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Analysis of Symmetrical Waveguide Junctions[1]

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Previous theory of consequences of symmetry in waveguide junctions has made limited use of mathematical tools available in the theory of group representations and has been limited to the consideration of nondissipative junctions. In this paper group-theoretical technique is utilized more fully (in much the same way as in the analysis of the vibration of symmetrical molecules) in the formulation of an improved and general technique for the analysis of symmetrical waveguide junctions subject merely to the requirement of linearity.

A waveguide junction, for the purpose of this paper, is a linear electromagnetic system possessing ideal waveguide leads, and is considered to be subject to excitation solely through the effects of nonattenuated modes in the waveguide leads. Under the conditions of the problem, an arbitrary electric (or magnetic) field in a waveguide junction is expressible linearly in terms of a finite number of linearly independent electric (or magnetic) basis fields. From any given ordered pair of electric (or magnetic) basis fields one can in principle calculate a complex number—an element of the admittance (or impedance) matrix characterizing the junction (relative to the choice of basis fields). The geometric concept of rotation and reflection of fields (and structures) is discussed in terms of a rotation-reflection operator, and the symmetry of a junction is characterized by a group of rotation-reflection operations under which the structure is invariant. A general procedure is given for the construction of a basis in which the basis fields transform according to irreducible representations of the symmetry group involved. Such basis fields are said to be of particular symmetry species and from the special properties of such fields follow the physical results, of which perhaps the most conspicuous is the vanishing of the matrix element between two fields of distinct symmetry species.

I. Introduction

This paper is concerned with symmetry properties in "waveguide junctions." A waveguide junction, for the purpose of this paper, is a linear electromagnetic system possessing ideal waveguide leads and is considered to be subject to excitation solely through the effects of nonattenuated waveguide modes in the waveguide leads. The electromagnetic boundary-value problem presented by a waveguide junction is, in general, impracticably difficult to solve. Nevertheless, important information concerning the characteristics of a waveguide junction in its primary function as a device for transferring power from one waveguide-mode to another is derivable with relatively little labor from general properties, such as reciprocity, losslessness, and, in particular, symmetry. Many waveguide junctions used in microwave practice do in fact possess useful and interesting properties in virtue of symmetry. A few simple examples of such junctions are shown in figure 1.

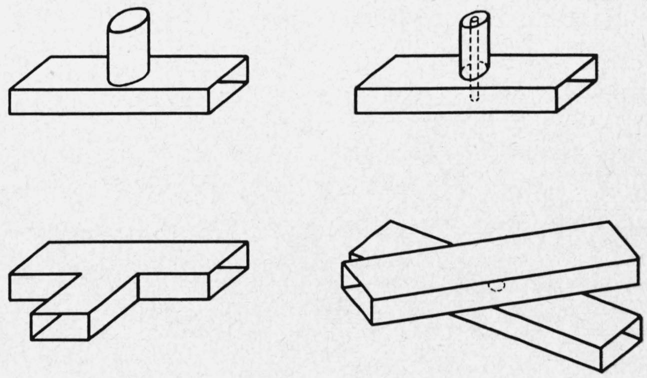

FIGURE 1. *Simple junctions possessing symmetry.*

The literature on the present subject is not extensive. The book, *Principles of microwave circuits*,[2] contains a valuable and fairly comprehensive treatment applying to nondissipative waveguide junctions. A report by Slater[3] is mainly concerned with the analysis of nondissipative T-junctions having essentially a single symmetry element. A paper by Chodorow, Ginzton, and Kane[4] deals with one particularly interesting junction, which is a waveguide analogue of a Wheastone-bridge network. There is to be found also the occasional and limited use of symmetry arguments in discussions primarily concerned with other matters.

The analytical technique employed in the book referred to in footnote 2 is partly formulated in general terms and partly indicated by the consideration of a series of examples. The technique used involves the restriction to nondissipative junctions as an explicit condition. The discussions contained in the papers referred to in footnotes 3 and 4 are of a more or less specific nature and, in the form given, are likewise subject to the restriction of no dissipation.

The object of the present paper is to develop a general theory of the consequences of symmetry in waveguide junctions of a general class: in the interior of a waveguide junction media that may be nonhomogeneous and anisotropic are permitted; dissipation, by reason of finite conductivity or radiation to infinity (or both), is permitted; fulfillment of the reciprocity condition is not required. Linear behavior and freedom from internal sources are as-

[1] A dissertation submitted to the faculty of the Graduate School of Arts and Sciences of the Catholic University of America in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

[2] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, Principles of microwave circuits, chapter 12 (McGraw-Hill Co., New York, N. Y., 1948).

[3] J. C. Slater, Technical Report No. 37, Electronics Research Lab., Mass. Inst. Tech. (Cambridge, Mass., 1947).

[4] M. Chodorow, E. L. Ginzton, and J. F. Kane, Proc. IRE **37**, 634 (1949).

sumed (the former being essential, the latter nonessential). A broad outline of the discussion follows.

The needed physical and mathematical formulation of the electromagnetic problem is given in section II. Under the conditions of the problem, an arbitrary electric (or magnetic) field in a waveguide junction is expressible linearly in terms of a finite number of linearly independent electric (or magnetic) basis fields. From any given ordered pair of basis fields of the same kind (electric or magnetic) one can in principle calculate a complex number—an element of the admittance (or impedance) matrix of the junction—which is a measure of the field of opposite kind associated with the second (or first) of the given pair of fields. The characterization of a junction by means of matrices is relative to the choice of basis fields; the basis fields first chosen might be described as "simple with respect to excitation." Formulas for change of basis are given.

The geometric concept of rotation and reflection of fields (and structures), taken up in section III, is discussed in terms of a rotation-reflection operator applicable to tensor point-functions.[5] The symmetry of a waveguide junction is characterized by a group of rotation-reflection operations under which the structure is invariant.

A method of symmetry analysis and the results obtained for the class of problems considered are presented in general terms in section IV (with some further results in appendix 2). The basis fields set up in section II do not necessarily exhibit particularly simple transformation properties under operations of the symmetry group; however, as is shown, it is possible to select linear combinations of the original basis fields to form new basis fields that do exhibit special transformation properties. Such fields are said to be of particular symmetry species. (If, for example, the symmetry group consists of only two operations, the identity and reflection in a plane, say, the two possible species are the familiar "even" and "odd." The general definition of symmetry species is provided by the theory of group representations, which theory indeed provides the natural mathematical tools for the analysis.) An important property of fields of the new basis is the vanishing of the matrix element between two fields of distinct symmetry species.

As might be expected, the method used here is in some respects very similar to methods used in the analysis of the vibration of symmetrical molecules. However, because dissipation (as well as failure of reciprocity) is permitted in the waveguide problem, the main part of the analysis here is formulated without reference to the question of eigenvalues of the matrices of a junction. The eigenvalue problem is discussed briefly in the latter part of section IV.

Three illustrative examples are considered in section V. In one of these examples the theoretical results previously obtained (footnote 4) for the waveguide Wheatstone bridge are presented in a more general context.

