Tables of the Functions $\int_0^\phi \sin^{1/3} x\,dx$ and $(4/3)\sin^{-4/3}\phi\int_0^\phi \sin^{1/3} x\,dx$

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This article concerns itself with the tabulation of the functions $\int_0^\phi \sin^{1/3} x\,dx$ and $4/3 \sin^{-4/3}\phi \int_0^\phi \sin^{1/3} x\,dx$, which are important in the theory of condensation of vapor at rest on cylindric surfaces. The series expansion for the latter function is given, and certain functional relations useful for interpolation are mentioned. Instructions for interpolation with Lagrangian interpolations are given for the user.

This article concerns itself with the tabulation of the functions

$$I_1(\phi)=\int_0^\phi \sin^{1/3} x dx, \qquad \phi=0°(1°)90°, \qquad (1)$$

$$I_2(\phi)=(4/3)\sin^{-4/3}\phi\int_0^\phi \sin^{1/3} x dx, \qquad \phi=0°(30')180°, \qquad (2)$$

which are important in the theory of condensation of vapor at rest on cylindric surfaces.[1 2]

The integral $I_1(\phi)$ may be expressed in terms of the Incomplete Beta function, namely $I_1(\phi)=\frac{1}{2}B_{\sqrt{\sin\phi}}(\frac{2}{3},\frac{1}{2})$. However, the existing tables of the Incomplete Beta function are not convenient for interpolation.

When $\phi=\pi/2$, we have

$$I_1(90°)=\int_0^{\pi/2}\sin^{1/3} x dx=\frac{\Gamma(2/3)\Gamma(3/2)}{\Gamma(7/6)}$$
$$=1.2935\,5478 \quad \text{(to 8 decimals).} \qquad (3)$$

Integrating by parts, one obtains for $I_2(\phi)$ the expression

$$I_2(\phi)=\sec\phi\left[1-1\left(\frac{3}{10}\right)\tan^2\phi+1.3\left(\frac{3}{10}\right)\left(\frac{3}{16}\right)\tan^4\phi\right.$$
$$\left.-1.3.5\left(\frac{3}{10}\right)\left(\frac{3}{16}\right)\left(\frac{3}{22}\right)\tan^6\phi+\ldots+R_n\right], \qquad (4)$$

where R_n is the remainder after n terms and is given by

$$R_n=\frac{(-1)^n3^{n-1}[1.3.5\ldots(2n-1)]}{10.16.22\ldots(6n-2)}$$
$$\cdot[\cos\phi\sin^{-4/3}\phi]\int_0^\phi\tan^{2n}x\sin^{1/3}x dx.$$

It may be shown that if $n\to\infty$, the resulting infinite series converges for $\phi\leq\pi/4$. When $\phi<\pi/4$ the convergence follows from the ordinary ratio test and when $\phi=\pi/4$ by application of Raabe's test since

$$\lim_{n\to\infty} n\{|U_n|/|U_{n+1}|-1\}=7/6, \text{ where } |U_n|$$
$$=\frac{1.3.5\ldots(2n-3)3^{n-1}}{10.16\ldots(6n-2)}.$$

Furthermore, it is clear that $|R_n|\to 0$ as $n\to\infty$; since

$$|R_n|<(2n-1)|U_n|\cos\phi\sin^{-4/3}\phi\int_0^{\pi/4}\tan^{2n}x dx$$
$$=(2n-1)|U_n|\cos\phi\sin^{-4/3}\phi\int_0^1\frac{t^{2n}}{1+t^2}dt$$

(setting $\tan x=t$), and therefore

$$|R_n|<(2n-1)|U_n|\cos\phi\sin^{-4/3}\phi\int_0^1 t^{2n}dt$$
$$<|U_n|\cos\phi\sin^{-4/3}\phi.$$

Thus if $0<\phi\leq\pi/4$, $R_n\to 0$, since $|U_n|\to 0$; establishing the fact that (4) represents $I_2(\phi)$ for $0<\phi\leq\pi/4$.

The function $I_1(\phi)$ has been tabulated only in the range from 0° to 90°, since the values for $90°\leq\phi\leq 180°$ may be obtained from the relation

$$I_1(\phi)=2I_1(90°)-I_1(180-\phi). \qquad (5)$$

The corresponding relation for $I_2(\phi)$ is

$$I_2(\phi)=2I_2(90°)\sin^{-4/3}\phi-I_2(180°-\phi). \qquad (6)$$

The table of $I_1(\phi)$ in the range from 0° to 20° was obtained with the aid of the expression for $I_2(\phi)$ given in (4) and multiplication by the factor $(3/4)\sin^{4/3}\phi$. From 20° to 90° the values were obtained by numerical integration. The function $I_2(\phi)$ in the range from 20° to 90° was obtained from the previously computed values of $I_1(\phi)$ and multiplication

[1] L. M. K. Boelter and others, Heat transfer notes (Univ. California Press, Berkeley, Calif., 1948).

