt - \frac{x-a}{b-a} \int_a^b \left[\delta g_i - \int_a^t \delta h_i ds \right] dt = 0 \qquad (i=1,2,\ldots,n) \tag{61}$$

and eq (40) of definition 5 are equivalent. Thus in the considerations of this section it will be assumed that (40) has been replaced by (61) in the definition of conjugate points.

If y is an arc such that $\delta T^{-1}(x,y;\eta)$, the inverse with respect to η of $\delta T(x,y;\eta)$, exists for all η in U, then y has nonconjugate end points, because the unique element of U which $\delta T(x,y;\eta)$ maps into θ is θ itself. In lemma 9 it was shown that if y has nonconjugate end points, $\delta T^{-1}(x,y;\eta)$ exists. Thus a necessary and sufficient condition for $\delta T^{-1}(x,y;\eta)$ to exist for all η in U is that y have nonconjugate end points. However, by lemma 3, if δT^{-1} exists at $y^{(0)}$ it must exist in some neighborhood of $y^{(0)}$. Therefore, the following lemma may be looked upon as a corollary of lemma 3.

LEMMA 14. *Let $y^{(0)}$ be an admissible element of Y having nonconjugate end points. Then there exists $\delta > 0$ such that no elements of $(y^{(0)})_\delta$ have conjugate end points.*

In this section we wish to give an alternate proof of lemma 14, which depends directly on the properties of functions with Lebesgue square integrable derivatives rather than on the Banach space theory of section 3.

To construct a proof of lemma 14 by contradiction, assume that there exists no neighborhood of $y^{(0)}$ in which all elements have nonconjugate end points. Then in every neighborhood of $y^{(0)}$ there exists at least one function y such that (61) has a nontrivial solution along y which vanishes at $x=a$ and $x=b$. Consequently, from consideration of the $1/q$-neighborhoods of $y^{(0)}$ for $q=1,2,\ldots$ one obtains the sequences $\{y^{(q)}\}$, $\{\eta^{(q)}\}$, which are such that

$$\left.\begin{aligned} &y^{(q)}(x) \to y^{(0)}(x) \quad \text{uniformly on } a \leq x \leq b, \quad (q=1,2,\ldots) \\ &y^{(q)\prime}(x) \to y^{(0)\prime}(x) \quad \text{uniformly on } a \leq x \leq b, \quad (q=1,2,\ldots) \end{aligned}\right\} \tag{62}$$

$$\left.\begin{aligned} &\delta T(x,y^{(q)};\eta^{(q)}) = \theta \qquad (q=1,2,\ldots) \\ &\eta^{(q)}(a) = \eta^{(q)}(b) = \theta \qquad (q=1,2,\ldots). \end{aligned}\right\} \tag{63}$$

Since all arcs η under consideration belong to the space U they must certainly belong to the class of the totality of arcs in $(x, \eta_1, \ldots, \eta_n)$-space defined by a set of n real valued functions

$$\eta_i(x) \qquad (a \leq x \leq b;\ i=1,2,\ldots,n)$$

that are absolutely continuous and have integrable square derivatives $\eta'(x)$ on $a \leq x \leq b$. Henceforth, we will think of the η's as belonging to this class.

Without loss of generality one can choose $\eta^{(q)}$ such that

$$\int_a^b \eta_i^{(q)\prime} \cdot \eta_i^{(q)\prime} dx = 1 \qquad (q=1,2,\ldots) \qquad (\text{no sum on } q). \tag{64}$$

Equations (64) imply the existence of a subsequence $\{\eta^{(q_j)\prime}\}$, which converges weakly in L^2 to some element $\eta^{(0)\prime}$. We can assume that this subsequence is the same as our original sequence. The weak convergence in L^2 of the sequence $\{\eta^{(q)\prime}\}$ implies

$$\begin{aligned} &\eta_i^{(q)} \to \eta_i^{(0)} \quad \text{uniformly on } a \leq x \leq b \qquad (i=1,2,\ldots,n) \\ &\eta_i^{(q)\prime} \to \eta_i^{(0)\prime} \quad \text{weakly in } L^2 \qquad (i=1,2,\ldots,n). \end{aligned} \tag{65}$$

It is also true that

$$\int_a^b \eta_i^{(0)\prime} \cdot \eta_i^{(0)\prime} dx \leq \liminf_{q\to\infty} \int_a^b \eta_i^{(q)\prime} \eta_i^{(q)\prime} dx \qquad (\text{no sum on } q). \tag{66}$$

A proof of this last fact has been given by McShane [4, p. 355].

Inequality (66) will be used in the proof of the following lemma.

LEMMA 15. *Let $\{\eta^{(q)\prime}\}$ be a sequence in L^2 such that (64) and (65) are satisfied and let $A_{ij}^{(q)}(x)$, $B_{ij}^{(q)}(x)$ $(i,j=1,2,\ldots,n)$ be continuous functions for almost all q such that*

$$A_{ij}^{(q)} \to A_{ij}^{(0)} \quad \text{uniformly on } a \leq x \leq b \qquad (i,j=1,2,\ldots,n) \tag{67}$$

$B_{ij}^{(q)} \to B_{ij}^{(0)}$ uniformly on $a \leq x \leq b$ $(i=1,2,\ldots,n)$

Then if

$$W_i^{(q)}(x) = \int_a^x (A_{ij}^{(q)} \eta_j^{(q)} + B_{ij}^{(q)} \eta_j^{(q)\prime})dt \quad (q=1,2,\ldots, \; i=1,2,\ldots,n) \quad \text{(no sum on } q),$$

it follows that

$$W_i^{(q)} \to W_i^{(0)} \quad \text{uniformly on } a \leq x \leq b \quad (i=1,2,\ldots,n)$$

$$(68)$$

$$W_i^{(q)\prime} \to W_i^{(0)\prime} \quad \text{weakly in } L^2 \quad (i=1,2,\ldots,n)$$

Consider

$$|W_i^q(x) - W_i^{(0)}(x)| =$$

$$\left|\int_a^x [(A_{ij}^{(q)}\eta_j^{(q)} - A_{ij}^{(0)}\eta_j^{(0)}) + (B_{ij}^{(q)}\eta_j^{(q)\prime} - B_{ij}^{(0)}\eta_j^{(0)\prime})]dt\right| \leq$$

$$\left|\int_a^x (A_{ij}^{(q)}\eta_j^{(q)} - A_{ij}^{(0)}\eta_j^{(q)})dt\right| + \left|\int_a^x (A_{ij}^{(0)}\eta_j^{(q)} - A_{ij}^{(0)}\eta_j^{(0)})dt\right| +$$

$$\left|\int_a^x (B_{ij}^{(q)}\eta_j^{(q)\prime} - B_{ij}^{(0)}\eta_j^{(q)\prime})dt\right| + \left|\int_a^x (B_{ij}^{(0)}\eta_j^{(q)\prime} - B_{ij}^{(0)}\eta_j^{(0)\prime})dt\right|.$$

