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Maximum Likelihood Estimates of Position Derived From Measurements Performed by Hyperbolic Instruments[1]

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Certain electronic surveying instruments—referred to as hyperbolic instruments—determine the difference between the distances from two fixed stations to a moving object. Data obtained from a single observation yield one relation between the three coordinates of the moving object and restrict it to a hyperboloid of revolution. To locate a position in space, observations from at least three pairs of stations are needed. If more than three observations are taken, more than three equations for the three coordinates are obtained. In general these equations will not be compatible; it is then a statistical problem to find an estimate for the unknown position. In this paper maximum likelihood estimates are obtained for positions in space derived from such hyperbolic data.

1. Introduction

Several electronic surveying instruments have been developed in recent years. These instruments utilize the Doppler effect [2, 3] and are designed to locate the position of a moving object. In this paper we are considering measurements obtained by using certain types of these instruments, which are often referred to as hyperbolic instruments. Hyperbolic instruments, such as Raydist (see footnote 2), determine the difference between the distances from two stations to a moving object. Data obtainable from one observation (using one pair of stations) yield one relation among the three coordinates of the moving object. To locate a position in space from hyperbolic data, at least three observations are needed. If more than three observations are taken, more than three equations for the three coordinates are obtained. In general, these equations will not be compatible; it is then a statistical problem to find an estimate for the unknown position. The use of exactly three observations is justified if it can be assumed that the data are not affected by any observational errors. In general, this assumption is not realistic and should not be made without a thorough investigation. It appears desirable, therefore, to use more than the minimum number of observations and to estimate the location of the moving object by statistical methods. The purpose of this paper is to obtain maximum likelihood estimates for positions in space derived from hyperbolic data.

2. Formulation of the Assumptions and the Problem

We denote by $N \geq 4$ the number of stations $S_\alpha(\alpha=1,2, \ldots N)$ and by $(a_{\alpha 1},a_{\alpha 2},a_{\alpha 3})$ the known coordinates of the station S_α. Let P be the unknown position of the moving object and $(\theta_1, \theta_2, \theta_3)$ be the coordinates of P. A single observation based on the stations S_α and S_β gives then a determination of the distance $\overline{PS}_\alpha - \overline{PS}_\beta$. If N stations are available it is possible to make at most

$$\binom{N}{2} = \frac{N(N-1)}{2}$$

observations based on different pairs of stations. However, not all these observations will be used in practice. Let $n \leqq [N(N-1)]/2$ be the number of observations actually taken. We make the following assumptions:

(I) The hyperbolic measurements $z_{\alpha\beta}$ are observations on n independent random variables $Z_{\alpha\beta}$. The variates $Z_{\alpha\beta}$ have rectangular distributions with mean $\mu_{\alpha\beta}$ and common range $2r$. Further $Z_{\beta\alpha} = -Z_{\alpha\beta}$ and $z_{\beta\alpha} = -z_{\alpha\beta}$.

(II) The parameters of the n rectangular distributions may be expressed in terms of the coordinates of the unknown point by means of the formula

$$E(Z_{\alpha\beta}) = \mu_{\alpha\beta} = \sqrt{\sum_{i=1}^{3}(\theta_i - a_{\alpha i})^2} - \sqrt{\sum_{i=1}^{3}(\theta_i - a_{\beta i})^2}. \tag{1}$$

(III) The distances d_α and d_β from the position of the moving object at time $t=0$ to the stations S_α and S_β, respectively, satisfy the inequality

$$d_\alpha - d_\beta > r. \tag{2}$$

The nature of the problem, as well as its solution are to a great extent determined by assumption (I). To justify this assumption it is necessary to describe briefly how hyperbolic data are obtained.

It is desired to determine the instantaneous location of a moving object. A transmitter is carried in the object and emits a continuous signal of frequency f_0. A second transmitter (the reference transmitter) emits a signal of a frequency f'_0, slightly different from f_0. Each station receives both signals simultaneously. A certain beat frequency n_α is

[1] An unclassified treatment of the mathematical portion of a classified Navord-report written by the author while a member of the staff of the U. S. Naval Ordnance Test Station, Inyokern, Calif.