[2] M. Jakob, Heat transfer 1 (John Wiley & Sons, Inc., New York, N. Y., 1949).

by the factor $(4/3)\sin^{-4/3}\phi$. The values at intervals of 30′ were obtained by interpolation. The relation (6) was then employed to extend the table to 180°.

The following information has been given so that interpolation with tables of Lagrangian interpolation coefficients [3] will yield the full accuracy provided by the table.

In the case of $I_1(\phi)$ in the range from 0° to 15° it will be advisable to interpolate in the table of $I_2(\phi)$ and multiply by $(3/4)\sin^{4/3}\phi$. From 15° to 90° the five-point Lagrangian interpolation formula will be adequate. In the case of $I_2(\phi)$, the three-point Lagrangian formula may be used from 0° to 15°, the four-point formula from 15° to 30°, the five-point formula from 30° to 135°, and the six-point formula from 135° to 165°. In the range from 165° to 180° the relation (6) given above should be used.

$$I_1(\phi)=\int_0^{\phi} sin^{\frac{1}{3}}x\,dx$$

$\phi°$	$I_1(\phi)$	$\phi°$	$I_1(\phi)$
0	0.0000 0000	45	0.5360 1006
1	.0033 9543	46	.5516 0397
2	.0085 5577	47	.5672 8574
3	.0146 9038	48	.5830 5283
4	.0215 5745	49	.5989 0273
5	.0290 2580	50	.6148 3300
6	.0370 1067	51	.6308 4124
7	.0454 5180	52	.6469 2507
8	.0543 0371	53	.6630 8218
9	.0635 3059	54	.6793 1028
10	.0731 0330	55	.6956 0713
11	.0829 9758	56	.7119 7050
12	.0931 9282	57	.7283 9822
13	.1036 7125	58	.7448 8812
14	.1144 1731	59	.7614 3808
15	.1254 1726	60	.7780 4600
16	.1366 5880	61	.7947 0980
17	.1481 3088	62	.8114 2743
18	.1598 2346	63	.8281 9686
19	.1717 2738	64	.8450 1607
20	.1838 3423	65	.8618 8308
21	.1961 3621	66	.8787 9590
22	.2086 2613	67	.8957 5260
23	.2212 9724	68	.9127 5122
24	.2341 4327	69	.9297 8985
25	.2471 5831	70	.9468 6658
26	.2603 3677	71	.9639 7950
27	.2736 7341	72	.9811 2674
28	.2871 6324	73	.9983 0643
29	.3008 0150	74	1.0155 1670
30	.3145 8368	75	1.0327 5571
31	.3285 0547	76	1.0500 2161
32	.3425 6272	77	1.0673 1258
33	.3567 5149	78	1.0846 2679
34	.3710 6796	79	1.1019 6242
35	.3855 08+6	80	1.1193 1767
36	.4000 6946	81	1.1366 9074
37	.4147 4754	82	1.1540 7981
38	.4295 3939	83	1.1714 8312
39	.4444 4181	84	1.1888 9886
40	.4594 5169	85	1.2063 2525
41	.4745 6601	86	1.2237 6052
42	.4897 8183	87	1.2412 0288
43	.5050 9630	88	1.2586 5056
44	.5205 0662	89	1.2761 0178
45	.5360 1006	90	1.2935 5478

[3] National Bureau of Standards, Tables of Lagrangian interpolation coefficients (Columbia University Press, New York, N. Y., 1944)