The first of the four terms on the right of the sign of inequality satisfies

$$\left|\int_a^x (A_{ij}^{(q)}\eta_j^{(q)} - A_{ij}^{(0)}\eta_j^{(q)})dt\right| \leq \int_a^b |A_{ij}^{(q)} - A_{ij}^{(0)}|\cdot|\eta_j^{(q)}|dt$$

$$\leq (b-a)\max|A_{ij}^{(q)} - A_{ij}^{(0)}|\cdot|\eta_j^{(q)}|.$$

By (67) $\max|A_{ij}^{(q)} - A_{ij}^{(0)}| \to 0$ for each $(i,j=1,2,\ldots,n)$. By (65) $|\eta_j^{(q)}|$ is uniformly bounded for all j, q under consideration. Thus the first term goes uniformly to zero on $a \leq x \leq b$. The second term on the right satisfies

$$\left|\int_a^x (A_{ij}^{(0)}\eta_j^{(q)} - A_{ij}^{(0)}\eta_j^{(0)})dt\right| \leq \int_a^b |A_{ij}^{(0)}|\cdot|\eta_j^{(q)} - \eta_j^{(0)}|dt$$

$$\leq \max|A_{ij}^{(0)}| \int_a^b |\eta_j^{(q)} - \eta_j^{(0)}|dt.$$

This goes uniformly to zero by (65). For the third term we have

$$\left|\int_a^x (B_{ij}^{(q)}\eta_j^{(q)\prime} - B_{ij}^{(0)}\eta_j^{(q)\prime})dt\right| \leq \int_a^b |B_{ij}^{(q)} - B_{ij}^{(0)}|\cdot|\eta_j^{(q)\prime}|dt$$

$$\leq \max|B_{ij}^{(q)} - B_{ij}^{(0)}| \int_a^b |\eta_j^{(q)\prime}|dt.$$

By the Schwartz inequality and (64)

$$\left(\int_a^b |\eta_j^{(q)\prime}|dt\right)^2 \leq (b-a)\int_a^b \eta_j^{(q)\prime 2}dt \leq (b-a) \quad (q=1,2,\ldots).$$

Thus $\int_a^b |\eta_j^{(q)\prime}|dt$ is uniformly bounded for $(j=1,2,\ldots,n)$ $(q=1,2,\ldots)$. Consequently, one may conclude from (67) that the third term goes uniformly to zero on $a \leq x \leq b$.

Only the fourth term remains to be discussed. Let j be temporarily held fixed. As is well known there exists for each arbitrary $\epsilon > 0$ a polynomial $P(x)$ such that

$$\int_a^b (B_{ij}^{(0)}(x) - P(x))^2 dx \leq \frac{\epsilon^2}{16}. \tag{69}$$

By (64), (66), and the Minkowski inequality it is true that

$$\left(\int_a^b [\eta_j^{(q)\prime} - \eta_j^{(0)\prime}]^2 dx\right)^{\frac{1}{2}} \leq$$

$$\left(\int_a^b (\eta_j^{(q)\prime})^2 dx\right)^{\frac{1}{2}} + \left(\int_a^b (\eta_j^{(0)\prime})^2 dx\right)^{\frac{1}{2}} \leq 2. \tag{70}$$

Thus by use of Schwarz' inequality, (69), and (70) one obtains

$$\left|\int_a^x B_{ij}^{(0)}(\eta_j^{(q)} - \eta_j^{(0)})' dt - \int_a^x P(t)(\eta_j^{(q)} - \eta_j^{(0)})' dt\right|$$

$$\leq \left(\int_a^x (B_{ij}^{(0)} - P)^2 dt\right)^{\frac{1}{2}} \left(\int_a^x (\eta_j^{(q)\prime} - \eta_j^{(0)\prime})^2 dt\right)^{\frac{1}{2}} \tag{71}$$

$$\leq \left(\int_a^b (B_{ij}^{(0)} - P)^2 dt\right)^{\frac{1}{2}} \left(\int_a^b (\eta_j^{(q)\prime} - \eta_j^{(0)\prime}) dt\right)^{\frac{1}{2}} \leq \frac{\epsilon}{2}$$

for all x on $a \leq x \leq b$. Integration by parts yields

$$\int_a^x P(t)(\eta_j^{(q)} - \eta_j^{(0)})' dt$$

$$= P(x)(\eta_j^{(q)} - \eta_j^{(0)}) - \int_a^x P'(t)(\eta_j^{(q)} - \eta_j^{(0)}) dt.$$

Therefore, it follows from the uniform convergence of $(\eta_j^{(q)} - \eta_j^{(0)})$ to zero that

$$\left|\int_a^x P(t)(\eta^{(q)\prime} - \eta^{(0)\prime}) dt\right| \leq \frac{\epsilon}{2} \quad \text{uniformly on } a \leq x \leq b \tag{72}$$

for q sufficiently large. Formulas (71) and (72) imply that

$$\left|\int_a^x B_{ij}^{(0)}(\eta_j^{(q)\prime} - \eta_j^{(0)\prime}) dt\right| \to 0 \quad \text{uniformly on } a \leq x \leq b.$$

This completes the proof of the first statement of (68).

Let l be any element of L^2 and consider

$$\left|\int_a^b W^{(q)\prime} l\,dx - \int_a^b W^{(0)\prime} l\,dx\right| \leq$$

$$\left|\int_a^b (A_{ij}^{(q)}\eta_j^{(q)} - A_{ij}^{(0)}\eta_j^{(0)})l\,dx\right| + \left|\int_a^b (B_{ij}^{(q)}\eta_j^{(q)\prime} - B_{ij}^{(0)}\eta_j^{(0)\prime})l\,dx\right|$$

$$\leq \max|l| \left\{\int_a^b |A_{ij}^{(q)}\eta_j^{(q)} - A_{ij}^{(0)}\eta_j^{(0)}|dx + \int_a^b |B_{ij}^{(q)}\eta_j^{(q)\prime} - B_{ij}^{(0)}\eta_j^{(q)\prime}|dx\right\} + \left|\int_a^b lB_{ij}^{(0)}(\eta_j^{(q)\prime} - \eta_j^{(0)\prime})\,dx\right|.$$

By a repetition of the arguments already given above the terms multiplied by $\max |l|$ go to zero. The factor l may be lumped with $B_{ij}^{(0)}$ without changing the reasoning used in analyzing term four. Thus the second statement of (68) is seen to be true.

The following corollary is a direct result of lemma 15.

Corollary 1. *Let the hypothesis of lemma 15 hold and let*

$$U_i^{(q)}(x) \equiv W_i^{(q)}(x) - \frac{x-a}{b-a} W_i^{(q)}(b) \tag{73}$$

$$V_i^{(q)}(x) \equiv \int_a^x W_i^{(q)}(t)dt - \frac{x-a}{b-a}\int_a^b W_i^{(q)}(t)dt. \tag{74}$$

Then,

$U_i^{(q)} \to U_i^{(0)}$ uniformly on $a \leq x \leq b$ $(i=1,2,\ldots,n)$

$V_i^{(q)} \to V_i^{(0)}$ uniformly on $a \leq x \leq b$ $(i=1,2,\ldots,n)$

$U_i^{(q)\prime} \to U_i^{(0)\prime}$ weakly in L^2 $(i=1,2,\ldots,n)$

$V_i^{(q)\prime} \to V_i^{(0)\prime}$ uniformly on $a \leq x \leq b$ $(i=1,2,\ldots,n)$.

