On the Derivation and Accuracy of Certain Formulas for Sample Sizes and Operating Characteristics of Nonsequential Sampling Procedures[1]

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Formulas are given that are needed for the computation of number of observations and operating characteristics of single sample acceptance sampling procedures based on tests of statistical hypotheses. Some of the same results may be obtained by reference to existing tables and curves located in widely scattered places. The hypotheses considered concern the means and standard deviations of certain populations (that is, binomial, Poisson, normal) where the test is made against a one-sided alternative. The comparison of two means or two variances as well as the test of a single mean or variance is discussed. The accuracy of the formulas is considered, and where approximations are involved, the results are compared with existing tables.

This paper develops and considers the question of the accuracy of certain formulas needed for the number of observations and operating characteristics of single-sample acceptance-sampling procedures based on statistical tests of hypotheses. For purposes of completeness the paper also contains formulas that are already known.

1. Introduction

It is now common knowledge among users of modern statistical tools that the characteristics of a sampling plan must be specified in terms of the risks of rejecting good material (Type I error: producer's risk) and accepting poor material (Type II error: consumer's risk). The problem of determining a sample size for a given probability of Type I error, which will minimize the probability of Type II error, has been considered by several authors, in particular [1 to 4].[3] This problem belongs to the category of that broad class of problems in the field of planned experimentation in which one is asked to provide adequate replication to detect treatment differences with desired amount of protection against taking wrong decisions. If for a certain sample size it is impossible to reduce simultaneously to small proportions the risks of two kinds it will be helpful to know this in advance.

Answers to most of the questions raised in this paper can be obtained from the existing published tables and curves. The effectiveness of these formulas in relation to assumptions and approximations that have been made in their derivation also has a theoretical interest. We shall restrict ourselves to the consideration of certain parametric hypotheses concerning means and standard deviations of certain populations, mainly against one-sided alternatives.

2. Preliminaries and Notation

In the ensuing sections H_1 denotes the null hypothesis, H_2 any one of a set of alternative hypo-

theses, α the probability of rejecting the null hypothesis H_1 when true, and β the probability of accepting H_1 when some alternative hypothesis, H_2, is true. In connection with the hypotheses concerning the means of certain populations in which the standard deviations are functionally related to the means and consequently unspecified, the reader will at once recognize that the acceptance-rejection criterion A used for a statistic T is not the best in the sense of the likelihood ratio test [5]. This difficulty, however, can be avoided by transformation of the original variables and has been indicated in the appropriate sections. In connection with the two-sample problem the formulas assume equal sample sizes. These formulas can obviously be extended to cases in which it is desired to take unequal sample sizes of N_1 and N_2 that are assumed in advance to be functionally related.

3. A General Formula Concerning Sample Size and Region of Rejection

Let $x=x(N)$ be a normally distributed variable with mean μ_1 and standard deviation (sd) $\sigma_1 = f(\mu_1)F(N)$ under H_1 and with mean μ_2 and sd $\sigma_2 = f(\mu_2)F(N)$ under $H_2(\mu_2 > \mu_1)$, where $F(N)$ is a certain function of the sample size N and is independent of σ. We assume $x > A$ as the critical region and obtain

$$\frac{1}{\sqrt{2\pi}} \int_{K_\alpha}^\infty e^{-t^2/2}dt = \alpha$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{K_{1-\beta}}^\infty e^{-t^2/2}dt = 1 - \beta,$$

where K_ϵ is the standardized normal deviate exceeded with probability ϵ and actually

$$\frac{A-\mu_1}{f(\mu_2)F(N)} = K_\alpha \tag{1}$$

$$\frac{A-\mu_2}{f(\mu_2)F(N)} = K_{1-\beta} = -K_\beta \tag{2}$$

Solving (1) and (2) for $F(N)$ and A we obtain

$$F(N) = \frac{\mu_2 - \mu_1}{K_\alpha f(\mu_1) + K_\beta f(\mu_2)} \qquad (3)$$

$$A = \frac{K_\alpha \mu_2 f(\mu_1) + K_\beta \mu_1 f(\mu_2)}{K_\alpha f(\mu_1) + K_\beta f(\mu_2)}, \qquad (4)$$

which may also be written as

$$1 = \frac{\mu_2 - \mu_1}{K_\alpha \sigma_1 + K_\beta \sigma_2} \qquad (5)$$

$$A = \frac{K_\alpha \mu_2 \sigma_1 + K_\beta \mu_1 \sigma_2}{K_\alpha \sigma_1 + K_\beta \sigma_2}, \qquad (6)$$

for later convenience in certain simplifications. We note in passing that in particular cases of application $f(\mu)$ will either be a function of μ (cf. binomial and Poisson), or it will not depend on μ, in which case it will be a certain function of the population sd σ (cf. normal) or a pure number (cf. transformed binomial and Poisson).

4. Tests Relating to the Parameter of the Binomial Distribution

4.1. Single Binomial

Consider a random sample of N items drawn from an infinite population in which a proportion P of the items possesses a definite attribute B and let p be the fraction of the items possessing B in the sample. Then $E(p) = P$ and $\sigma(p) = \sqrt{P(1-P)}/\sqrt{N}$. Our hypotheses are $H_1 : P = P_1$ and $H_2 : P = P_2 (P_2 > P_1)$. Using the normal approximation for the binomial variable p, (3) and (4) imply

$$N \cong \left(\frac{K_\beta \sqrt{P_2(1-P_2)} + K_\alpha \sqrt{P_1(1-P_1)}}{P_2 - P_1} \right)^2 \qquad (7)$$

$$A \cong \frac{K_\beta P_1 \sqrt{P_2(1-P_2)} + K_\alpha P_2 \sqrt{P_1(1-P_1)}}{K_\alpha \sqrt{P_1(1-P_1)} + K_\beta \sqrt{P_2(1-P_2)}} \qquad (8)$$

While (7) determines directly the sample size that will (approximately) guarantee a specified α and β, this may also be looked upon as providing the values of the probability of accepting H_1 for different values of P for given N and α. For example (7) yields

$$K_\beta \cong \frac{\sqrt{N}(P_2 - P_1) - K_\alpha \sqrt{P_1(1-P_1)}}{\sqrt{P_2(1-P_2)}}. \qquad (9)$$

The inverse sine transformation $\theta = 2 \sin^{-1} \sqrt{p}$ where θ is measured in radians [16] avoids the difficulty of the dependence of the standard deviation of of p on unknown P, since θ is approximately normally distributed about $2 \sin^{-1} \sqrt{P}$ with sd $\sigma(\theta) \cong \sqrt{(1/N)}$. In terms of the transformed quantities we obtain

$$N \cong \frac{1}{4} \left(\frac{K_\alpha + K_\beta}{\sin^{-1} \sqrt{P_2} - \sin^{-1} \sqrt{P_1}} \right)^2 \qquad (10)$$

$$A \cong 2 \left(\frac{K_\beta \sin^{-1} \sqrt{P_1} + K_\alpha \sin^{-1} \sqrt{P_2}}{K_\alpha + K_\beta} \right) \qquad (11)$$

$$K_\beta \cong 2 \sqrt{N} (\sin^{-1} \sqrt{P_2} - \sin^{-1} \sqrt{P_1}) - K_\alpha. \qquad (12)$$

We have derived (7) and (10) to illustrate the use of the results given in section 3. The comparison of the two formulas raises questions of quite a complicated nature. We have so far not found any convenient yardstick with which to compare their accuracy. In the light of the fact that the critical region $\theta > A$ (11) has certain theoretical advantages against $p > A$ (8) there is the temptation to recommend (10). As the following example will indicate ($\alpha = .05$, $P_1 = 0.1$ and $P_2 = 0.2$) the recommendation has nothing to do with the relative magnitude of the values of N given by (7) and (10).