$$I_2(\phi)=(4/3)\,sin^{-4/3}\phi\int_0^{\phi} sin^{1/3}x\,dx$$

$\phi°$	$I_2(\phi)$	$\phi°$	$I_2(\phi)$	$\phi°$	$I_2(\phi)$
0 00	1.0000 0000	45 00	1.1344 8391	90 00	1.7247 397
30	1.0000 1523	30	1.1377 6325	30	1.7364 633
1 00	1.0000 6093	46 00	1.1410 9202	91 00	1.7483 654
30	1.0001 3709	30	1.1444 7070	30	1.7604 493
2 00	1.0002 4373	47 00	1.1478 9982	92 00	1.7727 184
30	1.0003 8087	30	1.1513 7990	30	1.7851 763
3 00	1.0005 4851	48 00	1.1549 1149	93 00	1.7978 266
30	1.0007 4669	30	1.1584 9511	30	1.8106 731
4 00	1.0009 7542	49 00	1.1621 3134	94 00	1.8237 196
30	1.0012 3473	30	1.1658 2075	30	1.8369 700
5 00	1.0015 2466	50 00	1.1695 6390	95 00	1.8504 284
30	1.0018 4524	30	1.1733 6139	30	1.8640 988
6 00	1.0021 9651	51 00	1.1772 1382	96 00	1.8779 855
30	1.0025 7851	30	1.1811 2181	30	1.8920 928
7 00	1.0029 9130	52 00	1.1850 8597	97 00	1.9064 253
30	1.0034 3491	30	1.1891 0696	30	1.9209 875
8 00	1.0039 0942	53 00	1.1931 8542	98 00	1.9357 841
30	1.0044 1487	30	1.1973 2201	30	1.9508 199
9 00	1.0049 5134	54 00	1.2015 1740	99 00	1.9661 000
30	1.0055 1888	30	1.2057 7230	30	1.9816 294
10 00	1.0061 1757	55 00	1.2100 8739	100 00	1.9974 133
30	1.0067 4748	30	1.2144 6340	30	2.0134 572
11 00	1.0074 0870	56 00	1.2189 0106	101 00	2.0297 667
30	1.0081 0130	30	1.2234 0110	30	2.0463 473
12 00	1.0088 2537	57 00	1.2279 6430	102 00	2.0632 050
30	1.0095 8100	30	1.2325 9143	30	2.0803 458
13 00	1.0103 6829	58 00	1.2372 8327	103 00	2.0977 758
30	1.0111 8733	30	1.2420 4064	30	2.1155 015
14 00	1.0120 3823	59 00	1.2468 6434	104 00	2.1335 294
30	1.0129 2110	30	1.2517 5523	30	2.1518 662
15 00	1.0138 3605	60 00	1.2567 1415	105 00	2.1705 190
30	1.0147 8319	30	1.2617 4198	30	2.1894 948
16 00	1.0157 6265	61 00	1.2668 3960	106 00	2.2088 011
30	1.0167 7454	30	1.2720 0792	30	2.2284 453
17 00	1.0178 1899	62 00	1.2772 4787	107 00	2.2484 353
30	1.0188 9615	30	1.2825 6039	30	2.2687 793
18 00	1.0200 0614	63 00	1.2879 4644	108 00	2.2894 853
30	1.0211 4911	30	1.2934 0701	30	2.3105 621
19 00	1.0223 2521	64 00	1.2989 4308	109 00	2.3320 184
30	1.0235 3458	30	1.3045 5570	30	2.3538 633
20 00	1.0247 7737	65 00	1.3102 4589	110 00	2.3761 061
30	1.0260 5376	30	1.3160 1473	30	2.3987 566
21 00	1.0273 6390	66 00	1.3218 6330	111 00	2.4218 248
30	1.0287 0797	30	1.3277 9270	30	2.4453 208
22 00	1.0300 8613	67 00	1.3338 0407	112 00	2.4692 553
30	1.0314 9857	30	1.3398 9855	30	2.4936 394
23 00	1.0329 4546	68 00	1.3460 7734	113 00	2.5184 842
30	1.0344 2700	30	1.3523 4162	30	2.5438 016
24 00	1.0359 4339	69 00	1.3586 9264	114 00	2.5696 036
30	1.0374 9481	30	1.3651 3163	30	2.5959 027
25 00	1.0390 8147	70 00	1.3716 5988	115 00	2.6227 118
30	1.0407 0359	30	1.3782 7869	30	2.6500 443
26 00	1.0423 6137	71 00	1.3849 8940	116 00	2.6779 139
30	1.0440 5503	30	1.3917 9336	30	2.7063 350
27 00	1.0457 8481	72 00	1.3986 9197	117 00	2.7353 223
30	1.0475 5091	30	1.4056 8665	30	2.7648 912
28 00	1.0493 5359	73 00	1.4127 7885	118 00	2.7950 574
30	1.0511 9309	30	1.4199 7004	30	2.8258 373
29 00	1.0530 6964	74 00	1.4272 6174	119 00	2.8572 480
30	1.0549 8350	30	1.4346 5550	30	2.8893 070
30 00	1.0569 3493	75 00	1.4421 5289	120 00	2.9220 326
30	1.0589 2419	30	1.4497 5552	30	2.9554 436
31 00	1.0609 5156	76 00	1.4574 6505	121 00	2.9895 598
30	1.0630 1730	30	1.4652 8316	30	3.0244 013
32 00	1.0651 2169	77 00	1.4732 1157	122 00	3.0599 894
30	1.0672 6503	30	1.4812 5204	30	3.0963 459
33 00	1.0694 4761	78 00	1.4894 0636	123 00	3.1334 936
30	1.0716 6973	30	1.4976 7638	30	3.1714 561
34 00	1.0739 3169	79 00	1.5060 6397	124 00	3.2102 580
30	1.0762 3381	30	1.5145 7106	30	3.2499 247
35 00	1.0785 7641	80 00	1.5231 9961	125 00	3.2904 827
30	1.0809 5981	30	1.5319 5162	30	3.3319 597
36 00	1.0833 8435	81 00	1.5408 2915	126 00	3.3743.844
30	1.0858 5036	30	1.5498 3431	30	3.4177 866
37 00	1.0883 5820	82 00	1.5589 6925	127 00	3.4621 974
30	1.0909 0821	30	1.5682 3615	30	3.5076 492
38 00	1.0935 0076	83 00	1.5776 3727	128 00	3.5541 758
30	1.0961 3623	30	1.5871 7492	30	3.6018 123
39 00	1.0988 1497	84 00	1.5968 5144	129 00	3.6505 954
30	1.1015 3739	30	1.6066 6925	30	3.7005 633
40 00	1.1043 0386	85 00	1.6166 3082	130 00	3.7517 561
30	1.1071 1480	30	1.6267 3867	30	3.8042 153
41 00	1.1099 7060	86 00	1.6369 9538	131 00	3.8579 845
30	1.1128 7168	30	1.6474 0361	30	3.9131 093
42 00	1.1158 1848	87 00	1.6579 6606	132 00	3.9696 371
30	1.1188 1141	30	1.6686 8551	30	4.0276 177
43 00	1.1218 5092	88 00	1.6795 6480	133 00	4.0871 032
30	1.1249 3746	30	1.6906 0684	30	4.1481 481
44 00	1.1280 7150	89 00	1.7018 1463	134 00	4.2108 094
30	1.1312 5349	30	1.7131 9120	30	4.2751 469
45 00	1,1344 8391	90 00	1.7247 3971	135 00	4.3412 233