Let

$$u_i(x,y;\eta) \equiv \int_a^x \delta g_i dt - \frac{x-a}{b-a}\int_a^b \delta g_i dt \quad (i=1,2,\ldots,n)$$

$$v_i(x,y;\eta) \equiv \int_a^x \left(\int_a^t \delta h_i ds\right) dt - \frac{x-a}{b-a}\int_a^b \left(\int_a^t \delta h_i ds\right) dt \quad (i=1,2,\ldots,n).$$

Then u and v are elements of U satisfying

$$\delta T_i(x,y;\eta) = u_i - v_i \qquad (i=1,2,\ldots,n). \tag{75}$$

Since $\int_a^x \delta g_i(t,y^{(q)};\eta^{(q)})dt$ and $\int_a^x \delta h_i(t,y^{(q)};\eta^{(q)})dt$ are functions of the type $W_i^{(q)}(x)$ defined in lemma 15 for each admissible $y^{(q)}$ in Y and $\eta^{(q)\prime}$ in L^2, $u_i(x,y^{(q)};\eta^{(q)})$ and $v_i(x,y^{(q)};\eta^{(q)})$ are respectively of the types (73) and (74). Consequently, if we take $\{y^{(q)}\}$ and $\{\eta^{(q)}\}$ to be the sequences which result from contradicting lemma 14, the hypotheses of lemma 15 will be fulfilled because of (62) and (65), and corollary 1 will then imply

$$u_i(x,y^{(q)};\eta^{(q)}) \to u_i(x,y^{(0)};\eta^{(0)}) \text{ uniformly on } a \leq x \leq b \quad (i=1,2,\ldots,n) \tag{76}$$

$$v_i(x,y^{(q)};\eta^{(q)}) \to v_i(x,y^{(0)};\eta^{(0)}) \text{ uniformly on } a \leq x \leq b \quad (i=1,2,\ldots,n) \tag{77}$$

$$u_i'(x,y^{(q)};\eta^{(q)}) \to u_i'(x,y^{(0)};\eta^{(0)}) \text{ weakly in } L^2 \quad (i=1,2,\ldots,n) \tag{78}$$

$$v_i'(x,y^{(q)};\eta^{(q)}) \to v_i'(x,y^{(0)};\eta^{(0)}) \text{ uniformly on } a \leq x \leq b \quad (i=1,2,\ldots,n). \tag{79}$$

In view of (63), formulas (76) and (77) yield the result

$$\delta T(x,y^{(0)};\eta^{(0)}) = \theta. \tag{80}$$

The limit function $\eta^{(0)}(x)$ is clearly a continuous function on $a \leq x \leq b$ satisfying

$$\eta^{(0)}(a) = \eta^{(0)}(b) = \theta.$$

To show that $\eta^{(0)}$ is of class c' that is $\eta^{(0)}$ is in U, observe that by differentiation of (80)

$$g_{iy_j}\eta_j^{(0)} + g_{iy_j'}\eta_j^{(0)\prime} = \int_a^x \delta h_i dt + \frac{1}{b-a}\int_a^b \left[\delta g_i - \int_a^t \delta h_i ds\right] dt. \tag{81}$$

Since $|g_{iy_j'}| \neq 0$, (81) can be solved for $\eta^{(0)\prime}$. As all other quantities involved in (81) are continuous, $\eta^{(0)\prime}$ must have the same property.

By the hypothesis of nonconjugate end points the only element of U satisfying (80) is the zero element. Therefore $\eta^{(0)} \equiv \theta$ and (78) and (79) may be amended to read

$$u_i'(x,y^{(q)};\eta^{(q)}) \to u_i'(x,y^{(0)};\eta^{(0)}) = 0 \text{ weakly in } L^2 \quad (i=1,2,\ldots,n). \tag{82}$$

$$v_i'(x,y^{(q)};\eta^{(q)}) \to v_i'(x,y^{(0)};\eta^{(0)}) = 0 \text{ uniformly on } a \leq x \leq b \quad (i=1,2,\ldots,n). \tag{83}$$

Consider

$$\lim_{q\to\infty}\int_a^b \delta T_i'(x,y^{(q)};\eta^{(q)})\delta T_i'(x,y^{(q)};\eta^{(q)})dx$$

$$= \lim_{q\to\infty}\int_a^b u_i^{(q)\prime}u_i^{(q)\prime} - 2u_i^{(q)\prime}v_i^{(q)\prime} + v_i^{(q)\prime}v_i^{(q)\prime})dx. \tag{84}$$

Although we have not previously defined all the notations in (84) their meanings are obvious. By (82) and (83) the terms involving v_i' may be disregarded. By lemma 15

$$\lim_{q\to\infty}\int_a^b u_i^{(q)\prime} u_i^{(q)\prime}\,dx=$$

$$\lim_{q\to\infty}\left[\int_a^b \delta g_i^{(q)}\,\delta g_i^{(q)}\,dx-\frac{1}{b-a}\left(\int_a^b \delta g_i^{(q)}\,dx\right)\left(\int_a^b \delta g_i^{(q)}\,dx\right)\right]$$
$$=\lim_{q\to\infty}\int_a^b \delta g_i^{(q)}\,\delta g_i^{(q)}\,dx.$$

Multiplying out, one obtains after dropping the superscript (q) from the derivatives of g_i.

$$\lim_{q\to\infty}\int_a^b \delta g_i \delta g_i dx$$
$$=\lim_{q\to\infty}\int_a^b (g_{iy_j}\eta_i^{(q)}\, g_{iy_k}\eta_k^{(q)}+2\,g_{iy_j}\eta_i^{(q)}\, g_{iy'_k}\eta_k^{(q)\prime}+$$
$$g_{iy'_j}\eta_i^{(q)\prime}\, g_{iy'_k}\eta_k^{(q)\prime})dx$$
$$=\lim_{q\to\infty}\int_a^b g_{iy'_j} g_{iy_k}\eta_i^{(q)\prime}\,\eta_k^{(q)\prime}\,dx\equiv\lim_{q\to\infty}\int_a^b G_{jk}^{(q)}\,\eta_i^{(q)\prime}\,\eta_k^{(q)\prime}\,dx$$

since all terms involving $\eta^{(q)}$ go to zero.

Let $(\overbrace{A_{ij}})$ denote the transpose of the matrix (A_{ij}). Then (since by hypothesis $\det\left(g_{iy'_j}^{(q)}\right)\neq 0$).

$$\det(G_{jk}^{(q)})=\det\left(g_{iy'_j}^{(q)}\right)\det\left(\overbrace{g_{iy'_k}^{(q)}}\right)=\left[\det\left(g_{iy'_j}^{(q)}\right)\right]^2>0.$$

Hence $G_{jk}^{(q)}$ is a real, symmetric, positive definite matrix for each $q=1,2,\ldots$. Therefore, it follows that

$$\lim_{q\to\infty}\int_a^b \delta T_i'(x,y^{(q)};\eta^{(q)})\delta T_i'(x,y^{(q)};\eta^{(q)})dx=$$
$$\lim_{q\to\infty}\int_a^b G_{jk}^{(q)}\,\eta_j^{(q)\prime}\,\eta_k^{(q)\prime}\,dx\geq C>0.$$

However, (63) implies that this last limit is identically zero. Consequently, our assumption leads to a contradiction. Therefore there must exist at least one neighborhood of $y^{(0)}$ in which all elements have nonconjugate end points.