$\beta =$	0. 20	0. 10	0. 05	0. 01
(7)	68. 9	101. 2	132. 6	202. 8
(10)	76. 8	106. 3	134. 4	195. 8

4.2. Comparison of Two Binomials

Consider two binomial processes with P and P^* as the values of true proportions and let p and p^* be the observed proportions in a sample of N from each process. We have here $H_1 : P^* - P = 0$ and $H_2 : P^* - P > 0$. The test function $X = p^* - p$ under H_1 has an unspecified variance $V(X) = 2[\hat{P}(1-\hat{P})]/N$, where $\hat{P}$ is the common (unspecified) value of P and P^* under H_1. Under H_2 we have $V(X) = P(1-P)/N + [P^*(1-P^*)]/N$, where $P^* - P = \xi > 0$ say, with the value of ξ specified but not the values of P^* and P. We are then faced with the problem of comparing two means having unspecified and unequal variances under H_1 and H_2.

The only satisfactory solution to the problem of the comparison of two binomial means—see [4, chapter 7]—is usually given in terms of the transformed variables (section 4.1). Under the inverse sine transformation $\sin^{-1} \sqrt{p^*} - \sin^{-1} \sqrt{p}$ is approximately normally distributed (except when N is very small or the P's are close to 0 or 1) with mean $\sin^{-1} \sqrt{P^*} - \sin^{-1} \sqrt{P}$ and variance $(1/2N)$. We now use results of section 3 and obtain

$$N \cong \frac{1}{2} \left(\frac{K_\alpha + K_\beta}{\sin^{-1} \sqrt{P^*} - \sin^{-1} \sqrt{P}} \right)^2. \qquad (13)$$

$$A \cong \frac{K_\alpha}{\sqrt{2N}} \qquad (14)$$

$$K_\beta \cong \sqrt{2N} (\sin^{-1} \sqrt{P^*} - \sin^{-1} \sqrt{P}) - K_\alpha. \qquad (15)$$

5. Tests Relating to the Parameter of the Poisson Distribution

5.1. Single Poisson

Let $\bar{x}$ denote the mean of a random sample of size N from a Poisson population with parameter m. Let $H_1 : m = m_1$ and $H_2 : m = m_2$ $(m_2 > m_1)$. To apply

the results of Section 3 we assume normality for $\bar{x}$ with mean m and variance m/N and obtain

$$N \cong \left(\frac{\sqrt{m_1}K_\alpha + \sqrt{m_2}K_\beta}{m_2 - m_1} \right)^2 \qquad (16)$$

$$A \cong \frac{m_2\sqrt{m_1}K_\alpha + m_1\sqrt{m_2}K_\beta}{\sqrt{m_1}K_\alpha + \sqrt{m_2}K_\beta} \qquad (17)$$

$$K_\beta \cong \frac{(m_2 - m_1)\sqrt{N} - \sqrt{m_1}K_\alpha}{\sqrt{m_2}} \qquad (18)$$

We notice that under both H_1 and H_2 the variance of $\bar{x}$ is known given N but differs for H_1 and H_2. If we make use of the well known square-root transformation [17], the variance of the transformed variate is approximately independent of the unknown mean and is approximately equal to $1/4N$. We now obtain

$$N \cong \frac{1}{4} \left(\frac{K_\alpha + K_\beta}{\sqrt{m_2} - \sqrt{m_1}} \right)^2 \qquad (19)$$

$$A \cong \frac{K_\alpha \sqrt{m_2} + K_\beta \sqrt{m_1}}{K_\alpha + K_\beta} \qquad (20)$$

and

$$K_\beta \cong 2\sqrt{N}(\sqrt{m_2} - \sqrt{m_1}) - K_\alpha. \qquad (21)$$

For example for $m_1 = 3.0$, $m_2 = 4.0$, $\alpha = .05$, $\beta = .10$, formulas (16) and (19) yield $N = 29.3$ and $N = 29.8$, respectively. This is just an illustration; otherwise remarks made in connection with the single binomial in the last paragraph of section 4.1 apply here as well.

5.2. Comparison of Two Poissons

Let M and M^* be the parameters of two Poisson populations. Let $H_1: M = M^*$ and $H_2: M < M^*$, e.g. $\sqrt{M^*} - \sqrt{M} = \eta > 0$. Consider two independent random samples of size N drawn one from each of the two populations. Let $\bar{x}$ and $\bar{x}^*$ be the corresponding sample means. We may regard the quantity $\sqrt{\bar{x}^*} - \sqrt{\bar{x}}$ as approximately normally distributed with mean $m = \sqrt{M^*} - \sqrt{M}$ and variance $1/2N$ and consequently obtain

$$N = \frac{1}{2} \left(\frac{K_\alpha + K_\beta}{\sqrt{M^*} - \sqrt{M}} \right)^2 \qquad (22)$$

$$A \cong \frac{K_\alpha}{\sqrt{2N}} \qquad (23)$$

and

$$K_\beta \cong \sqrt{2N}(\sqrt{M^*} - \sqrt{M}) - K_\alpha. \qquad (24)$$

6. Tests Concerning the Mean of the Normal Population

6.1. Single Mean Test (σ known)

Let $\bar{x}$ denote the mean of a random sample of N observations from a normal population (μ, σ^2). We shall assume that σ is known from past experience.

Our hypotheses are $H_1: \mu = \mu_1$ and $H_2: \mu = \mu_2$ $(\mu_2 > \mu_1)$. Set $\mu_2 - \mu_1 = \Delta \sigma$. The test function $\bar{x}$ is normally distributed with mean μ and sd $\sigma/\sqrt{N}$. Using results of Section 3 we obtain

$$N = \left(\frac{K_\alpha + K_\beta}{\Delta} \right)^2 \qquad (25)$$

$$A = \frac{K_\alpha \mu_2 + K_\beta \mu_1}{K_\alpha + K_\beta} \qquad (26)$$

and

$$K_\beta = \Delta\sqrt{N} - K_\alpha. \qquad (27)$$

6.2. Comparison of Two Normal Means (σ's known)

Let $\bar{x}_1$, $\bar{x}_2$ be the sample means of two independent random samples of equal size N drawn one from each of the two normal populations $\pi_1(\mu_1, \sigma_1^2)$, $\pi_2(\mu_2, \sigma_2^2)$ respectively. We assume both σ_1^2 and σ_2^2 to be known from past experience. Our hypotheses are $H_1: \mu_1 = \mu_2$ and $H_2: \mu_1 < \mu_2$. The test function $\bar{x}_2 - \bar{x}_1$ is normally distributed $[(\mu_2 - \mu_1), (\sigma_1^2 + \sigma_2^2)/N]$ and, therefore, we obtain

$$N = \left(\frac{K_\alpha + K_\beta}{d} \right)^2 \qquad (28)$$

$$A = \frac{K_\alpha(\mu_2 - \mu_1)}{K_\alpha + K_\beta} \qquad (29)$$

and

$$K_\beta = \sqrt{N}d - K_\alpha \qquad (30)$$

where

$$d = \frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}.$$

If $\sigma_1^2 = \sigma_2^2 = \sigma^2$ we obtain

$$N = 2 \left(\frac{K_\alpha + K_\beta}{\Delta} \right)^2 \qquad (31)$$

$$A = \frac{K_\alpha(\mu_2 - \mu_1)}{K_\alpha + K_\beta} \qquad (32)$$

and

$$K_\beta = \Delta\sqrt{\frac{N}{2}} - K_\alpha. \qquad (33)$$

6.3. Single Mean Test (σ unknown)

For notation see section 6.1. In this case the well known Student's statistic $t = \sqrt{N}(\bar{x} - \mu_1)/s$ where s^2 is the unbiased estimate of σ^2, is used to accept or reject H_1. Tables for determining the sample size for detecting a given value of Δ with preassigned α and β are given in [1] and operating characteristic curves for the same are given in [2]. To obtain a convenient formula for the sample size we proceed as follows:

Consider $P\{t \geq k\} = P\{[(\bar{x} - \mu) - (ks/\sqrt{N})] > 0\}$ where k is a certain constant. The quantity $Z = (\bar{x} - \mu_1) - (ks/\sqrt{N})$ consists of two parts: $(\bar{x} - \mu_1)$ is normally distributed $(0, \sigma^2/N)$ under H_1, and

$(\mu_2-\mu_1,\ \sigma^2/N)$ under H_2; and for fixed N, $ks/\sqrt{N}$ is a constant multiple of s where s is approximately normally distributed $(C_1\sigma,\ C_2\sigma/\sqrt{2(N-1)})$ and C_1 and C_2 are certain constants less than 1.[4] It will be assumed for the purpose of this discussion that both C_1 and C_2 are equal to unity. The dual nature of these assumed approximations to the distribution of s should be noted. Therefore,

$$E(Z|H_1)=Z_1=-\frac{k}{\sqrt{N}}\,\sigma$$

$$E(Z|H_2)=Z_2=(\mu_2-\mu_1)-\frac{k}{\sqrt{N}}\,\sigma$$

$$V(Z)=\frac{\sigma^2}{N}+\frac{k^2}{N}\cdot\frac{\sigma^2}{2(N-1)}.$$

For $Z>A$ as the critical region we obtain (see section 3)

$$\frac{A-\left(-k\dfrac{\sigma}{\sqrt{N}}\right)}{\dfrac{\sigma}{\sqrt{N}}\sqrt{1+\dfrac{k^2}{2(N-1)}}}=K_\alpha$$

$$\frac{A-\left(\mu_2-\mu_1-\dfrac{k}{\sqrt{N}}\,\sigma\right)}{\dfrac{\sigma}{\sqrt{N}}\sqrt{1+\dfrac{k^2}{2(N-1)}}}=-K_\beta$$

Eliminating A from the above two equations we get

$$\frac{\Delta}{K_\alpha+K_\beta}=\frac{1}{\sqrt{N}}\sqrt{1+\frac{k^2}{2(N-1)}}\qquad(34)$$

where Δ has been previously defined (section 6.1). This is a quadratic in N and could be solved for N if k were known. We now determine k from the consideration that A is to be independent of σ. After simplification we obtain

$$A=\sigma\left(\frac{\Delta K_\alpha}{K_\alpha+K_\beta}-\frac{k}{\sqrt{N}}\right)$$

The right-hand side will be independent of σ if and only if the quantity in parentheses vanishes.

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[4] The general expressions for C_1 and C_2 are:

$$C_1=\frac{\sqrt{\dfrac{2}{N-1}}\ \Gamma\left(\dfrac{N}{2}\right)}{\Gamma\left(\dfrac{N-1}{2}\right)},\qquad C_2=\sqrt{2(N-1)(1-C_1^2)}.$$

For small values of N, C_1, and C_2 take the following values:

N	C_1	C_2
2	0.798	0.852
3	.886	.927
4	.921	.954
5	.940	.965
10	.973	.985
25	.990	.995

Therefore,

$$k=\frac{K_\alpha\Delta\sqrt{N}}{K_\alpha+K_\beta}\qquad(35)$$

whence $A=0$ and is not only independent of σ but actually takes the originally intended value in the inequality $P(Z>0)$. Substituting this value of k in (34) we finally obtain

$$N=\frac{b+\sqrt{b^2-4a}}{2a}\qquad(36)$$

where

$$a=\left(\frac{\Delta}{K_\alpha+K_\beta}\right)^2$$

and

$$b=\left(1+a\left(1+\frac{K_\alpha^2}{2}\right)\right)$$

Similarly

$$K_\beta=-K_\alpha+\frac{\Delta}{\sqrt{N-1}}\left(N^2-N\left(1+\frac{K_\alpha^2}{2}\right)\right)^{1/2}\quad(37)$$

If we replace $N-1$ by N, the above two formulas reduce to somewhat simpler expressions of the form

$$N=\left(\frac{1}{a}+\frac{1}{2}K_\alpha^2\right)+1\qquad(36a)$$

$$K_\beta=-K_\alpha+\Delta\left(N-\frac{K_\alpha^2}{2}\right)^{1/2}.\qquad(37a)$$

For any given Δ, α, and β, values of N from (36) are compared with the Neyman-Tokarska [5] Tables [1] are given in table 1. These values will be found to be approximately the same.

As pointed out in the previous paragraph (34) was obtained under the assumption that k is unknown. The classical procedure employs $k=t_\alpha$ (n) where $n=N-1$.[5] Therefore if the probability points of t are not available, (35) furnishes an approximation to such points t_α. We have considered the accuracy of such k points in relation to t_α in terms of $P[t>x]=\alpha(x)$. Values of t_α, k, $\alpha(t_\alpha)$, and $\alpha(k)$ are given in tables 2a and 2b for different N's and for different α and β. Strictly speaking $\alpha(t_\alpha)\equiv\alpha$, but when t_α to only three decimals is used, $\alpha(t_\alpha)$ may differ slightly from α as shown in tables 2a and 2b.

We notice that k values are in general conservative t_α estimators and that the values of $\alpha(k)$ are consistently greater than the corresponding value of $\alpha(t_\alpha)$. In this sense a user of our formulas is likely to declare slightly too many significant results. The danger, if it can be so called, is not very great, but it is still there.

The question is asked: is it possible to eliminate this "danger" and still utilize formula (36) for N? It should be noticed that we can not utilize the available percentage points of t in the derivation of (36). However, (34) can still be solved for N as-

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[5] Neyman-Tokarska's ρ is equal to our $\Delta\sqrt{N}$. Since the standard table of probability points of t, [18] table 4, gives the *two-tail* probability points of t, our $t_\alpha(n)$ corresponds to the entry given there for 2α and n degrees of freedom.

TABLE 1. *Comparisons among different methods for obtaining necessary sample sizes*

[Single sample one-sided mean test]

	$\alpha=.05$											
	$\beta=.20$				$\beta=.10$				$\beta=.05$			
Δ	$N-T$ [a]	(36)	(38)	(25)	$N-T$	(36)	(38)	(25)	$N-T$	(36)	(38)	(25)
2	3	3	4	2	4	4	4	2	5	4	4	3
1	8	8	8	6	10	10	10	9	12	12	12	11
.5	26	26	26	25	36	36	36	34	45	45	45	43
.25	100	100	100	99	139	138	138	137	175	175	175	173
.125	398	397	397	396	550	550	550	548	694	694	694	693
	$\alpha=.01$											
	$\beta=.10$				$\beta=.05$				$\beta=.01$			
Δ												
2	6	6	7	3	7	7	7	4	8	8	9	5
1	16	16	16	13	19	19	19	16	24	24	25	22
.5	55	55	55	52	66	66	66	63	89	89	89	87
.25	211	211	211	208	255	255	255	252	349	349	349	346
.125	837	836	836	833	1012	1012	1012	1009	1388	1388	1388	1388

[a] "$N-T$" refers to values obtained either directly or by interpolation from Neyman and Tokarska Tables [1]. Values under "(25)" are for σ known and are included here for purposes of comparison with the other three for σ unknown. For a relation between the comparison of sample sizes for σ known and unknown see section 6.5.

TABLE 2a.[a] *Comparison of the accuracy of the different percentage points of t and k values for the same Δ*

		$\alpha=.05$				
		Neyman and Tokarska			This paper	
	N	t_α	$\alpha(t_\alpha)$	N	k	$\alpha(k)$
$\beta=.50$	3	2.920	0.05000	3	3.338	0.04958
	4	2.353	.05000	4	2.110	.06270
	12	1.796	.05003	12	1.753	.05370
$\beta=.20$	3	2.920	.05000	3	2.458	.06659
	8	1.895	.04996	8	1.840	.05412
	26	1.708	.05012	26	1.691	.05163
$\beta=.10$	4	2.353	.05000	4	2.235	.05575
	10	1.833	.04999	10	1.783	.05414
$\beta=.05$	5	2.132	.04995	4	2.110	.06270
	12	1.796	.05003	12	1.753	.05370
$\beta=.01$	6	2.015	.05003	6	1.959	.05370
	17	1.746	.04995	17	1.718	.05250

[a] Values of $P(t>k)=\alpha(k)$ and $P(t>t_\alpha)=\alpha$ (t_α) for $N\leq21$ were obtained by interpolation in "Student's" Table I [6] and for $22\leq N\leq31$ by interpolation in Table XXV of [7].