[5] *Tensor* is used in the general sense, a scalar being a tensor of rank 0, etc.

II. Electromagnetic Formulation

A waveguide junction may be described briefly as an electromagnetic system comprising an arbitrary number, n, of ideal waveguide "leads", which individually may be of arbitrary cross section, and a "coupling region" from which the waveguide leads emerge. Various aspects of the theory of waveguide junctions have been considered in recent years by a number of authors, and a considerable body of systematic theory[6 7 8] centering on the use of impedance, admittance and scattering matrices, has been built up. Nevertheless, for the purposes of this paper a formulation, which can be brief, but which is in some respects new and more complete, is needed. Part of the formulation will depend, of course, on certain rather well-known general results of the theory of waveguides (see footnotes 2 and 7).

The domain of the electromagnetic field in a waveguide junction will be denoted by V, the complete boundary of V will be denoted by S, and the inward normal unit vector on S will be denoted by $\boldsymbol{k}$. The surface S and also, in part, the boundary conditions to be imposed may be described (in two typical cases) as follows. If the domain is of infinite

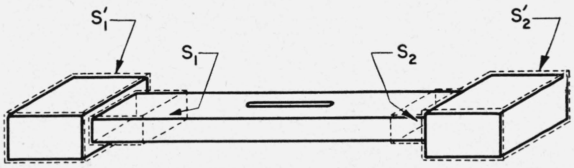

FIGURE 2. *Slotted waveguide.*

Illustrating S_m, S_m' ($n=2$); S_∞ is not shown.

extent (fig. 2), V is bounded internally by n closed surfaces S_m+S_m' ($m=1, 2, \cdot\cdot\cdot, n$), where S_m is a transverse surface (the *terminal surface*) in the m^{th} waveguide and S_m, S_m' together enclose the termination of the m^{th} waveguide. Although V in this case is externally unbounded, it is convenient to employ a large spherical bounding surface S_∞ (of radius r), appropriate limiting processes being implied. The complete boundary of V is then $S=S_\infty+S_1+S_1'+\cdot\cdot\cdot+S_m+S_m'$. On all parts of S except the terminal surfaces, the field is to satisfy homogeneous boundary-conditions: on S_∞, lim $(r\boldsymbol{E})$ is bounded, and $\boldsymbol{E}$ satisfies the outward-radiation condition; on S_m', the tangential component $\boldsymbol{E}_t$ of $\boldsymbol{E}$ vanishes ($\boldsymbol{E}$ denoting the electric field). If the field is confined to a finite domain by a perfectly conducting metal surface (fig. 3), then $S=S_0+S_1+\cdot\cdot\cdot+S_n$, where S_0 coincides with the metal surface. In this case the homogeneous boundary condition is simply $\boldsymbol{E}_t=0$ on S_0. (If metal walls are considered finitely conducting, but are sufficiently thick, S_0 may be taken on the outer surface where $\boldsymbol{E}$, hence $\boldsymbol{E}_t$, is substantially zero.)

The whole of the space and structure within V can be regarded as a linear, source-free medium,

[6] See footnote 2.
[7] J. C. Slater, Rev. Mod. Phys. **18**, 441 (1946).
[8] D. M. Kerns, J. Research NBS **42**, 515 (1949). (The references cited are believed to be the ones most useful in connection with the present paper.)

which is in general nonhomogeneous and anisotropic. The conductivity and the electric and magnetic inductive capacities of the medium are to be given by the real point-functions $\sigma=\sigma(\boldsymbol{r})$, $\epsilon=\epsilon(\boldsymbol{r})$, $\mu=\mu(\boldsymbol{r})$, respectively. Anisotropy is taken into account by considering σ, ϵ, μ to be, in general, tensors of rank two.

The interior of the m^{th} waveguide lead is a cylindrical, source-free domain τ_m (of finite length) in which $\sigma\equiv 0$, μ, ϵ=scalar constants. τ_m is bounded by a cylindrical surface (also of finite length) on which the conductivity becomes infinite. The terminal surface S_m is a transverse section of τ_m; S_m

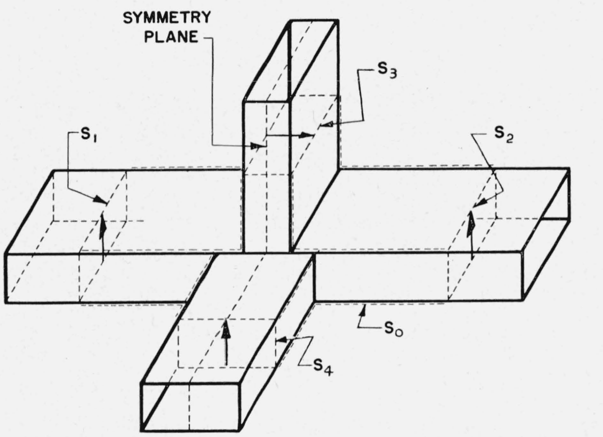

FIGURE 3. *Hybrid T-junction.*

Illustrating S_m and S_0 $(n=4)$.

is to be a closed, connected (but not necessarily simply connected) plane surface bounded by a curve C_m (which may consist of one or more distinct parts). As the figures indicate, S_m is to be located within its waveguide lead at some distance from any discontinuity. (A portion of a waveguide lead may be included bodily within V, as in fig. 2; in any case the interior of a waveguide lead from the inner portion of the junction out to the terminal surface is an integral part of the domain V.)

Harmonic time-dependence, at angular frequency ω, is assumed; we shall deal with the complex electric and magnetic field (amplitudes) $\boldsymbol{E}=\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{H}=\boldsymbol{H}(\boldsymbol{r})$, omitting the time-dependent factor $\exp(j\omega t)$. Within V, then, $\boldsymbol{E},\boldsymbol{H}$ satisfy Maxwell's equations in the form

$$\left.\begin{aligned}\nabla\times\boldsymbol{E}&=-j\omega\mu\cdot\boldsymbol{H},\\ \nabla\times\boldsymbol{H}&=(j\omega\epsilon+\sigma)\cdot\boldsymbol{E}.\end{aligned}\right\}\qquad(1)$$

The rationalized meter-kilogram-second system of units is assumed.

Under the specified conditions holding on S and in V the electromagnetic field within V will be determined by the boundary conditions on the terminal surfaces. The ultimate sources of the field, which are to be found within the waveguide terminations (pictured in fig. 2, implied in other figures), are of interest only insofar as certain fields, of a single frequency ω, are caused to appear on the terminal surfaces. If, as is assumed, the terminal surface S_m is sufficiently far from any discontinuity (or any other departure from the conditions defining a waveguide lead), the contributions of attenuated waveguide modes to the field on S_m will be negligible. Thus only nonattenuated modes need be considered in describing the field on a terminal surface.[9] It is understood that the field of a single nonattenuated mode may and in general will involve both incident and emergent progressive components. Waveguide and frequency being given, the number ν_m of nonattenuated modes supported in the m^{th} waveguide is necessarily finite or zero; it is naturally assumed that $\nu_m\gtreqless 1$. In general, *TEM*, *TM*, and *TE* modes will occur among the ν_m nonattenuated modes in waveguide m, but it will not be necessary to distinguish the several types of modes in the notation.