$I_2(\phi) = (4/3) sin^{-4/3}\, \phi \int_0^{\phi} sin^{1/3}\, x\, dx$—*Continued*

$\phi°$	$I_2(\phi)$	$\phi°$	$I_2(\phi)$	$\phi°$	$I_2(\phi)$
135 00	4. 3412 233	150 00	7. 6352 085	165 00	19. 8995 93
30	4. 4091 045	30	7. 8159 323	30	20. 8455 96
136 00	4. 4788 593	151 00	8. 0040 878	166 00	21. 8713 32
30	4. 5505 602	30	8. 2001 124	30	22. 9867 12
137 00	4. 6242 835	152 00	8. 4044 774	167 00	24. 2033 08
30	4. 7001 092	30	8. 6176 916	30	25. 5347 16
138 00	4. 7781 213	153 00	8. 8403 047	168 00	26. 9969 98
30	4. 8584 087	30	9. 0729 114	30	28. 6092 68
139 00	4. 9410 644	154 00	9. 3161 562	169 00	30. 3944 31
30	5. 0261 870	30	9. 5707 389	30	32. 3801 58
140 00	5. 1138 800	155 00	9. 8374 197	170 00	34. 6001 61
30	5. 2042 529	30	10. 1170 27	30	37. 0959 09
141 00	5. 2974 211	156 00	10. 4104 63	171 00	39. 9189 44
30	5. 3935 067	30	10. 7187 15	30	43. 1340 72
142 00	5. 4926 386	157 00	11. 0428 63	172 00	46. 8238 26
30	5. 5949 535	30	11. 3840 91	30	51. 0948 39
143 00	5. 7005 957	158 00	11. 7437 01	173 00	56. 0871 25
30	5. 8097 184	30	12. 1231 26	30	61. 9879 45
144 00	5. 9224 841	159 00	12. 5239 48	174 00	69. 0530 72
30	6. 0390 650	30	12. 9479 17	30	77. 6404 89
145 00	6. 1596 443	160 00	13. 3969 73	175 00	88. 2657 37
30	6. 2844 165	30	13. 8732 73	30	101. 6968 9
146 00	6. 4135 888	161 00	14. 3792 21	176 00	119. 1260 9
30	6. 5473 815	30	14. 9175 04	30	142. 4996 8
147 00	6. 6860 298	162 00	15. 4911 34	177 00	175. 2051 7
30	6. 8297 844	30	16. 1034 98	30	223. 6536 1
148 00	6. 9789 132	163 00	16. 7584 16	178 00	301. 4551 5
30	7. 1337 025	30	17. 4602 12	30	442. 8076 0
149 00	7. 2944 586	164 00	18. 2138 02	179 00	760. 9850 1
30	7. 4615 100	30	19. 0247 90	30	1918. 9846
150 00	7. 6352 085	165 00	19. 8995 93	180 00	∞

WASHINGTON, November 1, 1950.