TABLE 2b.[a] *Comparison of the accuracy of the different percentage points of t and k values for the same Δ*

		$\alpha=.01$				
		Neyman and Tokarska			This paper	
	N	t_α	$\alpha(t_\alpha)$	N	k	$\alpha(k)$
$\beta=.50$	4	4.541	0.00997	5	4.371	0.00594
	8	2.998	.01003	8	2.912	.01130
	24	2.500	.01250	24	2.473	.01322
$\beta=.20$	5	3.747	.01000	6	3.531	.00833
	13	2.681	.01002	13	2.644	.01070
$\beta=.10$	6	3.365	.00996	6	3.277	.01099
	16	2.602	.00996	16	2.571	.01064
$\beta=.05$	7	3.143	.01003	7	3.120	.01033
	19	2.552	.00974	19	2.528	.01043
$\beta=.01$	8	2.998	.01003	8	2.912	.01130
	24	2.500	.01250	24	2.473	.01322

[a] Values of $P(t>k)=\alpha(k)$ and $P(t>t_\alpha)=\alpha$ (t_α) for $N<21$ were obtained by interpolation in "Student's" Table I [6] and for $22\leq N\leq31$ by interpolation in Table XXV of [7].

suming k to be known, say equal to $t_\alpha(n)$, resulting in

$$N=\frac{(a+1)+\sqrt{(a-1)^2+2at_\alpha^2(n)}}{2a} \qquad (38)$$

Values of N from (38) corresponding to different $t_\alpha(n)$, Δ, α and β are also given in table 1. Since the values of N as given by (36), (38) and the Neyman-Tokarska Tables are approximately the same, it is recommended that (38) be used in conjunction with the t-tables. This will save the labor of calculating k values.

Since any symmetric two-sided test may be regarded as a combination of two one-sided tests, values of N and k may be obtained from the corresponding single-tailed formulas by substituting $K_\alpha/2$ for K_α and $t_\alpha/2$ for t_α, in which case the "β" deduced from the resulting value of K_β will over-estimate the true value of β, the probability of accepting H_1 when $\mu_2-\mu_1=\Delta\sigma$, by the amount $1-\beta'$ where β' is given by (37) or (37a) with the term in Δ taken with a negative sign. Values of N obtained from [8] and [9] and from the formulas (36) and (38) of this paper are given in table 3. The corresponding two-tailed values of t and k are also given in table 3.

6.4. Comparison of Two Means: (common σ unknown)

For notation refer to section 6.2. Consider two samples of equal size N. Our hypotheses are H_1: $\mu_1=\mu_2$; H_2: $\mu_1<\mu_2$. Let s^2 denote the unbiased estimate of the common variance σ^2. The statistic $t=\dfrac{\overline{x}_2-\overline{x}_1}{s\sqrt{\dfrac{2}{N}}}$ which under H_1 has "Student's" t-distribu-

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[Two-sided single mean test: σ unknown]

	$\alpha = .05$									
	$\beta = .30$					$\beta = .20$				
Δ	Tabu-lated	(36)	(38)	$t_\alpha/2$	k	Tabu-lated	(36)	(38)	$t_\alpha/2$	k
2	4	4	4	3.182	3.190	4	4	5	3.182	2.949
1	8	8	8	2.365	2.280	10	10	10	2.262	2.210
.5	27	27	27	2.056	2.038	33	33	33	2.037	2.021
.25	101	101	101	1.982	1.979	128	128	128	1.977	1.975
.125	397	397	397	1.966	1.965	504	504	504	1.964	1.964
	$\alpha = .01$									
2	6	6	6	4.032	4.184	6	7	7	4.032	3.933
1	13	13	13	3.055	3.019	15	15	15	2.977	2.941
.5	42	42	42	2.701	2.687	50	50	50	2.680	2.668
.25	157	157	157	2.607	2.604	190	190	190	2.601	2.599
.125	618	618	618	2.584	2.583	751	751	751	2.583	2.582

[a] "Tabulated" values were obtained from [8] and [9] and $t_\alpha/2$ indicates the two-tailed α value of t.   See footnote 5.

tion with $2(N-1)$ d.o.f. is used to accept or reject H_1. Consider $P\left(t \geq k_1\right) = P\left(\bar{x}_2 - \bar{x}_1 - k_1\sqrt{\dfrac{2}{N}}\, s > 0\right)$ where we assume k_1 to be a certain unknown constant.   Let

$$Z = (\bar{x}_2 - \bar{x}_1) - k_1\sqrt{\frac{2}{N}}\, s.$$

We assume that s is approximately normally distributed with mean σ and sd, $\sigma/\sqrt{4(N-1)}$.   Therefore Z is approximately normally distributed with

$$E(Z|H_1) = -k_1\sqrt{\frac{2}{N}}\,\sigma$$

$$E(Z|H_2) = \mu_2 - \mu_1 - k_1\sqrt{\frac{2}{N}}\,\sigma$$

$$\sigma(Z) = \sigma\sqrt{\frac{2}{N} + \frac{k_1^2}{2N(N-1)}}$$

where $\sigma(Z)$ denotes the standard deviation of Z. Proceeding as in section 3.3 we obtain

$$\frac{\Delta}{K_\alpha + K_\beta} = \sqrt{\frac{2}{N} + \frac{k_1^2}{2N(N-1)}} \quad (39)$$

and

$$A = \frac{K_\alpha \sigma \Delta}{K_\alpha + K_\beta} - \sigma\sqrt{\frac{2}{N}}\, k_1. \quad (40)$$

If we assume $k_1 = t_\alpha(n)$ where t_α is the *one-sided* α-point of t for $n = 2(N-1)$ d.o.f the equation (39) yields

$$N = \frac{a + 2 + \sqrt{(a-2)^2 + 2at_\alpha^2(n)}}{2a} \quad (41)$$

where a has been defined in section 6.3.   The reason for such an illogical assumption about the knowledge of $t_\alpha(n)$ before actually N is determined has been indicated in the previous section.   If k_1 is un-

known we determine k_1 from the consideration that A is to be independent of σ.   The relation (40) yields

$$k_1 = \sqrt{\frac{N}{2}} \cdot \frac{K_\alpha \Delta}{K_\alpha + K_\beta}. \quad (42)$$

We substitute this value of k_1 in (39) and obtain

$$N = \frac{b_1 + \sqrt{b_1^2 - 8a}}{2a} \quad (43)$$

where a has been previously defined (section 6.3) and

$$b_1 = 2 + a\left(1 + \frac{K_\alpha^2}{4}\right).$$

For determining the operating characteristic we similarly obtain

$$K_\beta = -K_\alpha + \Delta\left[\frac{N}{2}\left(1 - \frac{K_\alpha^2}{4(N-1)}\right)\right]^{1/2}. \quad (44)$$

We give in table 4 values of N as obtained from (43) and Tables of Neyman [6] and Tokarska [1]. While (42) provides approximate values of the per-

Table 4. *Comparisons among different methods for obtaining necessary sample sizes N* [a]

[Two means one-sided test; common σ unknown: $\Delta = 2$]

	$\alpha = .05$		$\alpha = .01$	
	Neyman-Tokarska	(43)	Neyman-Tokarska	(43)
$\beta = .50$	2	2	4	4
$\beta = .20$	4	4	6	7
$\beta = .10$	5	5	8	8
$\beta = .05$	6	6	9	9
$\beta = .01$	8	9	12	12

[a] N is the size of one of the two equal samples.