[2] J. F. McAllister, Measuring velocity of V-2 rockets by Doppler effect, Tele-Tech. **6**, No. 2, p. 56 to 59 and 129 (1947).

[3] Charles E. Hastings, Raydist—a radio navigation and tracking system, Tele-Tech. **6**, No. 6, p. 30 to 33 and 100 to 103 (June 1947).

thus associated with each station S_α; n_α depends on the velocity component of the object away from S_α and on the frequencies f_0 and f'_0. To obtain an observation two stations S_α and S_β have to be linked by wire or by radio The frequencies n_α and n_β give then a beat frequency; the quantity recorded is the number of cycles $n_{\alpha\beta}$ of the beat frequency signal occurring during the time of observation. It can be assumed that the observation $z_{\alpha\beta}$ on $Z_{\alpha\beta}$ can be satisfactorily approximated by a linear function of $n_{\alpha\beta}$ given by

$$z_{\alpha\beta}=\frac{c}{f_0}n_{\alpha\beta}+d_\alpha-d_\beta. \tag{3}$$

In this formula c is the speed of light. The frequency $n_{\alpha\beta}$ may be recorded continuously to permit reading $n_{\alpha\beta}$ to fractions of a cycle. Alternatively it is possible to devise a counter that registers full cycles only.[4] In either case it is easily possible to determine a range for the reading error of $n_{\alpha\beta}$ (and therefore also of $z_{\alpha\beta}$). Moreover, the recording devices are of such a nature that any error within the possible range is equally likely to occur. It seems, therefore, quite realistic to assume that the errors have a rectangular distribution about zero with a certain range. This, however, is equivalent to assumption (I). Equation (2) is a condition imposed on the location of the stations. As the half range r is rather small (2) implies hardly more than $d_\alpha>d_\beta$. This consideration shows that (2) is not a very serious restriction. The aim of this paper is the determination of a maximum likelihood estimate of the unknown position. That is, the unknown coordinates θ_1, θ_2, θ_3 have to be determined so that the greatest possible probability is given to the actual observations. This discussion will be based on assumptions (I), (II), and (III).

3. The Maximum Likelihood Estimate

Let $f_{\alpha\beta}(z)$ be the frequency function of the random variable $Z_{\alpha\beta}$. According to assumption (I)

$$f_{\alpha\beta}(z)=\begin{cases}\dfrac{1}{2r} \text{ if } \mu_{\alpha\beta}-r\leqq z\leqq\mu_{\alpha\beta}+r.\\ 0 \text{ otherwise}\end{cases}$$

The likelihood function[5] assumes, therefore, also only two values, the value $(1/2r)^n$ and the value zero. If one wants to determine the unknown parameters θ_1, θ_2, θ_3 so that the greatest possible probability is given to the actual observations, one has only to chose the parameters so that the likelihood function is not zero. This is the case if and only if each observation $z_{\alpha\beta}$ falls within the range of $Z_{\alpha\beta}$.
That is, the likelihood condition

$$\mu_{\alpha\beta}-r\leqq z_{\alpha\beta}\leqq\mu_{\alpha\beta}+r, \tag{4}$$

must be satisfied for the n pairs (α,β) used. From (1) it is seen that the likelihood condition (4) consists of a set of inequalities involving θ_1, θ_2, θ_3. A point $(\theta_1, \theta_2, \theta_3)$ of the parameter space is said to be a maximum likelihood estimate if its coordinates satisfy the maximum likelihood condition. In general, inequalities (4) will not have a unique solution. Either all the points of a certain region in the parameter space will satisfy (4), or it may happen that no maximum likelihood estimate exists. If maximum likelihood estimates exist, one may consider the set of all the maximum likelihood estimates. This set will be called the maximum likelihood region. Any point in the interior of the maximum likelihood region may be used as an estimate for the unknown position. The most important problem is the determination of the maximum likelihood region. This problem will be solved by expressing the boundaries of this region in terms of the actual observations. This requires certain transformations of the likelihood condition (4). If we substitute the value of $\mu_{\alpha\beta}$ from (1) and transpose we obtain

$$z_{\alpha\beta}-r\leqq\sqrt{\sum_{i=1}^{3}(\theta_i-a_{\alpha i})^2}-\sqrt{\sum_{i=1}^{3}(\theta_i-a_{\beta i})^2}\leqq z_{\alpha\beta}+r \tag{4a}$$