Of essential interest will be suitable expressions for the tangential (=transverse) components $\boldsymbol{E}_t,\boldsymbol{H}_t$ of the most general $\boldsymbol{E},\boldsymbol{H}$ on S_m consistent with the above conditions. Let the index μ $(\mu=1,2,\cdot\cdot\cdot,\nu_m)$ identify the nonattenuated modes supported at frequency ω in waveguide m. From waveguide theory we know that $\boldsymbol{E}_t$ on S_m can be expressed in the form

$$\boldsymbol{E}(\boldsymbol{r}_m)_t=\sum_{\mu=1}^{\nu_m}v_{m\mu}\boldsymbol{e}^0_{m\mu}(\boldsymbol{r}_m),\qquad(\boldsymbol{r}_m\text{ on }S_m),\qquad(2)$$

where $\boldsymbol{r}_m$ denotes $\boldsymbol{r}$ on S_m, the $v_{m\mu}$ are scalar coefficients, and the $\boldsymbol{e}^0_{m\mu}$ are derivable from eigenfunctions of certain two-dimensional boundary-value problems formulated for S_m and its perimeter C_m. The vector $\boldsymbol{e}^0_{m\mu}$, like $\boldsymbol{E}_t$, lies in the plane of S_m; $\boldsymbol{e}^0_{m\mu}$ is considered to be defined only for $\boldsymbol{r}$ on S_m. No coordinate-dependence is indicated for the $v_{m\mu}$ since S_m is considered to be in a fixed position in its waveguide lead. Similarly, $\boldsymbol{H}_t$ on S_m can be expanded in the form

$$\boldsymbol{H}(\boldsymbol{r}_m)_t=\sum_{\mu=1}^{\mu_m}i_{m\mu}\boldsymbol{h}^0_{m}(\boldsymbol{r}_m),\qquad(\boldsymbol{r}_m\text{ on }S_m);\qquad(3)$$

here the $i_{m\mu}$ are scalar coefficients and the $\boldsymbol{h}^0_{m\mu}$ are defined by

$$\boldsymbol{h}^0_{m\mu}(\boldsymbol{r}_m)\equiv(\zeta_0\eta_{m\mu})\boldsymbol{k}_m\times\boldsymbol{e}^0_{m\mu}(\boldsymbol{r}_m),\qquad(4)$$

where $\boldsymbol{k}_m$ denotes $\boldsymbol{k}$ on S_m, $\eta_{m\mu}$ is the wave-admittance of mode μ in waveguide m, and $\zeta_0=1$ ohm (see the following paragraph). The $\boldsymbol{e}^0_{m\mu}$ and the $\boldsymbol{h}^0_{m\mu}$ may be assumed to be real and to satisfy the orthogonality and normalization relation

$$\frac{1}{2}\int_{S_m}(\boldsymbol{e}^0_{m\mu}\times\boldsymbol{h}^0_{m\lambda})\cdot\boldsymbol{k}_m dS=\delta_{\mu\lambda}=\left\{\begin{matrix}1,\ \lambda=\mu\\ 0,\ \lambda\neq\mu\end{matrix}\right\}.\qquad(5)$$

Equations 4 and 5 each represent combinations of what is necessary with what is convenient. In particular, eq 5 is automatically satisfied if modes μ,λ are not mutually degenerate or if one is a *TE* and the other a *TM* mode.

[9] Attentuated modes are excluded mainly because these higher-mode interactions are usually avoided in practice. The finiteness of the number of modes involved is the essential point so far as the subsequent analysis is concerned.

Clearly, $v_{m\mu}$ and $i_{m\mu}$ are respectively linear measures, relative to the *standard terminal fields* $\boldsymbol{e}^0_{m\mu}$ and $\boldsymbol{h}^0_{m\mu}$, of the contributions of mode μ to $\boldsymbol{E}_t$ and $\boldsymbol{H}_t$ on the terminal surface S_m. The variables $a_{m\mu}$, $b_{m\mu}$ defined by

$$2a_{m\mu}=v_{m\mu}+\zeta_0 i_{m\mu}, \qquad 2b_{m\mu}=v_{m\mu}-\zeta_0 i_{m\mu}, \tag{6}$$

will be employed to a very limited extent in this paper. $a_{m\mu}$ and $b_{m\mu}$ are respectively linear measures (relative to $\boldsymbol{e}^0_{m\mu}$) of the electric fields of the incident and emergent progressive components of mode μ at S_m. The second of eq 6 shows that if $b_{m\mu}=0$, then the corresponding value of $v_{m\mu}/i_{m\mu}$ (called the characteristic impedance of mode μ in waveguide m) is equal to ζ_0. So far as the present paper is concerned, ζ_0 may be considered to be primarily a dimensional constant inserted to bring about an attractive dimensional scheme. Indeed, from eq 4 and 5 one finds (considering $\delta_{\mu\mu}$ and $\boldsymbol{k}_m$ to be dimensionless) $[\boldsymbol{e}^0_{m\mu}]=[\boldsymbol{h}^0_{m\mu}]=\text{meters}^{-1}$, and then from eq 2 and 3, $[v_{m\mu}]=\text{volts}$, $[i_{m\mu}]=\text{amperes}$.

It is appropriate to review the electromagnetic situation in the junction as a whole. For convenience let v and i denote the column matrices whose elements are respectively the N $v_{m\mu}$'s and the N $i_{m\mu}$'s, $N\equiv\nu_1+\nu_2+\cdots+\nu_n$. v uniquely determines and is uniquely determined by $\boldsymbol{E}_t$ on all the terminal surfaces; similarly, i determines and is determined by $\boldsymbol{H}_t$ on all the terminal surfaces. Now the specification of either $\boldsymbol{E}_t$ or $\boldsymbol{H}_t$ on all terminal surfaces (together with the homogeneous boundary condition holding elsewhere on S) is just sufficient to determine $\boldsymbol{E}$ and $\boldsymbol{H}$ throughout the domain V. Thus, if v is given, $\boldsymbol{E}$,$\boldsymbol{H}$, and i are determined; if i is given, $\boldsymbol{H}$,$\boldsymbol{E}$ and v are determined.[10] The existence of a homogeneous linear relation connecting v and i is implied. Moreover, there are (under the conditions of the problem) exactly N linearly independent electric fields possible in V; similarly, there are exactly N linearly independent magnetic fields possible in V. This fundamental property is expressed analytically in the following paragraph.