11. A Solution of the Fixed End Point Problem

To conclude the discussion of Newton's method, we wish to point out that with certain additional hypothesis we can make use of our previous results to obtain a solution of the fixed end point problem of the Calculus of Variations.

Let $f(x,y,y')$ be the integrand of the integral (1). A *strengthened Legendre condition* is said to hold in the region R if

$$f_{y'_i y'_j}(x,y,y')\pi_i\pi_j>0 \tag{85}$$

for all (x,y,y') in R and all $(\pi)\neq 0$. This implies that the determinant $|f_{y_i y_j}|\neq 0$ in R and is thus stronger than our previous condition that the determinant $|g_{iy'_j}|$ never vanishes in R.

The following terminology is well known and will be helpful. An arc y is said to satisfy a *strengthened Jacobi condition* if it has nonconjugate end points and has no subarc defined on $a\leq x\leq\zeta<b$ which has conjugate end points.

Let (2) be replaced by the Euler equation (13). Let the region R be convex in y' and let the strengthened Legendre condition hold everywhere in R. The following theorem is then true.

THEOREM 6. *Let $y^{(0)}$ be an admissible element of Y having nonconjugate end points. Then if there exists a sufficiently small constant $D\geq 0$ such that $||T(x,y^{(0)})||<D$, the sequence of estimates derived from Newton's method will converge quadratically to a unique arc $\bar{y}$ of class c^3 which lies within a predetermined neighborhood of $y^{(0)}$, joins the end joints of $y^{(0)}$, and satisfies the Euler equations. Moreover, if $\bar{y}$ is such that no interior point of $a\leq x\leq b$ is conjugate to $x=a$, $\bar{y}$ will afford a strong relative minimum to $I(y)$.*

By theorem 4, Newton's method converges quadratically to an arc $\bar{y}$ of class c^3, which joins the end points of $y^{(0)}$ and satisfies the Euler equations. The arc $\bar{y}$ also has the property of nonconjugate end points. This follows from lemma 14, which implies that the constant α_8 mentioned in the proof of theorem 4 can be chosen to have the additional property that no element of $(y^{(0)})_{\alpha_8}$ has conjugate end points. Since by hypothesis no subarc of $\bar{y}$ with initial point at $x=a$ has conjugate end points, a strengthened Jacobi condition is satisfied along $\bar{y}$. As it has also been assumed that a strengthened Legendre condition holds throughout R and that R is convex in y', we may conclude (see Bliss [2, p. 42, corollary 16.1]) that $\bar{y}$ affords a strong relative minimum to $I(y)$.

The following theorem gives further criteria that insure the convergence of the sequence of estimates derived from Newton's method to a solution of the fixed end point problem.

THEOREM 7. *Let a minimum y^* exist and have nonconjugate end points. Then there exists a constant $\alpha>0$ such that for any initial estimate $y^{(0)}$ taken from $(y^*)_\alpha$ the conclusions of theorem 6 hold.*

Upon recalling that a necessary condition for y^* to be a minimum is $T(x;y^*)=\theta$, one proceeds with the proof just as in theorem 5.

12. A Second Method for the Effective Solution of the Fixed End Point Problem

In section 2 it was remarked that in the application of Newton's method, one must solve the Euler equations of each term of a sequence of integrals whose integrands are quadratic in y and y'. Thus it is not inappropriate to introduce at this point a second method for solving fixed end point problems which will prove to be particularly applicable to the case of a quadratic integrand. In an actual computation a combination of Newton's method, and the method to be presented below, might prove to be advantageous.

Henceforth the symbol Y will denote the class of n-tuples of real valued functions on $a \leq x \leq b$ which have Lebesgue square integrable derivatives. Use was made of this class of functions in section 10. The symbol U will again denote the subclass of elements of Y, which vanish at the end points of $a \leq x \leq b$. Throughout the remainder of the paper we assume that all functions of Y are admissible and that the underlying region R is convex in y and y'. Our previous assumptions as to the nature of the integrand $f(x,y,y')$ are retained. Moreover, we assume that f and its derivatives are integrable along arcs in Y.

Assume that after starting from some initial estimate $y^{(0)}$ in Y one has obtained an "improved" estimate of a solution to the Euler equations $y^{(i)}$ also in Y. He then chooses the $i+1$st estimate to be

$$y^{(i+1)}=y^{(i)}-\alpha_i\eta^{(i)} \qquad \text{(no sum on } i\text{)}, \tag{86}$$

where $\eta^{(i)}$ is an element of U determined by the equation

$$\eta^{(i)}=T(x;y^{(i)}). \tag{87}$$

Here T is formed for the Euler equations as illustrated in section 2, and α_i is a real number.

A procedure similar to that just described has been announced by L. V. Kantorovitch [5] for the case of an integral whose integrand is quadratic in y and y'. However, his paper presents no proofs. Curry [6] has described an analogous process that can be used to minimize a function of n real variables. Other references may be found in the last paper cited.

In order to consider the convergence of the sequence defined by (86) it is helpful to examine the expression $I(y)-I(y-\alpha\eta)$. We may state the following lemma.

LEMMA 16. *Let y be an element of Y and let η be given by (87). Then,*

$$I(y)-I(y-\alpha\eta)=\alpha\int_a^b \eta_i'\eta_i' dx-\alpha^2\int_a^b W dx \tag{88}$$

where

$$W\equiv A_{ij}\eta_i\eta_j+2B_{ij}\eta_i\eta_j'+C_{ij}\eta_i'\eta_j'$$

$$A_{ij}=\int_0^1 (1-\mu)f_{y_iy_j}(x,y-\alpha\mu\eta,y'-\alpha\mu\eta')d\mu$$

and B_{ij}, C_{ij} are obtained from the formula for A_{ij} by replacing $f_{y_iy_j}$ by $f_{y_iy_j'}$ and $f_{y_i'y_j'}$ respectively.

An application of Taylor's theorem results in

$$I(y)-I(y=\alpha\eta)$$

$$=\alpha\int_a^b [f_{y_i}(x,y,y')\eta_i+f_{y_i'}(x,y,y')\eta_i']dx-\alpha^2\int_a^b W dx.$$

By integration by parts

$$\int_a^b f_{y_i}\eta_i dx=-\int_a^b\left[\int_a^x f_{y_i}dt+C_i\right]\eta_i' dx;$$

and since

$$\int_a^b C_i\eta_i' dx=C_i[\eta_i(b)-\eta_i(a)]=0,$$

the choice of C_i is arbitrary. Accordingly, we choose

$$C_i=\frac{1}{b-a}\int_a^b\left[f_{y_i}(t,y,y')-\int_a^t f_{y_i}(s,y,y')ds\right]dt.$$

Thus one may write

$$I(y)-I(y-\alpha\eta)=\alpha\int_a^b T_i'\eta_i' dx-\alpha_2\int_a^b W dx$$

$$=\alpha\int_a^b \eta_i'\eta_i' dx-\alpha^2\int_a^b W dx.$$

The next lemma follows directly from formula (88).