We next introduce the following notation

$$\left.\begin{aligned}n_0&=-\frac{f_0}{c}(d_\alpha-d_\beta-r)\\ n_1&=-\frac{f_0}{c}(d_\alpha-d_\beta+r)\end{aligned}\right\} \tag{4b}$$

From (4a) we see $n_1=n_0-2rf_0/c<n_0$.
In the following we distinguish three cases, namely,

Case (A): $n_{\alpha\beta}>n_0$
Case (B): $n_{\alpha\beta}<n_1$
Case (C): $n_1\leqq n_{\alpha\beta}\leqq n_0$.

From (3) it is seen that

$$\left.\begin{array}{l}z_{\alpha\beta}-r>0 \text{ and a fortiori } z_{\alpha\beta}+r>0 \text{ in case (A)}\\ z_{\alpha\beta}+r<0 \text{ and a fortiori } z_{\alpha\beta}-r<0 \text{ in case (B)}\\ z_{\alpha\beta}+r\geqq 0\geqq z_{\alpha\beta}-r \text{ in case (C).}\end{array}\right\} \tag{5}$$

We first consider the conditions imposed by a single observation $z_{\alpha\beta}$ based on an arbitrary pair of stations. It is no loss in generality if we assume $\alpha=1$ and $\beta=2$. To simplify the discussion we choose an appropriate system of coordinates. Let the position of the moving object at time $t=0$ be the origin O and choose the θ_1-axis so as to be parallel to the line joining S_1 and S_2 the θ_2-axis should be in the plane OS_1S_2 and per-

[4] Hastings Bulletin R–23, Complete Raydist tracking system for missiles (Sept. 1947).
[5] A description of the maximum likelihood method and a description of the likelihood function may be found in, H. Cramér, Mathematical Methods of Statistics, p. 498 (Princeton Univ. Press, Princeton, 1946).

pendicular to S_1S_2.[6] The θ_3-axis is perpendicular to the $\theta_1\theta_2$-plane; finally some orientation is given to the axes. With this choice of coordinates, we have

$$\left.\begin{aligned} &a_{11}\neq a_{21}\\ &a_{12}=a_{22}\\ &a_{13}=a_{23}=0\end{aligned}\right\} \tag{6}$$

We shall write $a_{11}=a_1$, $a_{21}=a_2$; $a_{12}=a_{22}=b$.

To make the formulae more concise, we introduce the following notation:

$$\underline{z}=|z_{12}-r|; \qquad \underline{z}^*=|z_{21}-r|, \tag{7}$$

$$\overline{z}=|z_{12}+r|; \qquad \overline{z}^*=|z_{21}+r|, \tag{7a}$$

$$R_i=\overline{PS}_i=\sqrt{(\theta_1-a_i)^2+(\theta_2-b)^2+\theta_3^2} \text{ for } i=1,2, \tag{7b}$$

$$m=(a_1+a_2)/2, \tag{7c}$$

$$d=(a_1-a_2)/2; \qquad d^*=(a_2-a_1)/2, \tag{7d}$$

$$T_1=m-\frac{\overline{z}^2}{4d}; \qquad T_1^*=m-\frac{(\overline{z}^*)^2}{4d^*}, \tag{7e}$$

$$T_2=m-\frac{\underline{z}^2}{4d}; \qquad T_2^*=m-\frac{(\underline{z}^*)^2}{4d^*}, \tag{7f}$$

$$A_1=\frac{2d}{\overline{z}}; \qquad A_1^*=\frac{2d^*}{\overline{z}^*}, \tag{7g}$$

$$A_2=\frac{2d}{\underline{z}}; \qquad A_2^*=\frac{2d^*}{\underline{z}^*}. \tag{7h}$$