[6] For this particular case ρ as defined in [1] is equal to $\Delta\sqrt{\dfrac{N}{2}}$.

centage points of t for $2(N-1)$ d.o.f, the formula (43) may be used advantageously by utilizing the available percentage points of t as acceptance-rejection criteria. Formulas obtained here for single-tailed comparisons can also be used for a two-tailed test by substituting appropriate two-tailed values of the quantities involved. If the two populations have unequal variances and their ratio is known we still can construct a test function similar to t and use the above formulas. (For the structure of the test function see [10].)

6.5. Comparison of Sample Sizes for Known and Unknown σ

It need hardly be emphasized that in situations in which large sample sizes are required, the normal test (sections 6.1 and 6.2) and t-test (sections 6.3 and 6.4) will both yield approximately the same N (see for example table 1). To determine the relations between the two, let N_m be the number of observations required when σ is known and N_t the corresponding number when σ is not known. After some simplification the following asymptotic relations between N_m and N_t are obtained. For the single-mean test, (25) and (36a) yield

$$N_t \sim N_m \left(1 + \frac{K_\alpha^2}{2N_m} \right) + 1 \qquad (45)$$

and for the two-means test, (31) and (43) yield

$$N_t \sim N_m \left(1 + \frac{K_\alpha^2}{4N_m} \right) + 1. \qquad (46)$$

7. Tests Concerning Variances of Normal Populations

Tests relating to population variances fall into two well defined categories. In one case we assume that the variability of a certain product is known and it is desired to find out whether a new product is more variable than this. In the other case we are asked to choose between two products on the basis of their variability which is unknown. We discuss these situations in the following sections.

7.1 Single Variance Test

Consider a random sample of size N $(=n+1)$ from a normal population (m, σ^2). Let s^2 be the unbiased sample estimate of σ^2. Our hypotheses are H_1: $\sigma^2 = \sigma_0^2$ and H_2: $\sigma^2 = \lambda \sigma_0^2 (\lambda > 1)$. For a given level of significance α, if $\chi^2 = ns^2/\sigma_0^2 > X_\alpha^2$, we reject H_1 and conclude that $\sigma^2 > \sigma_0^2$.

Let $\lambda(\alpha, \beta, n)$[7] denote the value of σ^2/σ_0^2 for which the probability of the decision $\sigma^2 = \sigma_0^2$ equals β when the test is conducted at the α level of significance with n d.o.f. It can be shown [2], [4] that the probability error of the second kind is exactly β if $\lambda(\alpha, \beta, n) = \chi_\alpha^2(n)/\chi_{1-\beta}^2(n))$. If we are testing H_1: $\sigma^2 = \sigma_0^2$ against H_2: $\sigma^2 = \lambda \sigma_0^2$ $(\lambda < 1)$ we have $\lambda(\alpha, \beta, n) = x_{1-\alpha}^2/\chi_\beta^2$.

Curves for the operating characteristics of such testing procedures are given in [2] and [4]. Eisenhart [4] has also given extensive tables for $\lambda(\alpha, \beta, n)$.

The problem of determining a direct relation between large n and any given set of values of λ, α, β was first considered by Wallis [1, footnote of p. 278]. Assuming normality of s (see section 6.3) and applying the results of Section 3 we obtain

$$n = \frac{1}{2} \left(\frac{K_\alpha + K_\beta \sqrt{\lambda}}{\sqrt{\lambda} - 1} \right)^2 \qquad (47)$$

$$A = \frac{\sqrt{\lambda} \sigma_0 (K_\alpha + K_\beta)}{K_\alpha + \sqrt{\lambda} K_\beta} \qquad (48)$$

$$\lambda(\alpha, \beta, n) = \left(\frac{K_\alpha + \sqrt{2n}}{\sqrt{2n} - K_\beta} \right)^2 \qquad (49)$$

To compare the accuracy of this formula with the Tables [4] consider the following situation: if a decision $\sigma = \sigma_0$ is a serious error from the practical viewpoint when $\sigma = 1.500 \ \sigma_0$ and it is desired to keep the risk of such an error below .05 when the test is conducted at the 5-percent level of significance, how many d.o.f. will be needed for s^2? The formula (47) gives $n = 33.8$ and from [4] we find that 34 d.o.f. are needed. Table 5 presents the calculated values of n from (47) and table 6 presents the calculated values of λ from (49). For a comparative discussion on the use of these formulas in relation to others see next section.

7.2. An Alternative Formula for the Single Variance Test Based on the Distribution of Log s[8]

As pointed out by Bartlett and Kendall [11] the distribution of $\log s^2$ depends on σ^2 only through the term σ^2 in its expected value. Consequently the choice of the critical region based on the distribution of $\log s$ in place of s has obvious advantages. In this section we explore the possibility of using some formulas based on the distribution of $\log s$.

The cumulant function $K(t)$ of $\log s$ is given [11] by

$$K(t) = \frac{it}{2} \left(\log \sigma^2 - \log \frac{n}{2} \right) + \log \Gamma \left(\frac{n + it}{2} \right) - \log \Gamma \left(\frac{n}{2} \right)$$

which yields the following expressions for the first two cumulants

$$\kappa_1 = \frac{1}{2} \left(\log \sigma^2 - \log \frac{n}{2} \right) + \frac{1}{2} \Psi \left(\frac{n}{2} \right)$$

$$\kappa_2 = \frac{1}{4} \Psi' \left(\frac{n}{2} \right),$$

where $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$. The results of Section 3

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[7] Our $\lambda(\alpha, \beta, n)$ is equivalent to $\rho(\alpha, \beta, n)$ of [4].

[8] All logarithms are to the base e.

applied to log s yields

$$\kappa_2(n) = \left(\frac{\log \dfrac{\sigma}{\sigma_0}}{K_\alpha + K_\beta} \right)^2 \tag{50}$$

$$A = \frac{K_\beta \log \sigma_0 + K_\alpha \log \sqrt{\lambda}\, \sigma_0}{K_\alpha + K_\beta} + \frac{1}{2}\left(\Psi\left(\frac{n}{2}\right) - \log \frac{n}{2} \right) \tag{51}$$

$$\lambda(\alpha, \beta, n) = e^{2\sqrt{\kappa_2(n)}(K_\alpha + K_\beta)}. \tag{52}$$

These formulas assume no other approximation except that of normality and in that sense may be regarded as exact relative to (47), which assumes dual type of approximation for the distribution of s (see section 6.3). While the accuracy of (50) and (52) does not depend upon the accuracy with which we estimate κ_1, it does depend upon the complicated expression $\Psi'\left(\dfrac{n}{2}\right)$. For a given α, β, and λ the only way in which we can utilize (50) for finding the necessary sample size is to approximate the

asymptotic expansion of $\Psi'\left(\dfrac{n}{2}\right)$ [12].

As a first approximation if we set $\kappa_2 \cong 1/[2(n-1)]$ we obtain

$$n = 1 + 2\left(\frac{K_\alpha + K_\beta}{\log \lambda} \right)^2 \tag{53}$$

$$\log \lambda = \sqrt{\frac{2}{n-1}}\, (K_\alpha + K_\beta). \tag{54}$$

In tables 5a and 5b values of n from [4], and values calculated from (47) and (53) for different values of α, β, and λ are compared. In table 6 we give values of $\lambda(\alpha, \beta, n)$ to reemphasize the nature of approximations based on the distribution of s and log s. In this connection for the application of (52) values of $\kappa_2(n)$ were taken from [11].