LEMMA 17. *Assume that for every number ν there exists a number $K>0$ and a number α^* such that if $I(y)\leq\nu$ and $0<\alpha\leq\alpha^*$ then*

$$\int_a^b W dx\leq K\int_a^b \eta_i'\eta_i' dx. \tag{89}$$

Then in case $I(y)\leq\nu$ and $0<\alpha\leq\alpha^*$ one has the relation

$$I(y)-I(y-\alpha\eta)\geq\alpha(1-\alpha K)\int_a^b \eta_i'\eta_i' dx. \tag{90}$$

Formula (90) helps to prove the following theorem.

THEOREM 8. *Let there exist finite numbers ν, $K>0$, α^* such that if $I(y)\leq\nu$ and $0<\alpha\leq\alpha^*$, formula (89) holds. Let $y^{(0)}$ be such that $I(y^{(0)})\leq\nu$ and let α and $\bar{\alpha}$ be chosen so that $0<\bar{\alpha}\leq\alpha<1/K$. Then if $I(y)$ is bounded below on the class of arcs in Y joining the end points of $y^{(0)}$, the sequence of estimates $\{y^{(i)}\}$ defined by (86) and (87) is such that $\{I(y^{(i)})\}$ is convergent. Furthermore,*

$$T(x;y^{(i)})\rightarrow\theta \quad \text{uniformly on } a\leq x\leq b \tag{91}$$

$$T'(x,y^{(i)})\rightarrow\theta \quad \text{in } L^2. \tag{92}$$

That α may be chosen as specified in the hypothesis is a consequence of the fact that K is finite. As a result of such a choice of α, one obtains the inequality

$$I(y^{(0)})>I(y^{(0)}-\alpha_0\eta^{(0)})\equiv I(y(^{(1)})$$

from (90) by setting $y\equiv y^{(0)}$, $\alpha\equiv\alpha_0$. Since $I(y^{(1)})<\nu$, the argument can be repeated provided $\eta^{(i)}\not\equiv\theta$. Hence a sequence $\{y^{(i)}\}$ such that

$$I(y^{(i)})>I(y^{(i+1)}) \qquad (i=0,\ 1,2,\ \ldots)$$

is constructed. Due to the hypothesis that $I(y)$ is bounded below the sequence of definite integrals converges.

By (90) and the way $\{\alpha_i\}$ is chosen

$$I(y^{(i)})-I(y^{(i+1)})\geq\alpha_1(1-\alpha_i K)\int_a^b \eta_i^{(i)\prime}\eta_i^{(i)\prime}dx$$
$$\geq\bar{\alpha}(1-K\sup\alpha_i)\int_a^b \eta_i^{(i)\prime}\eta_i^{(i)\prime}dx\geq 0, \tag{93}$$

where here and in similar expressions that occur below there is no sum on the index that denotes which step of the iteration process we are considering. As the left-hand side of (93) can be made as small as desired by choosing i sufficiently large, we may conclude that

$$\lim_{i\to\infty}\left(\int_a^b \eta_i^{(i)\prime}\eta_i^{(i)\prime}dx\right)^{\frac{1}{2}}=0\cdot \tag{94}$$

Conclusion (92) follows at once from (94). Since (91) is a consequence of (92), the theorem is proved.

The problem of determining general conditions under which an assumption of the type made in lemma 17 is valid seems to be a difficult one. However, special cases may be considered in which K is independent of ν and α^* is arbitrary. A rather trivial example of this occurs when the integrand is given by

$$f(x,y,y')=g(x)\sqrt{1+y'^2}, \qquad g(x)\geq 0.$$

If $2K\geq|g(x)|$ on $a\leq x\leq b$, we have

$$\int_a^b W dx=\int_a^b\int_0^1(1-\mu)\,\frac{g(x)\eta'^2}{[1+(y'-\alpha\mu\eta')^2]^{\frac{3}{2}}}\,d\mu dx$$
$$\leq K\int_a^b \eta'^2 dx\cdot$$

A much more important case is that in which the integrand is quadratic, that is,

$$f(x,y,y')=A_{ij}(x)y_iy_j+2B_{ij}(x)y_iy_j'+C_{ij}(x)y_i'y_j'+D_i(x)y_i+E_i(x)y_i'+F_i(x). \tag{95}$$

For $f(x,y,y')$ given by (95) we obtain upon integrating out μ

$$\int_a^b W dx=\int_a^b[A_{ij}(x)\eta_i\eta_j+B_{ij}(x)\eta_i\eta'_j+C_{ij}(x)\eta_i'\eta_j']dx$$
$$\leq K\int_a^b \eta_i'\eta_i'dx$$

for a suitably chosen constant K provided $\eta_i(a)=\eta_i(b)=0$. This last result follows from the Schwarz inequality and the fact that the integrand is a quadratic form in η and η' having continuous coefficients.

By setting $y\equiv y^{(0)}$, $\eta\equiv\bar{y}-y^{(0)}$, $\alpha\equiv-1$ in the computations employed to establish lemma 16, one obtains the formula

$$I(\bar{y})=I(y^{(0)})+\int_a^b T_i'(x;y^{(0)})\eta_i'dx+\int_a^b W dx, \tag{96}$$

which is true for all $\bar{y}$ in Y which join the end points of $y^{(0)}$. This equation is helpful in the proofs to be given below.

Lemma 18. *Let there exist a constant $k>0$ such that*

$$\int_a^b W dx\geq k\int_a^b \eta_i'\eta_i'dx \tag{97}$$

for all η in U and all y in Y. Then $I(y)$ is bounded below on the class of elements of Y joining the end points of some arbitrary fixed $y^{(0)}$ in Y. Furthermore, given a constant ν there exists a constant M such that for each y in Y joining the end points of $y^{(0)}$ for which $I(y)\leq\nu$

$$\int_a^b \eta_i'\eta_i'dx\leq M. \tag{98}$$

In (98) we have $\eta\equiv y-y^{(0)}$.

By Schwarz' inequality

$$\int_a^b T_i'(x;y^{(0)})\eta_i'\leq$$
$$\left(\int_a^b T_i'(x;y^{(0)})T_i'(x;y^{(0)})dx\right)^{\frac{1}{2}}\left(\int_a^b \eta_i'\eta_i'dx\right)^{\frac{1}{2}}$$
$$\equiv h\left(\int_a^b \eta_i'\eta_i'dx\right)^{\frac{1}{2}}\cdot$$

Thus by (96) and (97)

$$I(y)\geq I(y^{(0)})-h\left(\int_a^b \eta_i'\eta_i'dx\right)^{\frac{1}{2}}+k\int_a^b \eta_i'\eta_i'dx$$
$$=I(y^{(0)})-\frac{h^2}{4k}+\left[\frac{h}{2\sqrt{k}}-\left(k\int_a^b \eta_i'\eta_i'dx\right)^{\frac{1}{2}}\right]^2$$
$$\geq I(y^{(0)})-\frac{h^2}{4k},$$

provided y joins the end points of $y^{(0)}$. This proves that $I(y)$ is bounded below on the class of functions joining the end points of $y^{(0)}$. The remainder of the lemma is also a ready consequence of this last inequality.