We define the *electric basis-field* $\boldsymbol{e}_{m\mu}$ as the electric field [11] in V corresponding to the special boundary condition implied by $v_{m\mu}=1$, $v_{l\lambda}=0$ $(l\lambda\neq m\mu)$. The connection, as well as the difference, between $\boldsymbol{e}_{m\mu}$ and $\boldsymbol{e}^0_{m\mu}$ is to be noted: $\boldsymbol{e}_{m\mu}$ is defined throughout V, its tangential component $(\boldsymbol{e}_{m\mu})_t$ reduces to $\boldsymbol{e}^0_{m\mu}$ on S_m, to zero on other terminal surfaces. From the boundary conditions it follows that the N $\boldsymbol{e}_{m\mu}$'s are linearly independent. The $\boldsymbol{E}$ in V corresponding to arbitrary v may be written

$$\boldsymbol{E}(\boldsymbol{r})=\sum_{m\mu}\boldsymbol{e}_{m\mu}(\boldsymbol{r})v_{m\mu}=\boldsymbol{e}v, \tag{7}$$

where $\boldsymbol{e}$ is the row matrix $\boldsymbol{e}=(\boldsymbol{e}_{11}\cdots\boldsymbol{e}_{m\mu}\cdots\boldsymbol{e}_{n\nu_n})$. Similarly, the *magnetic basis-field* $\boldsymbol{h}_{l\lambda}$ is defined as the magnetic field in V corresponding to the boundary condition $i_{l\lambda}=1$, $i_{k\kappa}=0$ $(k\kappa\neq l\lambda)$. The $\boldsymbol{H}$ in V corresponding to arbitrary i is then

$$\boldsymbol{H}(\boldsymbol{r})=\sum_{l\lambda}\boldsymbol{h}_{l\lambda}(\boldsymbol{r})i_{l\lambda}=\boldsymbol{h}i, \tag{8}$$

where $\boldsymbol{h}$ is the row matrix $\boldsymbol{h}=(\boldsymbol{h}_{11}\cdots\boldsymbol{h}_{l\lambda}\cdots\boldsymbol{h}_{n\nu_n})$. The possible $\boldsymbol{E}$'s and the possible $\boldsymbol{H}$'s are elements of linear vector spaces,[12] of dimension N. $\boldsymbol{E}$ has the *coordinates* v relative to the *basis* $\boldsymbol{e}$; $\boldsymbol{H}$ has the *coordinates* i relative to the *basis* $\boldsymbol{h}$. The particular basis fields introduced in eq 7 and 8 are the simplest ones to start out with; $\boldsymbol{e}$, $\boldsymbol{h}$, or both together will accordingly be called a primitive basis.

The usefulness of the following definition will become apparent. The *bracket* $[\boldsymbol{E}^1, \boldsymbol{H}^2]$ of any electric field $\boldsymbol{E}^1$ in V and any magnetic field $\boldsymbol{H}^2$ in V is defined by

$$[\boldsymbol{E}^1, \boldsymbol{H}^2]=\frac{1}{2}\sum_{m=1}^{n}\int_{S_m}(\boldsymbol{E}^1\times\overline{\boldsymbol{H}}^2)\cdot\boldsymbol{k}_m dS, \tag{9}$$

where, as always in this paper, the superposed bar denotes the complex conjugate. It is to be emphasized that $\boldsymbol{E}^1$ and $\boldsymbol{H}^2$ are by no means necessarily associated electric and magnetic components of the same electromagnetic field. The most important algebraic property of the bracket is exemplified by

$$[(a\boldsymbol{E}^1+b\boldsymbol{E}^2), \boldsymbol{H}^3]=a[\boldsymbol{E}^1, \boldsymbol{H}^3]+b[\boldsymbol{E}^2, \boldsymbol{H}^3],$$

$$[\boldsymbol{E}^1, (c\boldsymbol{H}^2+d\boldsymbol{H}^3)]=[\boldsymbol{E}^1, \boldsymbol{H}^2]\overline{c}+[\boldsymbol{E}^1, \boldsymbol{H}^3]\overline{d},$$

where a, b, c, d are any constants. (There will be no need to define or use brackets of the type $[\boldsymbol{H}, \boldsymbol{E}]$.)

The orthonormalization of the standard terminal fields (as expressed in eq 5) has as an immediate consequence a corresponding property of the basis fields:

$$[\boldsymbol{e}_{m\mu}, \boldsymbol{h}_{l\lambda}]=\delta_{m\mu,\,l\lambda}, \tag{10}$$

where $\delta_{m\mu,\,l\lambda}=1$ for $m\mu=l\lambda$, $\delta_{m\mu,\,l\lambda}=0$ for $m\mu\neq l\lambda$. The coordinates (relative to the given basis) of arbitrary $\boldsymbol{E}$ and $\boldsymbol{H}$ in V may be defined by

$$[\boldsymbol{E}, \boldsymbol{h}_{l\lambda}]=v_{l\lambda}, \qquad [\boldsymbol{e}_{m\mu}, \boldsymbol{H}]=\overline{i}_{m\mu}, \tag{11}$$

for if $\boldsymbol{E}=\boldsymbol{e}v$ and $\boldsymbol{H}=\boldsymbol{h}i$, then, with the aid of eq 10, one finds that eq 11 do in fact yield $v_{l\lambda}$ and $\overline{i}_{m\mu}$. Let $\mathfrak{H}(\boldsymbol{e}_{l\lambda})$ denote the magnetic field associated with $\boldsymbol{e}_{l\lambda}$, and let $\mathfrak{E}(\boldsymbol{h}_{m\mu})$ denote the electric field associated with $\boldsymbol{h}_{m\mu}$ (so that $\nabla\times\boldsymbol{e}_{l\lambda}=-j\omega\mu\cdot\mathfrak{H}(\boldsymbol{e}_{l\lambda})$ and $\nabla\times\boldsymbol{h}_{m\mu}=(j\omega\epsilon+\sigma)\cdot\mathfrak{E}(\boldsymbol{h}_{m\mu})$). Replacing $\boldsymbol{E}$ and $\boldsymbol{H}$ in eq 11 by $\mathfrak{E}(\boldsymbol{h}_{m\mu})$ and $\mathfrak{H}(\boldsymbol{e}_{l\lambda})$, respectively, we write

$$[\mathfrak{E}(\boldsymbol{h}_{m\mu}), \boldsymbol{h}_{l\lambda}]=Z_{l\lambda,\,m\mu}, \qquad [\boldsymbol{e}_{m\mu}, \mathfrak{H}(\boldsymbol{e}_{l\lambda})]=\overline{Y}_{m\mu,\,l\lambda}, \tag{12}$$

thereby defining the N-dimensional square matrices Z and Y. (The first index-pair attached to the

[10] It may be assumed, with no appreciable loss of generality, that the electromagnetic field in V corresponding to arbitrarily prescribed v or i exists and is unique, and that $E\equiv H\equiv 0$ (throughout V) corresponds to $v=0$ or $i=0$. See paper in footnote 2, p. 134; also paper in footnote 8, p. 535.

[11] Possibly one might prefer to say that $\boldsymbol{e}_{m\mu}$ is a field of electric type inasmuch as the units of $\boldsymbol{e}_{m\mu}$ are those of $\boldsymbol{e}^0_{m\mu}$ and not those of $\boldsymbol{E}$.