It appears that for the customary values of the probabilities of errors of the two kinds $\alpha = \beta = .05$ and $\alpha = \beta = .01$, formulas (47) and (53) provide very good approximations to n for small sample sizes. If the percentage points of the χ^2-distribution are available, (47) is preferable because it is easier to compute n from (47) than from (53). Even for such a small value as $n = 5$, (47) errs on the safe side in this sense that it gives (at least for $\alpha = \beta$) a sample size which will be always sufficient to detect this difference. The formula (53) also shares this property with (47). In the absence of the percentage points of the χ^2-distribution it perhaps ought to be emphasized that on comparison of the critical regions for s and log s (cf. (48) and (51)) there is not much basis for choice. The choice of the critical region based on the distribution of log s has certain theoretical advantages, but the computation of the critical region is somewhat more complicated since it involves the approximation of $\Psi(n/2)$.

The effectiveness of formulas (47) and (53) varies when α and β are not equal. It appears (see table 5 and table 6) that for $\beta > \alpha$ it is safer to use (53) because it is always likely to err on the safe side in the sense of the previous paragraph. However, if $\beta < \alpha$ it appears that it is safer to use (47) because (53) is likely to give a value of n, which will actually be less than the necessary sample size.

7.3. Comparison of Two Population Variances

Let σ_1^2 and σ_2^2 denote the variances of the two normal populations and let s_1^2 and s_2^2 be their inde-

TABLE 5a. *Comparison of the sample sizes $(n = N-1)$ for the single variance test*

	$\alpha = .05$					
	$\beta = .05$			$\beta = .10$		
λ	(47)	(53)	Tabulated	(47)	(53)	Tabulated
2	46.0	46.1	46.4	34.9	36.7	36.0
2.5	26.7	26.8	26.6	20.0	21.4	20.5
3	18.8	18.9	18.8	13.9	15.2	14.5
3.5	14.7	14.8	14.6	10.8	11.9	11.1

TABLE 5b. *Comparison of the sample sizes $(n = N-1)$ for the single variance test*

	$\alpha = .01$					
	$\beta = .01$			$\beta = .05$		
λ	(47)	(53)	Tabulated	(47)	(53)	Tabulated
2	91.9	91.1	91.4	63.1	66.6	64.3
2.5	53.4	52.6	53.1	35.9	38.6	37.2
3	37.7	36.9	37.6	25.0	27.1	25.6
3.5	29.4	28.6	28.7	19.3	21.1	19.8

TABLE 6. *Comparison of the tabulated and calculated values of $\lambda\,(\alpha, \beta, n)$*

	$\alpha = .05$											
	$\beta = .25$				$\beta = .05$				$\beta = .01$			
n	(49)	(52)	(54)	Tabulated	(49)	(52)	(54)	Tabulated	(49)	(52)	(54)	Tabulated
5	3.734	5.073	5.155	4.139	10.037	10.01	10.239	9.664	33.065	16.14	16.577	19.972
10	2.595	2.977	2.984	2.717	4.681	4.700	4.715	4.646	8.126	6.476	6.501	7.156
15	2.199	2.401	2.403	2.265	3.454	3.463	3.468	3.442	5.109	4.479	4.487	4.780
20	1.990	2.122	2.122	2.033	2.900	2.906	2.907	2.895	3.973	3.625	3.627	3.802

pendent sample estimates based on n_1 and n_2 d.o.f. respectively. Our hypotheses are H_1: $\sigma_1^2 = \sigma_2^2$ and H_2: $\sigma_1^2 > \sigma_2^2$. The statistic $F = s_1^2/s_2^2$ is used to accept or reject H_1. For a given level of significance α, if $F \leq F_\alpha$ we accept H_1 and accept H_2 when $F > F_\alpha$. Let ϕ denote the true value of σ_1^2/σ_2^2. It has been shown in [2] and [4] that the probability of an error of the second kind will be exactly β if

$$\phi(\alpha, \beta, n_1, n_2) = \frac{F_\alpha(n_1, n_2)}{F_{1-\beta}(n_1, n_2)} = F_\alpha(n_1, n_2) F_\beta(n_2, n_1). \quad (55)$$

The operating characteristics (ϕ, β) for one-sided alternatives have been given in [2] and [4]. We shall develop here certain approximate formulas for ϕ in terms of α, β, n_1, and n_2.

By taking one-half the logarithm of (55) we obtain

$$\frac{1}{2} \log \phi = z_\alpha(n_1, n_2) + z_\beta(n_2, n_1) \quad (56)$$

where z_α denotes the α-probability point of Fisher's z-distribution. For purposes of approximation, (56) has decided advantage over (55) in that it enables us to make better use of the Cornish-Fisher approximation [13] since the z-distribution approaches normality relatively faster than the F-distribution. We shall employ the following approximations [14] for the cumulants of z:

(i) $\kappa_1 = \dfrac{1}{2} \left(\dfrac{1}{n_2} - \dfrac{1}{n_1} \right)$

(ii) $\kappa_2 = \dfrac{1}{2} \left(\dfrac{1}{n_1} + \dfrac{1}{n_2} \right)$

(iii) $\kappa_s = \dfrac{(s-2)!}{2} \left(\dfrac{1}{(n_2-1)^{s-1}} + \dfrac{(-1)^s}{(n_1-1)^{s-1}} \right) (s > 1)$.

Let z_1 and z_2 denote the mathematical expectation of z under H_1 and H_2, respectively. Consequently we have

$$z_1 \cong \frac{1}{2} \left(\frac{1}{n_2} - \frac{1}{n_1} \right)$$

$$z_2 \cong \frac{1}{2} \left(\frac{1}{n_2} - \frac{1}{n_1} \right) + \frac{1}{2} \log \phi$$

$$V(z) \cong \frac{1}{2} \left(\frac{1}{n_1} + \frac{1}{n_2} \right)$$

By applying the results of section 3 to z, it is found that

$$2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right) = \left(\frac{\log \phi}{K_\alpha + K_\beta} \right)^2, \quad (57)$$

which may be rewritten as

$$\log \phi = \frac{2(K_\alpha + K_\beta)}{\sqrt{h}}, \quad (58)$$

where $h = \dfrac{2 n_1 n_2}{n_1 + n_2}$. By using $\kappa_2 = \left(\dfrac{1}{n_2 - 1} + \dfrac{1}{n_1 - 1} \right)$ there results

$$2 \left(\frac{1}{n_1 - 1} + \frac{1}{n_2 - 1} \right) = \left(\frac{\log \phi}{K_\alpha + K_\beta} \right)^2 \quad (59)$$

Directly applying to (56) the Cornish-Fisher approximation [13] for the percentage points of the z-distribution in conjunction with Cochran's λ-corrections [15], we obtain (58) and the following two expressions for $\log \phi$:

$$\log \phi = 2 \left(\frac{K_\alpha}{\sqrt{h - \lambda_\alpha}} + \frac{K_\beta}{\sqrt{h - \lambda_\beta}} \right) \quad (60)$$

$$\log \phi = 2 \left(\frac{K_\alpha}{\sqrt{h - \lambda_\alpha}} + \frac{K_\beta}{\sqrt{h - \lambda_\beta}} \right) + \frac{1}{8} \left(\frac{1}{n_2} - \frac{1}{n_1} \right) (K_\alpha^2 - K_\beta^2) \quad (61)$$

where λ_ϵ is given by

ϵ	.75	.50	.25	.05	.01
λ_ϵ	.5758	.5000	.5758	.9509	1.4020

Formulas (58), (59), and (60) are not changed when n_1 and n_2 are interchanged. Formulas (60) and (61) are identical when either $\alpha = \beta$ or $n_1 = n_2$, or both.