Considering that according to assumption (I) $z_{21}=-z_{12}$, we obtain from (7) and (7a)

$$\underline{z}^*=\overline{z} \text{ and } \overline{z}^*=\underline{z}. \tag{8}$$

We see also

$$d^*=-d, \tag{8a}$$

and therefore

$$T_1^*+T_2=T_1+T_2^*=2m, \tag{8b}$$

from these relations we see

$$A_1^*=-A_2 \text{ and } A_2^*=-A_1. \tag{8c}$$

A simple computation gives $R_1{}^2-R_2{}^2=-4d(\theta_1-m)$ so that we have with the aid of (7e) and (7f)

$$\left.\begin{aligned} &R_1^2-R_2^2-\underline{z}^2=4d(T_2-\theta_1)\\ &R_1^2-R_2^2-\overline{z}^2=4d(T_1-\theta_1)\\ &R_2^2-R_1^2-\underline{z}^2=4d(\theta_1-T_1^*)\\ &R_2^2-R_1^2-\overline{z}^2=4d(\theta_1-T_2^*)\end{aligned}\right\} \tag{9}$$

We next rewrite the maximum likelihood condition for each of the three cases (A), (B) and (C).

Case (*A*)

From (5), (7) and (7a) it is seen that $\underline{z}=z_{12}-r$, $\overline{z}=z_{12}+r$. It follows then from (4a) that $0<\underline{z}+R_2\leqq R_1\leqq\overline{z}+R_2$. By squaring and transposing we obtain the inequalities

$$R_2\leqq\frac{R_1^2-R_2^2-\underline{z}^2}{2\underline{z}} \text{ and } R_2\geqq\frac{R_1^2-R_2^2-\overline{z}^2}{2\overline{z}}.$$

Considering (9), (7g), and (7h), we obtain from these relations the final form for the maximum likelihood conditions

(A.1) $R_2\leqq A_2(T_2-\theta_1)$ and (A.2) $R_2\geqq A_1(T_1-\theta_1)$.

In an analogous manner we obtain the likelihood conditions in Case (B).

(B.1) $R_1\leqq A_2^*(T_2^*-\theta_1)$ and (B.2) $R_1\geqq A_1^*(T_1^*-\theta_1)$.

It is seen that the inequalities (B.1) and (B.2) may be obtained from (A.1) and (A.2) by writing R_1 instead of R_2 and by attaching an asterisk to the A and T symbols.

Case (*C*) Similarly, we obtain the likelihood conditions

(C.1) $R_2\geqq A_1(T_1-\theta_1)$ and (C.2) $R_1\geqq A_1^*(T_1^*-\theta_1)$.

It is seen that (C.1)≡(A.2) and (C.2)≡(B.2)

4. Discussion of the Likelihood Condition

Before discussing the likelihood condition, it is desirable to derive a simple relation.

From (2) we have $d_1>d_2$; in our coordinate system this means

$$a_1^2>a_2^2 \text{ or } |a_1|>|a_2|.$$

From this we see easily

$$\text{sign } d=\text{sign }(a_1-a_2)=\text{sign } a_1=-\text{sign } d^*. \tag{10}$$

Here sign x stands for "the sign of x", that is, sign $x=+1$ if $x>0$ and sign $x=-1$ if $x<0$.

We first discuss the case (A) and start with condition (A.1). It is seen immediately that (A.1) can only be satisfied if $A_2(T_2-\theta_1)\geqq 0$.