[12] For the mathematical postulates defining such spaces, see, e. g., F. D. Murnaghan, Theory of group representations, p. 11 (Johns Hopkins Press, Baltimore, Md., 1938).

matrix element labels the row in both cases.) Now if $\boldsymbol{E}$ and $\boldsymbol{H}$ are associated ($\boldsymbol{E}=\mathfrak{E}(\boldsymbol{H})$, $\boldsymbol{H}=\mathfrak{H}(\boldsymbol{E})$), it is clear that

$$\boldsymbol{E}=\mathfrak{E}(\sum_{k\kappa}\boldsymbol{h}_{k\kappa}i_{k\kappa})=\sum_{k\kappa}\mathfrak{E}(\boldsymbol{h}_{k\kappa})\,i_{k\kappa},$$

$$\boldsymbol{H}=\mathfrak{H}(\sum_{k\kappa}\boldsymbol{e}_{k\kappa}v_{k\kappa})=\sum_{k\kappa}\mathfrak{H}(\boldsymbol{e}_{k\kappa})\,v_{k\kappa},$$

since $\mathfrak{E}$ and $\mathfrak{H}$ are linear functions of their arguments. By inserting these forms for $\boldsymbol{E}$ and $\boldsymbol{H}$ into eq 11 one finds

$$v_{l\lambda}=\sum_{k\kappa}Z_{l\lambda,\,k\kappa}i_{k\kappa},\qquad \bar{i}_{m\mu}=\sum_{k\kappa}\overline{Y}_{m\mu,\,k\kappa}\bar{v}_{k\kappa}. \tag{13}$$

Thus the coordinates of associated $\boldsymbol{E}$ and $\boldsymbol{H}$ are related by $v=Zi$ or, equivalently, by $i=Yv$. Z and Y are, respectively, the impedance and the admittance matrices (relative to the primitive basis) characteristic of the waveguide junction.

The bracket $[\boldsymbol{E},\boldsymbol{H}]$ of (associated) $\boldsymbol{E}$ and $\boldsymbol{H}$ is (directly from eq 9) the integral of the inward normal component of the complex Poynting's vector extended over the aggregate of the terminal surfaces. Thus the total (complex) power influx W across the terminal surface is

$$W=[\boldsymbol{E},\boldsymbol{H}]=\sum_{m\mu,l\lambda}v_{m\mu}[\boldsymbol{e}_{m\mu},\boldsymbol{h}_{l\lambda}]\,\bar{i}_{l\lambda}=i^{*}v. \tag{14}$$

Here and subsequently the star is used to denote the Hermitian conjugate (=transposed complex-conjugate) of a matrix. The additional expressions $W=i^{*}Zi=v^{*}Y^{*}v$ follow immediately from eq 13.

Consider the introduction of a new electric-field basis $\boldsymbol{e}'=(\boldsymbol{e}'_1\cdots\boldsymbol{e}'_N)$ related to the primitive basis $\boldsymbol{e}=(\boldsymbol{e}_{11}\cdots\boldsymbol{e}_{n\nu_n})$ by means of the linear transformation $\boldsymbol{e}'=\boldsymbol{e}\alpha$, where α is a unitary matrix ($\alpha^{*}=\alpha^{-1}$).[13] The postulate that eq 7, 14, 8 shall have invariant meaning determines the transformations of v, i, and $\boldsymbol{h}$ relative to that of $\boldsymbol{e}$. Namely, from

$$\boldsymbol{E}=\boldsymbol{e}v=\boldsymbol{e}'v', \tag{7}$$

$$W=i^{*}v=(i')^{*}v', \tag{14}$$

$$\boldsymbol{H}=\boldsymbol{h}i=\boldsymbol{h}'i', \tag{8}$$

and the given $\boldsymbol{e}'=\boldsymbol{e}\alpha$ one finds easily

$$\left.\begin{aligned}\boldsymbol{e}'&=\boldsymbol{e}\alpha, & v'&=\alpha^{-1}v,\\ \boldsymbol{h}'&=\boldsymbol{h}\alpha, & i'&=\alpha^{-1}i.\end{aligned}\right\} \tag{15}$$

And from eq 12

$$Z'=\alpha^{-1}Z\alpha,\qquad Y'=\alpha^{-1}Y\alpha. \tag{16}$$

(It should be noted that the formulas for $\boldsymbol{h}'$, i', Z', and Y' are written for unitary α and do not hold unless α is unitary.)

[13] Transformations with an arbitrary nonsingular α could be considered but will not be needed in this paper.

With the above provision for *change of basis* the electromagnetic formulation, as far as needed here, is essentially complete. It may be remarked that Z and Y can be regarded as metric tensors of the vector spaces of eq 7 and 8, respectively (the metric is not in general Hermitian, to be sure).[14] The scheme acquires additional meaning when it is recognized that for each bracket there is an expression involving volume integrals extended throughout the domain V (appendix, 1).

Although the discussion will relate primarily to Z, Y, and the corresponding basis fields, the results to be obtained for Z and Y will hold also for the scattering matrix S, which is defined as follows. Let

$$2a=v+\zeta_0 i,\qquad 2b=v-\zeta_0 i, \tag{17}$$

be the matrix form of eq 6. Then the scattering matrix furnishes the relation

$$b=Sa, \tag{18a}$$

and a simple calculation shows that

$$S=(Z-\zeta_0)(Z+\zeta_0)^{-1}, \tag{18b}$$

where ζ_0 is to be interpreted as a multiple of the N-dimensional unit matrix. Equations 17 and 18 will have invariant meaning under a change of basis provided

$$a'=\alpha^{-1}a,\qquad b'=\alpha^{-1}b,\qquad S'=\alpha^{-1}S\alpha. \tag{19}$$

The matrix ζ_0, which must transform like Z (from eq 17), is invariant under a unitary change of basis.

So far as the subsequent symmetry analysis is concerned, Z, Y, or S is an arbitrary nonsingular matrix subject only to the consequences of structural symmetry of the waveguide junction. (It may be understood that Z, Y, and S are such that the real part of W, $\mathrm{Re}(W)$, can not be negative, but this condition is not used in the analysis.) The following special conditions (of electrical origin) are of interest.

I. The nondissipative condition ($\mathrm{Re}(W)\equiv 0$): Z, Y are skew-Hermitian; S is unitary: $Z^{*}=-Z$, $Y^{*}=-Y$, $S^{*}=S^{-1}$

II. The nonreactive condition ($\mathrm{Im}(W)\equiv 0$): Z, Y, S are Hermitian: e.g., $Z^{*}=Z$.

III. The reciprocity condition: Z, Y, S are symmetric: e.g., $Z_{m\mu,\,l\lambda}=Z_{l\lambda,\,m\mu}$, or, in the matrix notation to be used, $\tilde{Z}=Z$.