TABLE 7A.[a] *Comparison of the calculated (formula 61) and tabulated values of φ (α, β, n_1, n_2)*

n_2 \ n_1	5	10	15	20	30	60
	$\alpha = .05$, $\beta = .05$					
5	26.312	15.674	13.082	11.918	10.844	9.839
	25.51	**15.75**	**13.40**	**12.36**	**11.39**	**10.49**
10	15.674	8.910	7.240	6.486	5.787	5.130
	15.75	**8.870**	**7.237**	**6.513**	**5.844**	**5.223**
15	13.082	7.240	5.787	5.130	4.516	3.937
	13.40	**7.237**	**5.777**	**5.128**	**4.527**	**3.967**
20	11.918	6.486	5.130	4.516	3.937	3.390
	12.36	**6.513**	**5.128**	**4.512**	**3.939**	**3.402**
30	10.844	5.787	4.516	3.937	3.390	2.866
	11.39	**5.844**	**4.527**	**3.939**	**3.389**	**2.869**
60	9.839	5.130	3.937	3.390	2.866	2.355
	10.49	**5.223**	**3.967**	**3.402**	**2.869**	**2.354**

[a] Figures in bold face type are taken from [4].

TABLE 7B.[a] *Comparison of calculated (formula 61) and tabulated values of φ (α, β, n_1, n_2)*

n_2 \ n_1	5	10	15	20	30	60
	$\alpha = .01$, $\beta = .01$					
5	135.047	57.720	43.294	37.428	32.304	27.816
	120.3	**56.65**	**44.29**	**39.19**	**34.69**	**30.72**
10	57.720	23.893	17.434	14.791	12.474	10.436
	56.65	**23.51**	**17.35**	**14.84**	**12.65**	**10.74**
15	43.294	17.434	12.474	10.436	8.650	7.082
	44.29	**17.34**	**12.41**	**10.41**	**8.679**	**7.168**
20	37.428	14.791	10.436	8.650	7.082	5.697
	39.19	**14.84**	**10.41**	**8.630**	**7.082**	**5.731**
30	32.304	12.474	8.650	7.082	5.697	4.471
	34.69	**12.65**	**8.679**	**7.082**	**5.693**	**4.479**
60	27.816	10.436	7.082	5.697	4.471	3.372
	30.72	**10.74**	**7.168**	**5.731**	**4.479**	**3.372**

[a] Figures in bold face type are taken from [4].

Comparison of calculated (formula 61) and tabulated values of φ $(\alpha, \beta, n_1, n_2)$

n_2 \ n_1	5	10	15	20	30	60
	$\alpha=.05, \beta=.25$					
5	9.742	7.371	6.674	6.338	6.012	5.693
	9.569	7.507	6.900	6.609	6.327	6.052
10	6.345	4.632	4.112	3.837	3.607	3.359
	6.285	4.620	4.123	3.882	3.646	3.614
15	5.464	3.192	3.432	3.195	2.959	2.722
	5.469	3.902	3.428	3.197	2.969	2.741
20	5.061	3.578	3.116	2.886	2.655	2.421
	5.102	3.577	3.113	2.885	2.658	2.429
30	4.682	3.264	2.815	2.590	2.361	2.125
	4.759	3.273	2.815	2.589	2.361	2.127
60	4.325	2.964	2.526	2.303	2.072	1.828
	4.439	2.987	2.533	2.306	2.073	1.828

TABLE 8. *Values of φ $(\alpha, \beta, n_1, n_2)$ from* $\log \varphi = (K_\alpha + K_\beta) \sqrt{2 \left(\frac{1}{n_1-1} + \frac{1}{n_2-1} \right)}$

n_2 \ n_1	5	10	15	20	30	60
	$\alpha=.05, \beta=.05$					
5	26.83	16.37	13.98	12.93	11.96	11.06
10	16.37	8.963	7.299	6.570	5.901	5.284
15	13.98	7.299	5.803	5.148	4.545	3.987
20	12.93	6.570	5.148	4.524	3.948	3.412
30	11.96	5.901	4.545	3.948	3.393	2.872
60	11.06	5.284	3.987	3.393	2.872	2.355
	$\alpha=.01, \beta=.01$					
5	104.9	52.15	41.70	37.33	33.43	29.95
10	52.15	22.24	16.63	14.33	12.31	10.53
15	41.70	16.63	12.02	10.15	8.511	7.072
20	37.33	14.33	10.15	8.455	6.973	5.673
30	33.43	12.31	8.511	6.973	5.629	4.447
60	29.95	10.53	7.072	5.673	4.447	3.358
	$\alpha=.05, \beta=.25$					
5	10.17	7.178	6.421	6.076	5.752	5.445
10	7.178	4.694	4.061	3.771	3.496	3.234
15	6.421	4.061	3.455	3.175	2.908	2.651
20	6.076	3.771	3.175	2.898	2.633	2.376
30	5.752	3.496	2.908	2.633	2.366	2.104
60	5.445	3.234	2.651	2.376	2.104	1.829

It appears on the basis of several computations (not given here) that formula (58) is likely to give values of ϕ that are much lower than its tabulated values [4], and consequently the sample sizes given by it will fall below the minimum desired. For $\alpha=\beta$ formula (61), then equivalent to (60), gives values of ϕ which are much closer to its tabulated values (table 7). This is also true of (61) for $\beta>\alpha$. If in addition to $\alpha=\beta$, $n_1=n_2=n$, then from (60)

$$n = \lambda\alpha + \Psi \left(\frac{K_\alpha + K_\beta}{\log \phi} \right)^2. \qquad (62)$$

We have not found any formula which will give an approximately correct answer for degrees of freedom as small as, say, $n_1=n_2=5$. The question of finding appropriate sample sizes for $n_1 \neq n_2$ cannot ordinarily be answered without the help of tables.

In such a situation, however, the experimentalist has no choice in the determination of n_1 and n_2. If it is decided in advance to maintain a certain ratio between n_1 and n_2 (this appears to be more often the case in practice) a formula would be more practical to use than the existing tables. Since formula (61) seems to be very complicated to use, we recommend the use of (59). Values of ϕ as given by this formula are given in table 8. It appears that for $\alpha=\beta=.05$, formula (59) will always give sufficient sample sizes; but for $\alpha=\beta=.01$, it will give values slightly less than actually needed.

8. References

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9. Appendix. Tabular summary of single-sampling formulas for testing hypotheses