Considering (7h) and (10) this necessary condition can be written

$$T_2 \text{ sign } a_1\geqq\theta_1 \text{ sign } a_1. \tag{11}$$

This condition restricts the point P to a half space determined by the observations and the location of the stations. In the following we assume that (11) is satisfied. We can, therefore, square (A.1) and obtain

[6] If the origin is located on the line S_1S_2 the three points OS_1S_2 do not determine a plane. In this case the θ_2-axis may be chosen in an arbitrary plane so as to intersect the θ_1-axis perpendicularly.

$R_2^2 \leqq A_2^2(T_2-\theta_1)^2$. As $R_2^2=(\theta_1-a_2)^2+(\theta_2-b)^2+\theta_3^2$, it follows that

$$(\theta_2-b)^2+\theta_3^2 \leqq (A_2^2-1)\theta_1^2-2(A_2^2T_2-a_2)\,\theta_1+(A_2^2T_2^2-a_2^2). \tag{12}$$

We have to distinguish two possibilities (a) If $A_2^2=1$ then from (7h) and (7f) $T_2=m-d=a_2$. The right-hand side of (12) is then zero so that (12) can only be satisfied for the points of the line $\theta_2=b$, $\theta_3=0$.

We consider next the case where (b) $A_2^2 \neq 1$. From (7f) and (7h) it is seen easily that

$$A_2^2T_2-a_2=m(A_2^2-1). \tag{12a}$$

Using this formula we obtain by a simple computation from (12)

$$(A_2^2-1)(\theta_1-m)^2-(\theta_2-b)^2-\theta_3^2 \geqq \frac{A_2^2(T_2-a_2)^2}{A_2^2-1}. \tag{12b}$$

The equality sign in (12b) determines a quadric surface of revolution Q_1. The center of this quadric is the point $(m,b,0)$; the axis of revolution is parallel to the θ_1-axis and is therefore the line joining the two stations S_1 and S_2. The surface Q_1 is a hyperboloid if $A_2^2>1$ and an ellipsoid if $A_2^2<1$. Denote by u the real (resp. major) semiaxis and by v the imaginary (resp. minor) semiaxis of Q_1. Then it is seen from (12b), (7h), and (7f) that

$$u^2=\frac{\underline{z}^2}{4} \text{ and } v^2=\frac{\underline{z}^2}{4}\,|A_2^2-1|. \tag{13}$$

Let $2e$ be the distance between the two focal points of Q_1, then $e^2=u^2+v^2\text{sign}(A_2^2-1)$ or from (13), e=d and u=$\underline{z}/2$.

$$e=d \text{ and } u=\frac{1}{2}\,\underline{z}. \tag{13a}$$

The focal points of the quadric Q_1 are therefore the stations S_1 and S_2. The quadric surface Q_1 divides the space into two regions, we call the region containing the center the interior of Q_1 and its complement the exterior of Q_1. We can then say that condition (A.1) is satisfied outside Q_1 if $A_2^2>1$ and inside Q_1 if $A_2^2<1$. If $A_2^2=1$ then (A.1) is satisfied only on points of the line $\theta_2=b$, $\theta_3=0$ for which $A_2=\text{sign}(T_2-\theta_1)$.

We next consider condition (A.2). This inequality is satisfied in two different regions. In the first region the right-hand side of (A.2) is positive, in the second the right-hand side of (A.2) is negative or zero. We assume first that the right-hand side of (A.2) is positive. By squaring and transposing we obtain then

$$(\theta_2-b)^2+\theta_3^2 \geqq (A_1^2-1)\theta_1^2-2(A_1^2T_1-a_2)\theta_1+A_1^2T_1^2-a_2^2. \tag{14}$$

Again we distinguish two possibilities: (a) If $A_1^2=1$ then from (7e) and (7g) $T_1=m\text{-}d=a_2$. The right-hand side of (14) is then zero; in this case (14) is satisfied for all points; (b) if $A_1^2 \neq 1$ we see easily from (7e) and (7g) that

$$A_1^2T_1-a_2=m(A_1^2-1). \tag{14a}$$

Using this formula (14) reduces by a simple computation to

$$(A_1^2-1)(\theta_1-m)^2-(\theta_2-b)^2-\theta_3^2 \leqq \frac{A_1^2(T_1-a_2)^2}{A_1^2-1}. \tag{14b}$$