(20)

These conditions may be incorporated, at will, after the main results have been obtained; of course, the more usual cases are III (alone) and the combination of III and I. The matrix conditions in I and II are

[14] For the mathematical postulates leading to a positive-definite Hermitian metric-tensor, see, e. g., Murnaghan, p. 17. To make connection with Murnaghan's notation, one may define

$$(\boldsymbol{E}^1|\boldsymbol{E}^2)=[\boldsymbol{E}^1,\mathfrak{H}(\boldsymbol{E}^2)]\ (\text{or } (\boldsymbol{H}^1|\boldsymbol{H}^2)=[\mathfrak{E}(\boldsymbol{H}^1),\boldsymbol{H}^2]).$$

invariant under transformations of the type needed, but the property $\tilde{Z}=Z$ is not (unless α is real).[15] In general $\tilde{Z}'$ is not simply equal to Z' but rather, from eq 16,

$$\tilde{Z}'=\tilde{\alpha}Z\tilde{\alpha}^{-1}=\tilde{\alpha}(\alpha Z'\alpha^{-1})\tilde{\alpha}^{-1}=(\tilde{\alpha}\,\alpha)Z'(\tilde{\alpha}\,\alpha)^{-1}. \quad (21)$$

Obviously Y' and S' are subject to the same condition when Y and S are symmetric.

III. The Rotation Operator; Symmetry Groups

The geometric concept of spatial rotation and reflection of a waveguide junction or of an electromagnetic field is represented and in fact analytically defined by certain transformations of the tensor point-functions involved (e. g., $\sigma(\boldsymbol{r})$, $\boldsymbol{e}_{m\mu}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$, etc.). These transformations are basic for what follows and will be set down presently. In the present context tensor components will be distinguished by the use of the letters x, y, z as indices, and a fixed, orthogonal three-dimensional basis is to be understood. The three unit vectors of such a basis plus an arbitrary origin O define a rectangular Cartesian coordinate-system Oxyz ($\boldsymbol{r}_x=x$).

For the present purpose the intrinsic geometric properties of a particular rotation are conveniently characterized by a tensor $\boldsymbol{R}$ (independent of $\boldsymbol{r}$) whose components R_{xy} are such that $R=(R_{xy})$ is a real orthogonal matrix. The determinant of R may be $+1$ or -1; a rotation (as the term is used here for convenience) may be termed proper or improper according as the determinant of R is $+1$ or -1.

As a preliminary to the complete expression of the desired transformation, we consider the transformation $\boldsymbol{F}(\boldsymbol{r})\rightarrow\boldsymbol{F}'(\boldsymbol{r})$ of a tensor $\boldsymbol{F}$ (of rank 0,1, or 2) furnished by

$$F'(\boldsymbol{r})=F(\boldsymbol{r}), \quad (22a)$$

$$F'_x(\boldsymbol{r})=\sum_y R_{xy}F_y(\boldsymbol{r}), \quad (22b)$$

or

$$F'_{xy}(\boldsymbol{r})=\sum_{\hat{x},\hat{y}} R_{x\hat{x}}R_{y\hat{y}}F_{\hat{x}\hat{y}}(\boldsymbol{r}). \quad (22c)$$

This may be called "local" rotation, inasmuch as the components of $\boldsymbol{F}'$ at the point $\boldsymbol{r}$ are related directly to those of $\boldsymbol{F}$ at the same point. It will be convenient to use the notation $\boldsymbol{r}'=\boldsymbol{Rr}$ for eq 22b when $\boldsymbol{F}(\boldsymbol{r})=\boldsymbol{r}$.

It may be noted that in electromagnetic theory it is customary (perhaps invariably so) to consider electric charge a scalar, thereby determining $\boldsymbol{E}$ as a vector and $\boldsymbol{H}$ as a pseudovector. Since a pseudovector is a (antisymmetric) tensor of rank two, eq 22c applies; but in terms of the usual pseudovectorial components, say H_x of $\boldsymbol{H}$, eq 22c becomes

$$H'_x(\boldsymbol{r})=\det(R)\sum_y R_{xy}H_y(\boldsymbol{r}), \quad (23)$$

where det (R) is the determinant of R. It is well to note also that in the present framework $\boldsymbol{E}\times\boldsymbol{H}$ is a vector and $\boldsymbol{E}\times\boldsymbol{H}\cdot\boldsymbol{k}$ is a scalar.

By combining a local rotation and a suitably related functional transformation, which may be called "parallel transport", we obtain the transformation to be termed *rotation of a tensor field* (or, briefly, rotation). In the sense of this term, the rotation corresponding to $\boldsymbol{R}$ applied to the tensor point-function $\boldsymbol{F}$ produces a new tensor point-function, to be denoted by $P_R\boldsymbol{F}$, whose components at the point $\boldsymbol{r}'=\boldsymbol{Rr}$ are equal respectively to those of the locally rotated $\boldsymbol{F}$ at the point $\boldsymbol{r}$. Thus, for all $\boldsymbol{r}$ in the domain V of definition of $\boldsymbol{F}$,

$$P_RF(\boldsymbol{r}')=F(\boldsymbol{r}), \quad (24a)^{16}$$

$$[P_R\boldsymbol{F}(\boldsymbol{r}')]_x=\sum_y R_{xy}F_y(\boldsymbol{r}), \quad (24b)$$

$$[P_R\boldsymbol{F}(\boldsymbol{r}')]_{xy}=\sum_{\hat{x},\hat{y}} R_{x\hat{x}}R_{y\hat{y}}F_{\hat{x}\hat{y}}(\boldsymbol{r}). \quad (24c)$$

$P_R\boldsymbol{F}$ is defined in the rotated domain conveniently denoted by P_RV, P_RV being such that $\boldsymbol{r}'$ is in P_RV if $\boldsymbol{r}$ is in V. It should be observed that the meaning of a rotation depends upon the location of the origin O of Oxyz (at which point $\boldsymbol{F}$ undergoes only local rotation). By what amounts to a change in the designation of the independent variable, one may write, say for eq 24b,

$$[P_R\boldsymbol{F}(\boldsymbol{r})]_x=\sum_y R_{xy}F_y(\boldsymbol{R}^{-1}\boldsymbol{r}), \quad (25)$$

which holds for $\boldsymbol{r}$ in P_RV. In the later applications of eq 24 it will in fact generally be convenient to have $\boldsymbol{r}$ as the argument of $P_R\boldsymbol{F}$.

The analytical construction of eq 24 belies the simplicity of the underlying geometric picture: for proper rotations, at least, the transformation defines what might be described as a rigid motion (in which one point is fixed) of a tensor field; and for scalars and vectors, at least, the transformation is easily visualized. (Simple illustrations are given in fig. 4 and in connection with example A, section V.)

We impart operational meaning to the symbol P_R by saying that it stands for the operation (local rotation plus parallel transport) by which the function $P_R\boldsymbol{F}$ is produced from the function $\boldsymbol{F}$. It is convenient to describe the properties of the transformations 24 by describing the properties of P_R.