Condition (97) does not seem to be very widely applicable. However, in case the integrand is quadratic the strengthened Jacobi condition in conjunction with the strengthened Legendre condition implies the existence of a constant $k>0$ such that (97) holds.

Lemma 19. *Let $\{y^{(q)}\}$ be a sequence in Y such that*

$$y_i^{(q)}\to\bar{y}_i \text{ uniformly on } a\leq x\leq b, \quad (i=1,2,\ldots,n) \tag{99}$$

$$y_i^{(q)\prime}\to\bar{y}_i' \text{ almost uniformly} \quad (i=1,2,\ldots,n) \tag{100}$$

$$T_i'(x;y^{(q)})\to 0 \text{ almost uniformly} \quad (i=1,2,\ldots,n). \tag{101}$$

Then

$$T_i'(x;\bar{y})=0 \quad \textit{almost everywhere} \quad (i=1,2,\ldots,n). \tag{102}$$

Let $\boldsymbol{I}$ denote the interval $a \leq x \leq b$. Since $\bar{y}'$ is in L^2 it is summable and hence finite almost everywhere. Hence given $\epsilon > 0$ there exists by (100) a set S_1 of measure less than $\epsilon/2$ such that $y_i^{(q)\prime} \to \bar{y}_i'$ uniformly on $\boldsymbol{I} - S_1$ for $(i = 1, 2, \ldots n)$. Of course, S_1 must contain all the infinites of $\bar{y}'$. The summability of $\bar{y}'$ also implies the existence of a set S_2 of measure less than $\epsilon/4$ such that $\bar{y}_i'$ is bounded on $\boldsymbol{I} - S_2$ for $(i = 1, 2, \ldots, n)$. Therefore, if S_3 is an open set of measure less than ϵ containing $S_1 + S_2$, the following statements are valid:

(a) $\boldsymbol{I} - S_3$ is closed and bounded,
(b) $\bar{y}_i'$ is bounded on $\boldsymbol{I} - S_3$ $(i = 1, 2, \ldots, n)$,
(c) $y_i^{(q)} \to \bar{y}_i$ uniformly on $\boldsymbol{I} - S_3$ $(i = 1, 2, \ldots, n)$,
(d) $y_i^{(q)\prime} \to \bar{y}_i'$ uniformly on $\boldsymbol{I} - S_3$ $(i = 1, 2, \ldots, n)$.

By (a) through (d) it can be concluded that

$$f_{y_i'}(x, y^{(q)}(y^{(q)\prime}) \to f_{y_i'}(x, \bar{y}, \bar{y}) \quad \text{uniformly on } \boldsymbol{I} - S_3$$

and similarly for f_{y_i}. Hence it readily is verified that $T_i'(x; y^{(q)}) \to T_i'(x, \bar{y})$ uniformly on $\boldsymbol{I} - S_3$ for $(i = 1, 2, \ldots, n)$, that is, almost uniformly and thus almost everywhere. Conclusion (102) is then obtainable from (101).

If one wishes, he may express (102) in the form $T(x; \bar{y}) = \theta$. This follows directly by integration.

Lemma 20. *Let $\bar{y}$ be an element of Y satisfying (102). Then if there exists $k > 0$ such that (97) holds, $\bar{y}$ affords a unique minimum to $I(y)$ on the class of functions in Y which join the end points of $\bar{y}$.*

Let y be any element of Y joining the end points of $\bar{y}$. By (96)

$$I(y) - I(\bar{y}) = \int_a^b T_i'(x; \bar{y})(y_i' - \bar{y}_i')\,dx + \int_a^b W\,dx.$$

By hypothesis $T_i'(x; \bar{y}) = 0$ almost everywhere and $y_i' - \bar{y}_i'$ is in L^2. Thus the first integral on the right is zero by Hölder's inequality. Therefore, if $y \neq \bar{y}$,

$$I(y) - I(\bar{y}) = \int_a^b W\,dx \geq k \int_a^b (y_i' - \bar{y}_i')(y_i' - \bar{y}_i')\,dx > 0.$$

Consequently,

$$I(y) > I(\bar{y}). \tag{103}$$

Assume that y also satisfies (102). Then by reversing the roles of y and $\bar{y}$ one obtains $I(\bar{y}) > I(y)$. This contradicts (103), and so implies that the assumption that $y \neq \bar{y}$ satisfies (102) is false. Hence y cannot be a minimum.

Theorem 9. *Let there exist finite numbers ν, $K > 0$, α^* such that if $I(y) \leq \nu$ and $0 < \alpha < \alpha^* < 1/K$, formula (89) holds. Let $y^{(0)}$ be such that $I(y^{(0)}) \leq \nu$ and let α and $\bar{\alpha}$ be chosen so that $0 < \bar{\alpha} \leq \alpha < \alpha^*$. Then if there exists a constant $k > 0$ such that (97) is satisfied, the conclusions of theorem 8 are satisfied. Furthermore, there exists a function $\bar{y}$ such that for the sequence $\{y^{(i)}\}$ described in theorem 8 we have*

$$y^{(i)} \to \bar{y} \quad \text{uniformly on } a \leq x \leq b$$

$$y^{(i)\prime} \to \bar{y}' \quad \text{in the mean of order 2.}$$

The limit $\bar{y}$ affords a unique minimum to $I(y)$ on the class of functions in Y joining the end points of $y^{(0)}$.

By lemma 18, $I(y)$ is bounded below on the class of y in Y joining the end points of $y^{(0)}$. Since all of the other hypotheses of theorem 8 have been assumed, the conclusions of that theorem must be valid.

As the sequence $\{y^{(i)}\}$ of theorem 8 is such that each $y^{(i)}$ joins the end points of $y^{(0)}$ and

$$I(y^{(i)}) < I(y^{(0)}) \leq \nu \qquad (i = 1, 2, \ldots),$$

formula (98) of lemma 18 is applicable to each term. This formula implies the existence of $\overline{M} \geq 0$ such that for all n, m

$$\left(\int_a^b \eta_i^{(n,m)\prime} \eta_i^{(n,m)\prime}\,dx\right)^{\frac{1}{2}} \leq \overline{M}, \tag{104}$$

where $\eta^{(n,m)} \equiv y^{(n)} - y^{(m)}$. For $n > m$, $I(y^{(n)}) - I(y^{(m)})$ is negative. Thus from substitution in the main formula of the proof of lemma 18 it follows that

$$0 \leq k \int_a^b \eta_i^{(n,m)\prime} \eta_i^{(n,m)\prime}\,dx \leq h_m \left(\int_a^b \eta_i^{(n,m)\prime} \eta_i^{(n,m)\prime}\,dx\right)^{\frac{1}{2}}, \qquad n > m,$$

where

$$h_m \equiv \left(\int_a^b T_i'(x; y^{(m)})\, T_i'(x; y^{(m)})\,dx\right)^{\frac{1}{2}} \to 0, \qquad m \to \infty.$$