Section in text	Form of population	H_1 null hypothesis	H_2 alternative hypothesis	Criterion for rejecting H_1	Formula for determining N	Formula for determining K_β	Notes
4.1	Single binomial, mean P Sample mean p	$P=P_1$	$P=P_2$ $(P_2>P_1)$	$\theta>2\dfrac{K_\beta\sin^{-1}\sqrt{P_1}+K_\alpha\sin^{-1}\sqrt{P_2}}{K_\alpha+K_\beta}$ where $\theta=2\sin^{-1}\sqrt{p}$	$N=\dfrac{1}{4}\left(\dfrac{K_\alpha+K_\beta}{\sin^{-1}\sqrt{P_2}-\sin^{-1}\sqrt{P_1}}\right)^2$	$K_\beta=2\sqrt{N}(\sin^{-1}\sqrt{P_2}-\sin^{-1}\sqrt{P_1})-K_\alpha$	
4.2	Two binomials, means P and P^* Sample means p and p^*	$P=P^*$	$P<P^*$	$\theta'>\dfrac{K_\alpha}{\sqrt{2N}}$ where $\theta'=\sin^{-1}\sqrt{p^*}-\sin^{-1}\sqrt{p}$	$N=\dfrac{1}{2}\left(\dfrac{K_\alpha+K_\beta}{\sin^{-1}\sqrt{P^*}-\sin^{-1}\sqrt{P}}\right)^2$	$K_\beta=\sqrt{2N}(\sin^{-1}\sqrt{P^*}-\sin^{-1}\sqrt{P})-K_\alpha$	
5.1	Single Poisson, mean m Sample mean $\bar{x}$	$m=m_1$	$m=m_2$ $(m_2>m_1)$	$\sqrt{\bar{x}}>\dfrac{K_\alpha\sqrt{m_2}+K_\beta\sqrt{m_1}}{K_\alpha+K_\beta}$	$N=\dfrac{1}{4}\left(\dfrac{K_\alpha+K_\beta}{\sqrt{m_2}-\sqrt{m_1}}\right)^2$	$K_\beta=2\sqrt{N}(\sqrt{m_2}-\sqrt{m_1})-K_\alpha$	
5.2	Two Poissons, means M and M^* Sample means $\bar{x}$ and $\bar{x}^*$	$M=M^*$	$M<M^*$	$\sqrt{\bar{x}^*}-\sqrt{\bar{x}}>\dfrac{K_\alpha}{\sqrt{2N}}$	$N=\dfrac{1}{2}\left(\dfrac{K_\alpha+K_\beta}{\sqrt{M^*}-\sqrt{M}}\right)^2$	$K_\beta=\sqrt{2N}(\sqrt{M^*}-\sqrt{M})-K_\alpha$	
6.1	Single normal, mean μ	$\mu=\mu_1$ $\sigma^2=\sigma_0^2$ (known)	$\mu=\mu_2$ $(\mu_2>\mu_1)$ $\sigma^2=\sigma_0^2$ (known)	$\bar{x}>\dfrac{K_\alpha\mu_2+K_\beta\mu_1}{K_\alpha+K_\beta}$	$N=\left(\dfrac{K_\alpha+K_\beta}{\Delta}\right)^2$	$K_\beta=\Delta\sqrt{N}-K_\alpha$	$\Delta=\dfrac{\mu_2-\mu_1}{\sigma_0}$
6.2	Two normals, means μ_1 and μ_2	$\mu_1=\mu_2$ $\sigma_2^2\neq\sigma_1^2$ (both known)	$\mu_1<\mu_2$ $\sigma_2^2\neq\sigma_1^2$ (both known)	$\bar{x}_2-\bar{x}_1>\dfrac{K_\alpha(\mu_2-\mu_1)}{K_\alpha+K_\beta}$	$N=\left(\dfrac{K_\alpha+K_\beta}{d}\right)^2$	$K_\beta=\sqrt{N}d-K_\alpha$	$d=\dfrac{\mu_2-\mu_1}{\sqrt{\sigma_1^2+\sigma_2^2}}$
6.2	Two normals, means μ_1 and μ_2	$\mu_1=\mu_2$ $\sigma^2=\sigma_1^2=\sigma_2^2$ (known)	$\mu_1<\mu_2$ $\sigma^2=\sigma_1^2=\sigma_2^2$ (known)	$\bar{x}_2-\bar{x}_1>\dfrac{K_\alpha(\mu_2-\mu_1)}{K_\alpha+K_\beta}$	$N=2\left(\dfrac{K_\alpha+K_\beta}{\Delta}\right)^2$	$K_\beta=\Delta\sqrt{\dfrac{N}{2}}-K_\alpha$	$\Delta=\dfrac{\mu_2-\mu_1}{\sigma}$
6.3	Single normal, mean μ; variance σ^2 $s^2=$estimate of σ^2 (unknown)	$\mu=\mu_1$	$\mu=\mu_2$ $(\mu_2>\mu_1)$	$(\bar{x}-\mu_1)-k\dfrac{s}{\sqrt{N}}>0,$ where $k=\dfrac{K_\alpha\Delta\sqrt{N}}{K_\alpha+K_\beta}$	$N=\dfrac{b+\sqrt{b^2-4a}}{2a},$ or $N=\dfrac{a+1+\sqrt{(a-1)^2+2at}}{2a}$	$K_\beta=\dfrac{\Delta}{\sqrt{N-1}}\left[N^2-N\left(1+\dfrac{K_\alpha^2}{2}\right)\right]^{1/2}-K_\alpha$	$a=\left(\dfrac{\Delta}{K_\alpha+K_\beta}\right)^2$ $b=1+a\left(1+\dfrac{K_\alpha^2}{2}\right)$
6.4	Two normals, means μ_1 and μ_2	$\mu_1=\mu_2$	$\mu_1<\mu_2$	$(\bar{x}_2-\bar{x}_1)-k_1\sqrt{\dfrac{2}{N}}s>0,$ where $k_1=\sqrt{\dfrac{N}{2}}\cdot\dfrac{K_\alpha\Delta}{K_\alpha+K_\beta}$	$N=\dfrac{b_1+\sqrt{b_1^2-8a}}{2a},$ or $N=\dfrac{a+2+\sqrt{(a-2)^2+2at\frac{2}{\alpha}(n)}}{2a}$	$K_\beta=\Delta\left[\dfrac{N}{2}\left(1-\dfrac{K_\alpha^2}{4(N-1)}\right)\right]^{1/2}-K_\alpha$	$a=\left(\dfrac{\Delta}{K_\alpha+K_\beta}\right)^2$ $b_1=2+a\left(1+\dfrac{K_\alpha^2}{4}\right)$
7.1	Single normal, variance σ^2 $s^2=$estimate of σ^2	$\sigma^2=\sigma_0^2$	$\sigma^2=\lambda\sigma_0^2$ $(\lambda>1)$	$s>\dfrac{\sqrt{\lambda}\,\sigma_0(K_\alpha+K_\beta)}{K_\alpha+\sqrt{\lambda}\,K_\alpha}$	$n=N-1=\dfrac{1}{2}\left(\dfrac{K_\alpha+K_\beta\sqrt{\lambda}}{\sqrt{\lambda}-1}\right)^2$	$\lambda(\alpha,\beta,n)=\left(\dfrac{K_\alpha+\sqrt{2n}}{\sqrt{2n}-K_\beta}\right)^2$	$n=N-1$ $\lambda=\dfrac{\sigma^2}{\sigma_0^2}$
7.2	do	do	do	$\log s>\dfrac{K_\beta\log\sigma_0+K_\alpha\log\sqrt{\lambda}\,\sigma_0}{K_\rho+K_\beta}+\dfrac{1}{2}\left(\psi\left(\dfrac{n}{2}\right)-\log\dfrac{n}{2}\right)$	$\kappa_2(n)=\left(\dfrac{\log\frac{\sigma}{\sigma_0}}{K_\alpha+K_\beta}\right)^2$	$\lambda(\alpha,\beta,n)=e^{2\sqrt{\kappa_2'(n)}\,(K_\alpha+K_\beta)}$	
7.2	do	do	do	do	$n=1+2\left(\dfrac{K_\alpha+K_\beta}{\log\lambda}\right)^2$	$\log\lambda=\sqrt{\dfrac{2}{n-1}}\,(K_\alpha+K_\beta)$	
7.3	Two normals, variances σ_1^2 and σ_2^2 s_1^2 and $s_2^2=$estimates of σ_1^2 and σ_2^2	$\sigma_1^2=\sigma_2^2$	$\sigma_1^2>\sigma_2^2$	$\log\dfrac{s_1^2}{s_2^2}>\dfrac{1}{n_2}-\dfrac{1}{n_1}+\dfrac{K_\alpha}{K_\alpha+K_\beta}\log\phi$	$\log\phi=(K_\alpha+K_\beta)\times\left[2\left(\dfrac{1}{n_1-1}+\dfrac{1}{n_2-1}\right)\right]^{1/2}$		$\phi=\dfrac{\sigma_1^2}{\sigma_2^2}$
7.3	do	do	do	do	$\log\phi=2\left(\dfrac{K_\alpha}{\sqrt{h-\lambda_\alpha}}+\dfrac{K_\beta}{\sqrt{h-\lambda_\beta}}\right)$		$h=\dfrac{2n_1n_2}{n_1+n}$
7.3	do	do	do	do	$\log\phi=2\left(\dfrac{K_\alpha}{\sqrt{h-\lambda_\alpha}}+\dfrac{K_\beta}{\sqrt{h-\lambda_\alpha}}\right)+\dfrac{1}{3}\left(\dfrac{1}{n_2}-\dfrac{1}{n_1}\right)K_\alpha^2-K_\beta^2$		