The properties of P_R to be listed now are all rather evident from the geometric picture corresponding to eq 24 and to some extent from eq 24 themselves. No proofs will be needed here. (a) P_R is linear:

$$P_R(c\boldsymbol{F}+d\boldsymbol{G})=cP_R\boldsymbol{F}+dP_R\boldsymbol{G},$$

[15] The fact that reciprocity is manifested in $\tilde{Z}=Z$ (in the primitive basis) depends upon the fact that the standard terminal fields are (by hypothesis) real.

[16] Equation 24a essentially reproduces Wigner's definition; eq 24b,c give the generalization needed here. E. Wigner, Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren, p. 113 (Vieweg and Sohn, Braunschweig, 1931).

where c, d are scalar constants and $\boldsymbol{F}$, $\boldsymbol{G}$ are tensors. (b) P_R is distributive with respect to tensor products in that

$$P_R(\boldsymbol{FG})=(P_R\boldsymbol{F})(P_R\boldsymbol{G}), \tag{26}$$

where on both sides the same type of tensor product is implied (e. g., $\mu\cdot\boldsymbol{H}, \boldsymbol{E}\times\overline{\boldsymbol{H}}$). (c) If the product

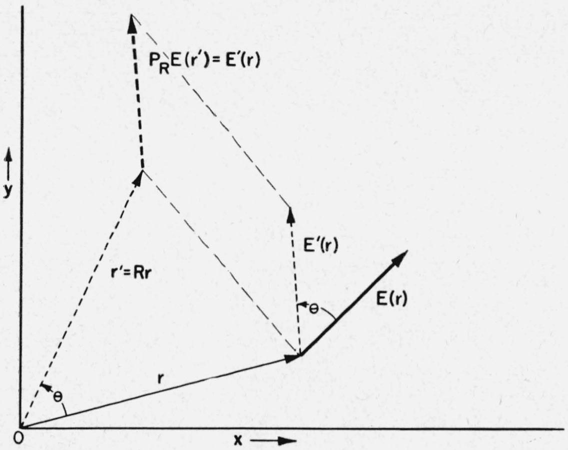

FIGURE 4. *Proper rotation of a vector.*

$\boldsymbol{FG}$ is a scalar (e. g., $\boldsymbol{F}=\boldsymbol{E}\cdot\sigma$, $\boldsymbol{G}=\overline{\boldsymbol{E}}$, $\boldsymbol{FG}=\boldsymbol{E}\cdot\sigma\cdot\overline{\boldsymbol{E}}$) then

$$\int_V \boldsymbol{FG}\,dV=\int_{P_RV}(P_R\boldsymbol{F})\,(P_R\boldsymbol{G})dV, \tag{27}$$

and in this sense P_R is unitary. The same property obviously holds also for surface integrals. (d) The identities

$$\left.\begin{aligned}\nabla_r[P_Rf(\boldsymbol{r})]&=P_R[\nabla_rf(\boldsymbol{r})],\\ \nabla_r\cdot[P_R\boldsymbol{E}(\boldsymbol{r})]&=P_R[\nabla_r\cdot\boldsymbol{E}(\boldsymbol{r})],\\ \nabla_r\times[P_R\boldsymbol{E}(\boldsymbol{r})]&=P_R[\nabla_r\times\boldsymbol{E}(\boldsymbol{r})],\end{aligned}\right\} \tag{28}$$

in which the subscript r denotes differentiations with respect to $\boldsymbol{r}$, lead to the statement that P_R commutes with the ∇-operations.

The *product* P_SP_R (of P_R *followed* by P_S) is defined by $(P_SP_R)\boldsymbol{F}\equiv P_S(P_R\boldsymbol{F})$ for arbitrary $\boldsymbol{F}$, where $P_S(P_R\boldsymbol{F})$ denotes the function produced from $\boldsymbol{F}$ by first applying P_R to $\boldsymbol{F}$ and then P_S to $P_R\boldsymbol{F}$. The product of two P-operations is itself a P-operation and in fact $P_SP_R=P_{SR}$, where the matrix representing $\boldsymbol{SR}$ is the matrix product SR. It is worth while to verify this multiplication law, say in the vector case. Letting $\boldsymbol{r}''=\boldsymbol{Sr}'=\boldsymbol{SRr}$ and thrice using eq 24b, one may write

$$\begin{aligned}\{P_S[P_R\boldsymbol{F}(\boldsymbol{r}'')]\}_x&=\sum_y S_{xy}[P_R\boldsymbol{F}(\boldsymbol{r}')]_y\\ &=\sum_{y,z}S_{xy}R_{yz}F_z(\boldsymbol{r})\\ &=\sum_z(SR)_{xz}F_z(\boldsymbol{r})=[P_{SR}\boldsymbol{F}(\boldsymbol{r}'')]_x.\end{aligned}$$

The equality of the first and last members, holding identically in $\boldsymbol{r}''$ for an arbitrary vector function $\boldsymbol{F}$, shows in the vector case that P_SP_R as defined is equal to P_{SR}. Since the collection $\{R\}$ of all three-dimensional real orthogonal matrices, upon matrix multiplication, constitutes a group, it is clear that the collection $\{P_R\}$ of all rotations of a given tensor function, upon multiplication as defined, also constitutes a group; $\{R\}$ and $\{P_R\}$ are abstractly identical and may be identified as the three-dimensional rotation-reflection group. The groups of interest in the present discussion are subgroups [17] of this group, as will become clear from the content of the next few paragraphs.

A tensor function $\boldsymbol{F}$ will be said to be *invariant* with respect to the rotation P_R (in which O is a fixed point) if

$$P_R\boldsymbol{F}(\boldsymbol{r})=\boldsymbol{F}(\boldsymbol{r}), \tag{29}$$

for all $\boldsymbol{r}$ in the domain V of definition of $\boldsymbol{F}$. (For this equation to hold it is necessary that $P_RV=V$.) When eq 29 holds, P_R will be said to be a *covering operation of* $\boldsymbol{F}$.

For any choice of O, the corresponding set of covering operations $P_Q,P_R,P_S, \cdots$ of a given function constitutes a group. For, associative multiplication is defined within the set: $(P_RP_S)\boldsymbol{F}=P_R(P_S\boldsymbol{F})=P_R\boldsymbol{F}=\boldsymbol{F}$; moreover, the set contains the identity operation P_I (obviously) and the inverse of each of its operations: $P_S^{-1}\boldsymbol{F}=P_S^{-1}P_S\boldsymbol{F}=\boldsymbol{F}$. It may happen, of course, that a group of covering operations actually consists only of the single element P_I; this trivial case is obviously of no interest in what follows.

It may be recalled that the structure of a waveguide junction is described by three functions, $\sigma(\boldsymbol{r})$, $\mu(\boldsymbol{r})$, $\epsilon(\boldsymbol{r})$, which are in general tensors of rank two, defined in a region V. If σ,μ,ϵ are invariant with respect to P_R, we say that the structure is invariant with respect to P_R and that P_R is a covering operation of the structure. For any choice of O, the set of covering operations of a given structure will evidently be a group. If O is chosen so as to make the group of covering operations as large as possible, then this group may be said to characterize the symmetry of the structure and be called the symmetry group of the structure.[18] The symbol $\{P_R\}$, used above for the three-dimensional rotation-reflection group, will be used also for symmetry groups.