Consequently, the existence of $\bar{y}$ such that $y^{(i)\prime} \to \bar{y}'$ in the mean of order 2 follows from 104. This last result implies $y^{(i)} \to \bar{y}$ uniformly on $a \leq x \leq b$. By recalling (92) and taking subsequences one finally obtains a subsequence $\{y^{(q)}\}$ such that

$$y^{(q)} \to \bar{y} \quad \text{uniformly on } a \leq x \leq b$$

$$y^{(q)\prime} \to \bar{y}' \quad \text{almost uniformly}$$

$$T'(x; y^{(q)}) \to \theta \quad \text{almost uniformly.}$$

Consequently (by lemma 19) $\bar{y}$ satisfies (102). That $\bar{y}$ is unique is a consequence of lemma 20. Hence the theorem is proved. It is not difficult to show that $\bar{y}(x)$ is of class c^2.

It is clear that these results apply to the case when f is a polynomial of degree two in y_i'.

13. An Example of Numerical Computation

Consider the problem of minimizing the integral

$$I(y) = \int_{x_1}^{x_2} y\sqrt{1 + y'^2}\,dx, \qquad y > 0,$$

whose Euler equation is

$$y''=\frac{1+y'^2}{y}. \tag{105}$$

To apply Newton's method write (105) as

$$T=1+y'^2-yy''=0.$$

Then

$$\delta T=2y'\eta'-y''\eta-y\eta'',$$

and the differential equation $T+\delta T=0$ can be put into the form

$$\eta''=A\eta+B\eta'+C,$$

where

$$A=\frac{y''}{y},\quad B=\frac{2y'}{y},\quad C=\frac{1+y'^2}{y}-y''.$$

The general solution of (105) is given by

$$y=b\cosh\frac{x-a}{b}. \tag{106}$$

Given two points (x_1, y_1) and (x_2, y_2) three cases can occur.

(1) Two catenaries (106) join the given points. In this case one will be minimizing and the other will not.

(2) One catenary (106) joins the given points. This catenary is not minimizing.

(3) No catenary (106) joins the given points. The examples for computation were selected with this in mind.

EXAMPLE I:

The points (x_1, y_1), (x_2, y_2) are respectively $(-2, 3.086)$ and $(2, 3.086)$. The solutions are

$$y=2\cosh\frac{x}{2}$$

$$y=1.404\cosh\frac{x}{1.404},$$

where the first is minimizing and the second is not. The results of computation are presented below in tables 1, 2, and 3. In these tables as well in tables 4, 5, y denotes the exact solution, $y^{(0)}$ the initial approximation, and $y^{(i)}$ the approximation at the ith step. The symbol c_i denotes the value of

$$c=\frac{1+y'^2}{y}-y''$$

along the curve $y^{(i)}$. Along a solution of (105) we have $c=0$. The function v_i found in tables 2 to 5 is a solution of the equation $\delta c=0$, which results from taking the variation of the equation $c_i=0$. The evaluation of v_i can be arranged to be a by-product of the computation of $y^{(i+1)}$. It follows from the theory of conjugate points that y will be a minimizing arc if for c_i sufficiently small v_i does not vanish more than once. If v_i vanishes twice, y is not minimizing.

It is of interest to note that convergence to a solution of the Euler equations that was not a minimum was obtained. This shows the necessity of making additional assumptions (as was done in section 11) in order to insure convergence to a minimum.

EXAMPLE II:

The points (x_1, y_1), (x_2, y_2), are, respectively, $(0, 2.5894)$ and $(5, 5.9284)$. The solutions are

$$y=2\cosh\frac{x-1.5}{2}$$

$$y=1.8883\cosh\frac{x-1.5806}{1.8883}.$$

The first arc is minimizing and the second is not. It will be noticed that although the two solutions are quite close to each other convergence to each one was obtained.

EXAMPLE III:

The points (x_1,y_1) and (x_2,y_2) are, respectively, $(0,2)$, $(5,3)$. In this case there is no solution. After an initial estimate consisting of the line joining $(0,2)$ and $(5,3)$, the second iteration resulted in points (x,y) lying outside of the domain in which $y>0$.

The material in this section was supplied to the author by M. R. Hestenes, under whose auspices the computations were carried out.

TABLE 1

$y=2\cosh x/2$, $\quad y^{(0)}=3.086 \qquad (-2\leq x\leq 2)$

x	$y^{(0)}$	$y^{(1)}$	$y^{(2)}$	$y^{(3)}$	$y^{(4)}$	y	c_0	c_1	c_2	c_3	c_4
0	3.086	2.438	2.132	2.016	1.994	2.000	.324	.086	.018	.002	.000
±.2	3.086	2.447	2.140	2.026	2.004	2.010	.324	.087	.019	.002	.000
±.4	3.086	2.464	2.167	2.056	2.034	2.040	.324	.087	.020	.003	.000
±.6	3.086	2.497	2.213	2.106	2.085	2.090	.324	.092	.021	.003	.000
±.8	3.086	2.542	2.277	2.177	2.157	2.162	.324	.096	.023	.003	.000
±1.0	3.086	2.600	2.360	2.269	2.251	2.255	.324	.101	.025	.004	.000
±1.2	3.086	2.672	2.463	2.383	2.367	2.371	.324	.107	.028	.004	.000
±1.4	3.086	2.756	2.586	2.320	2.507	2.510	.324	.114	.031	.005	.000
±1.6	3.086	2.853	2.730	2.682	2.673	2.675	.324	.121	.034	.006	.000
±1.8	3.086	2.963	2.897	2.870	2.865	2.866	.324	.128	.038	.006	.000
±2.0	3.086	3.086	3.086	3.086	3.086	3.086	.324	.136	.042	.007	.000

TABLE 2

$y=2\cosh(x/2)$, $\quad y^{(0)}=1.886+.3x^2 \qquad (-\leq x\leq 2)$

x	$y^{(0)}$	$y^{(1)}$	$y^{(2)}$	y	c_0	c_1	c_2	v_1
0	1.886	2.053	2.003	2.000	−.070	.010	.001	1.000
±.2	1.898	2.063	2.013	2.010	−.066	.010	.001	.995
±.4	1.934	2.092	2.043	2.040	−.053	.010	.001	.981
±.6	1.994	2.140	2.093	2.091	−.034	.009	.001	.957
±.8	2.078	2.208	2.165	2.162	−.008	.009	.001	.923
±1.0	2.186	2.297	2.258	2.255	−.022	.008	.001	.877
±1.2	2.318	2.407	2.373	2.371	.055	.007	.001	.818
±1.4	2.474	2.540	2.512	2.510	.089	.006	.001	.745
±1.6	2.654	2.696	2.676	2.675	.124	.005	.001	.656
±1.8	2.858	2.878	2.867	2.866	.158	.003	.001	.548
±2.0	3.086	3.086	3.086	3.086	.191	.002	.001	.419