The equality sign in (14b) determines again a quadric surface of revolution Q_2. The center of this quadric is the point $(m, b, 0)$; the axis of revolution is again the line joining the stations S_1 and S_2. The surface Q_2 is a hyperboloid if $A_1{}^2>1$ and an ellipsoid if $A_1{}^2<1$. Denote by p the real (major) semiaxis and by q the imaginary (minor) semiaxis of Q_2. Then it is seen from (14b), (7g) and (7e) that

$$p^2=\frac{\overline{z}^2}{4} \text{ and } q^2=\frac{\overline{z}^2}{4}\,|A_1^2-1|. \tag{15}$$

Let $2f$ be the distance between the two focal points of Q_2. By an argument similar to the one used before, we obtain

$$f=d \text{ and } p=\frac{\overline{z}}{2}, \tag{15a}$$

so that the quadrics Q_1 and Q_2 are confocal. We summarise these results in the following statement.

If the right-hand side of (A.2) is positive, the condition (A.2) is satisfied inside the quadric Q_2 when $A_1{}^2>1$, but outside Q_2 when $A_1{}^2<1$. In case $A_1{}^2=1$, condition (A.2) is satisfied for all points for which $A_1=\text{sign }(T_1-\theta_1)$.

We investigate next the second region in which (A.2) is valid by assuming that the right-hand side of (A.2) is nonpositive; that is

$$A_1(T_1-\theta_1) \leqq 0. \tag{15b}$$

From (7g) and (10) we see that sign A_1=sign a_1 so that (A.2) is satisfied if

$$T_1 \text{ sign } a_1 \leqq \theta_1 \text{ sign } a_1. \tag{15c}$$

From (15c) and the necessary condition (11) it is seen that (A.2) is satisfied if θ_1 belongs to the interval (T_1, T_2) in case (15b) holds. Therefore, (A.2) is satisfied either in the above-mentioned region determined by Q_2 respectively in a half space or in a region bounded by two planes perpendicular to the line joining S_1 and S_2.

In case (A) we have $\underline{z}=z_{12}-r>0$ so that $\overline{z}>\underline{z}$. Therefore, we see from (7g) and (7h) that $A_2{}^2>A_1{}^2$ and using also (11a) and (15a), $u<p$. We are now ready to determine the region where (A.1) and (A.2)

are satisfied simultaneously. We have to distinguish five possibilities.

1. $A_2^2 > A_1^2 > 1$. It is easily seen that the point $(T_2, b, 0)$ [resp. the point $(T_1, b, 0]$ is on the line segment joining the vertices of Q_1 (resp. Q_2). From our previous results we see that (A.1) and (A.2) are satisfied in one-half of the hyperbolic shell bounded by Q_1 and Q_2.

2. $A_2^2 > 1 = A_1^2$. The point $(T_1, b, 0)$ coincides with S_2, while $(T_2, b, 0)$ is on the line segment joining the vertices of Q_1. Conditions (A.1) and (A.2) are satisfied in one-half of the outside of the hyperboid Q_1.

3. $A_2^2 > 1 > A_1^2$. The point $(T_1, b, 0)$ is then outside Q_1 and outside Q_2, while $(T_2, b, 0)$ is inside Q_1. The region consists again of one-half of the outside of the hyperboloid Q_1.

4. $A_2^2 = 1 > A_1^2$. In this case conditions (A.1) and (A.2) are satisfied only by points of the line S_1S_2, which obey (11).

5. If $1 > A_2^2 > A_1^2$ no region exists where (A.1) and (A.2) are simultaneously satisfied. It is worth while to remark that cases (2), (3), (4), and (5) are of little practical interest. From (7g), (7h), and (10) it is seen that $A_1^2 \leqq 1$ (respectively $A_2^2 \leqq 1$) is equivalent to

$$|2d| \leqq |z_{12} - r| \quad \text{(respectively } |2d| \leqq |z_{12} + r|).$$

A situation where these relations hold is conceivable but unlikely as long as r is small. This is seen if we remember that the difference between two sides of a triangle is always less than its third side.