TABLE 3

$y=1.404 \cosh \frac{x}{1.404}, \quad y^{(0)}=1.286+.45x^2 \quad (-2 \leq x \leq 2)$

x	$y^{(0)}$	$y^{(1)}$	$y^{(2)}$	y	c_0	c_1	c_2	v_1
0	1.286	1.367	1.407	1.404	−.123	.010	.000	1.000
±.2	1.304	1.381	1.421	1.418	−.108	.010	.000	.989
±.4	1.358	1.425	1.464	1.462	−.068	.010	.000	.957
±.6	1.448	1.499	1.537	1.534	−.008	.009	.000	.900
±.8	1.574	1.604	1.640	1.638	.065	.008	.000	.816
±1.0	1.736	1.744	1.778	1.776	.143	.007	.001	.697
±1.2	1.934	1.921	1.951	1.949	.220	.006	.001	.542
±1.4	2.168	2.138	2.163	2.162	.293	.005	.001	.337
±1.6	2.438	2.401	2.420	2.419	.361	.006	.001	.073
±1.8	2.744	2.715	2.725	2.725	.421	.009	.001	−.265
±2.0	3.086	3.086	3.086	3.086	.474	.017	.001	−.693

TABLE 4

$y=2 \cosh \frac{x-1.5}{2}, \quad y^{(0)}=2.5894-.8322x+.3x^2 \quad (0 \leq x \leq 5)$

x	$y^{(0)}$	$y^{(1)}$	$y^{(2)}$	$y^{(3)}$	y	c_0	c_1	c_2	c_3	v_2
.0	2.589	2.589	2.589	2.589	2.589	.054	.012	.004	.001	.000
.2	2.435	2.468	2.449	2.439	2.438	.019	.011	.004	.001	.189
.4	2.304	2.368	2.331	2.313	2.310	−.014	.011	.004	.001	.357
.6	2.198	2.287	2.235	2.210	2.206	−.044	.011	.003	.001	.507
.8	2.116	2.226	2.160	2.128	2.124	−.069	.010	.003	.001	.642
1.0	2.057	2.184	2.106	2.068	2.062	−.088	.010	.003	.001	.762
1.2	2.023	2.160	2.072	2.029	2.022	−.099	.010	.003	.001	.871
1.4	2.012	2.154	2.057	2.010	2.002	−.103	.005	.002	.000	.970
1.6	2.026	2.167	2.061	2.010	2.002	−.098	.009	.002	.000	1.058
1.8	2.063	2.198	2.085	2.031	2.022	−.086	.009	.002	.000	1.138
2.0	2.125	2.248	2.129	2.072	2.062	−.066	.009	.002	.000	1.209
2.2	2.211	2.316	2.192	2.133	2.124	−.040	.009	.002	.000	1.272
2.4	2.320	2.404	2.277	2.215	2.206	−.010	.009	.002	.000	1.328
2.6	2.454	2.513	2.383	2.320	2.310	.023	.009	.002	.000	1.374
2.8	2.611	2.643	2.511	2.448	2.438	.058	.008	.002	.000	1.412
3.0	2.793	2.795	2.663	2.599	2.590	.093	.008	.002	.000	1.441
3.2	2.998	2.970	2.840	2.777	2.768	.128	.008	.002	.000	1.459
3.4	3.221	3.171	3.043	2.982	2.972	.162	.007	.002	.001	1.464
3.6	3.482	3.398	3.276	3.217	3.208	.194	.007	.002	.000	1.456
3.8	3.759	3.654	3.539	3.483	3.474	.224	.007	.002	.001	1.431
4.0	4.061	3.941	3.836	3.785	3.776	.252	.008	.003	.001	1.388
4.2	4.386	4.261	4.169	4.124	4.116	.277	.009	.003	.001	1.323
4.4	4.736	4.616	4.541	4.503	4.428	.300	.011	.004	.001	1.232
4.6	5.109	5.011	4.955	4.928	4.924	.323	.013	.004	.001	1.111
4.8	5.507	5.447	5.416	5.401	5.400	.343	.017	.004	.001	.956
5.0	5.928	5.928	5.928	5.928	5.928	.361	.022	.005	.001	.760

TABLE 5

$y=1.888 \cosh \frac{x-1.581}{1.888} \quad y^{(0)}=2.5894-1.2322x+.38x^2 \quad (0 \leq x \leq 5)$

x	$y^{(0)}$	$y^{(1)}$	$y^{(2)}$	$y^{(3)}$	y	c_0	c_1	c_2	c_3	v_2
0	2.589	2.589	2.589	2.589	2.589	.212	.002	.009	.002	.000
.2	2.358	2.375	2.403	2.414	2.416	.159	−.008	.008	.001	.185
.4	2.157	2.194	2.245	2.267	2.270	.103	−.003	.007	.001	.344
.6	1.987	2.044	2.114	2.145	2.149	.046	.003	.006	.001	.479
.8	1.847	1.923	2.009	2.047	2.052	−.008	.008	.005	.001	.595
1.0	1.737	1.828	1.927	1.927	1.978	−.056	.012	.005	.001	.693
1.2	1.658	1.758	1.869	1.920	1.927	−.095	.016	.004	.001	.777
1.4	1.609	1.712	1.833	1.889	1.897	−.121	.019	.003	.001	.847
1.6	1.591	1.690	1.819	1.880	1.888	−.131	.021	.003	.001	.905
1.8	1.603	1.690	1.827	1.892	1.901	−.124	.022	.002	.001	.953
2.0	1.645	1.714	1.857	1.926	1.935	−.102	.022	.002	.001	.989
2.2	1.718	1.761	1.910	1.981	1.990	−.065	.021	.002	.001	1.015
2.4	1.821	1.831	1.986	2.059	2.068	−.018	.019	.002	.001	1.030
2.6	1.954	1.927	2.085	2.160	2.170	.035	.017	.002	.001	1.034
2.8	2.118	2.049	2.210	2.285	2.295	.091	.015	.002	.001	1.024
3.0	2.313	2.198	2.361	2.437	2.447	.147	.012	.003	.001	1.000
3.2	2.538	2.378	2.541	2.616	2.626	.201	.009	.003	.001	.960
3.4	2.793	2.590	2.751	2.824	2.834	.252	.006	.004	.001	.900
3.6	3.078	2.839	2.995	3.065	3.074	.300	.005	.004	.001	.818
3.8	3.394	3.126	3.274	3.340	3.349	.342	.004	.005	.001	.710
4.0	3.741	3.456	3.593	3.654	3.662	.381	.006	.006	.001	.570
4.2	4.117	3.834	3.955	4.008	4.015	.416	.009	.006	.001	.394
4.4	4.524	4.264	4.364	4.408	4.414	.447	.016	.007	.001	.174
4.6	4.962	4.752	4.826	4.858	4.862	.474	.025	.007	.001	−.096
4.8	5.430	5.305	5.345	5.363	5.365	.499	.038	.008	.002	−.426
5.0	5.925	5.928	5.928	5.928	5.928	.521	.053	.008	.002	−.826

14. References

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