Case (B) can be discussed independently by repeating the arguments used for case (A).

We can then summarize the results by distinguishing the following five cases:

1. $A_1^2 > A_2^2 > 1$. In this case (B.1) and (B.2) are satisfied in one half of the hyperbolic shell bounded by Q_1 and Q_2.

2. $A_1^2 > 1 = A_2^2$.
3. $A_1^2 > 1 > A_2^2$.

} conditions (B.1) and (B.2) are satisfied in one half of the outside of Q_2.

4. $A_1^2 = 1 > A_2^2$. Then (B.1) and (B.2) are satisfied only by points of the line S_1S_2 that obey (11).

5. If $1 > A_1^2 > A_2^2$ no region exists where both (B.1) and (B.2) are simultaneously satisfied. Cases (2), (3), (4), and (5) offer again little practical interest.

We finally consider Case (C) with the conditions (C.1)$\equiv$(A.2) $R_2 \geqq A_1(T_1 - \theta_1)$ and (C.2)$\equiv$(B.2) $R_1 \geqq A_1{}^*(T_1{}^* - \theta_1)$. It is seen that (C.1) is identical with (A.2) and (C.2) with (B.2), therefore, the previous results can be applied immediately. It is advisable to remark that no necessary conditions [similar to (11)] exist; however, (C.1) as well as (C.2) are satisfied if the right-hand sides are nonpositive. Considering the previous results, case (C) may be summarized by distinguishing the following five cases.

1. $A_2^2 > A_1^2 > 1$ [or $A_1^2 > A_2^2 > 1$]. Conditions (C.1) and (C.2) are satisfied in a simply connected region bounded by one nappe of Q_1 and by one nappe of Q_2 so that the convex side of one nappe is turned toward S_1, while the convex side of the other nappe is turned toward S_2.

2. $A_2^2 > A_1^2 = 1$ [or $A_1^2 > A_2^2 = 1$]. The conditions (C.1) and (C.2) are satisfied in the region that is bounded by one nappe of one of the quadrics and that contains the center.

3. $A_2^2 > 1 > A_1^2$ [or $A_1^2 > 1 > A_2^2$]. The conditions are again satisfied in the region that is bounded by one nappe of one of the quadrics and that contains the center.

4. $A_2^2 = 1 > A_1^2$ [or $A_1^2 = 1 > A_2^2$]
5. $1 > A_2^2 > A_1^2$ [or $1 > A_1^2 > A_2^2$]

} the conditions are everywhere satisfied.

In case (C) the observation gives only little information about the location of the moving object. However, case (C) can occur only if $|z_{12}| < r$. This will happen only for a small portion of the path of the moving object if r is small. Moreover, it should be possible to locate the stations in such a way that case (C) occurs only for one observation at a time so that this possibility should not cause any difficulty in locating the moving object.

5. The Maximum Likelihood Region

In section 2, it was assumed that n hyperbolic observations (based on n pairs of stations) were taken to determine the position of the moving object. In section 3 we derived the likelihood condition imposed by a single observation and discussed in section 4 the resulting information on the location of the moving object. It was shown that the moving object was confined to a certain region in space with known boundaries. We finally consider all the n observations. Each observation determines a region; if there exists a point set in the parameter space common to all these regions then a maximum likelihood region R exists and is identical with this point set. The boundaries of R are hyperboloids of revolution. Any point inside the region R can be considered to be a maximum likelihood estimate.[7]

With an appropriate choice of stations it is easy to devise a computational procedure that leads always to an interior point of R. If the maximum likelihood region becomes small, then it is practically possible to use any point interior to R as an estimate for the unknown position of the moving object.

[7] The maximum likelihood region cannot increase in size when the number of observations increases. On the contrary, it is to be expected that it will decrease.

WASHINGTON, August 18, 1950.