IV. Symmetry Analysis

Consider a waveguide junction having the symmetry group $\{P_R\}$, and let $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy

$$\left.\begin{aligned}\nabla\times\boldsymbol{E}&=-j\omega\mu\cdot\boldsymbol{H}\\ \nabla\times\boldsymbol{H}&=(j\omega\epsilon+\sigma)\cdot\boldsymbol{E}\end{aligned}\right\}\text{ in } V,\qquad \boldsymbol{E}_t=\boldsymbol{F}\text{ on } S, \tag{30}$$

[17] For a discussion of these groups, see, e. g., J. Rosenthal and G. M. Murphy, Rev. Mod. Phys. 8, 317 (1936).

[18] If μ,ϵ are scalar constants and $\sigma\equiv O$ in V, then the necessary condition $P_RV=V$ contained in eq 29 becomes also sufficient and the symmetry group will be determined by V. In this way the present formulation includes cases in which symmetry is determined by geometrical figure alone.

where the vector function $\boldsymbol{F}$, defined on the complete boundary S of V, denotes in a unified notation any prescribed set of electric fields on the terminal surfaces and the homogeneous boundary condition holding on all other parts of S. Let the whole system—structure and field—be subjected, in imagination, at least, to the (proper or improper) rotation in space corresponding to P_R. One obtains an analytical description of the resulting situation by applying P_R to V, to S, and to both sides of each of the three equations above.

$$\left.\begin{aligned}\nabla\times(P_R\boldsymbol{E})&=-j\omega(P_R\mu)\cdot P_R\boldsymbol{H}\\ \nabla\times(P_R\boldsymbol{H})&=(j\omega P_R\epsilon+P_R\sigma)\cdot P_R\boldsymbol{E}\end{aligned}\right\}\text{in } P_RV,$$

$$P_R(\boldsymbol{E}_t)=P_R\boldsymbol{F} \text{ on } P_RS. \tag{31}$$

These equations hold formally whether or not the structure is invariant with respect to P_R; indeed, if P_R is a proper rotation, the configuration of the system relative to itself is in no way altered. However, the structure is by hypothesis invariant, so that eq 31 is equivalent to

$$\left.\begin{aligned}\nabla\times(P_R\boldsymbol{E})&=-j\omega\mu\cdot P_R\boldsymbol{H}\\ \nabla\times(P_R\boldsymbol{H}&=(j\omega\epsilon+\sigma)\cdot P_R\boldsymbol{E}\end{aligned}\right\}\text{in } V,\quad (P_R\boldsymbol{E})_t=P_R\boldsymbol{F}\text{ on } S. \tag{32}$$

The differential equations here are the same as those in eq 30; that is to say, the macroscopic Maxwell's equations of the problem are invariant with respect to P_R. The field $P_R\boldsymbol{E}$, $P_R\boldsymbol{H}$ appearing in eq 32, although closely related to the original $\boldsymbol{E}$, $\boldsymbol{H}$, is nevertheless a new field relative to the fixed axes Oxyz; and since the rotated structure is indistinguishable from the nonrotated one, $P_R\boldsymbol{E}$, $P_R\boldsymbol{H}$ may be counted as a new field relative to the structure. One may, of course, consider that only the field, and not the structure, is rotated.[19] The essential observations to be made here are: (a) If $\boldsymbol{E}$, $\boldsymbol{H}$ is a possible field in V, so also is $P_R\boldsymbol{E}$, $P_R\boldsymbol{H}$ (so that the vector spaces of eq 7, 8 are closed with respect to symmetry operations), (b) $P_R\boldsymbol{E}$, $P_R\boldsymbol{H}$ is essentially determined by the transformed terminal fields prescribed in eq 32, since the stated boundary conditions holding on all parts of S other than the terminal surfaces are not affected by P_R.

It follows from (a) above that the set of electric fields coinciding initially with the primitive basis $\boldsymbol{e}$ spans a representation of the symmetry group $\{P_R\}$. Namely, since $\boldsymbol{e}_{m\mu}$ is a possible electric field, $P_R\boldsymbol{e}_{m\mu}$ is also a possible electric field, and so $P_R\boldsymbol{e}_{m\mu}$ is expressible as a linear combination of primitive electric fields:

$$P_R\boldsymbol{e}_{m\mu}(\boldsymbol{r})=\sum_{l\lambda}\boldsymbol{e}_{l\lambda}(\boldsymbol{r})D(R)_{l\lambda,\,m\mu}. \tag{33}$$

(This equation is not to be regarded as defining a change of basis; the coefficient $D(R)_{l\lambda,\,m\mu}$ is to be regarded as a tensor component defined relative to the basis $\boldsymbol{e}$ of the N-dimensional vector space introduced in section II.) We take $D(R)_{l\lambda,\,m\mu}$ as the element in the $(l\lambda)^{\text{th}}$ row and the $(m\mu)^{\text{th}}$ column of an $N\times N$ matrix $D(R)$. These matrices constitute a representation of $\{P_R\}$. For, calculation of

$$\begin{aligned}P_Q(P_R\boldsymbol{e}_{m\mu})&=\sum_{l\lambda}(P_Q\boldsymbol{e}_{l\lambda})D(R)_{l\lambda,\,m\mu}\\&=\sum_{l\lambda,\,k\kappa}\boldsymbol{e}_{k\kappa}D(Q)_{k\kappa,\,l\lambda}D(R)_{l\lambda,\,m\mu}\\&=\sum_{k\kappa}\boldsymbol{e}_{k\kappa}[D(Q)D(R)]_{k\kappa,\,m\mu},\end{aligned}$$

and comparison of this result with

$$(P_QP_R)\boldsymbol{e}_{m\mu}=\sum_{k\kappa}\boldsymbol{e}_{k\kappa}D(QR)_{k\kappa,\,m\mu},$$

shows that the matrix associated with the product P_QP_R is in fact the product, in the proper order, of the matrices individually associated with P_Q and P_R.

To determine further essential properties of $D(R)$ we utilize observation (b) above and consider the transformation of the terminal-surface boundary conditions that define $\boldsymbol{e}_{m\mu}$. Upon transformation by P_R, the standard terminal-field $\boldsymbol{e}^0_{m\mu}$, defined on terminal surface S_m in waveguide m, goes over into the field $P_R\boldsymbol{e}^0_{m\mu}$ defined on (and tangential to) the terminal surface in an equivalent waveguide whose index may be denoted by $R(m)$. $P_R\boldsymbol{e}^0_{m\mu}$ must accordingly be expressible linearly in terms of the standard terminal fields in waveguide $R(m)$. Thus, letting $R(m)=k$ to simplify the typography,

$$P_R\boldsymbol{e}^0_{m\mu}(\boldsymbol{r}_k)=\sum_{\kappa=1}^{\nu_\kappa}\boldsymbol{e}^0_{k\kappa}(\boldsymbol{r}_k)D(R)_{k\kappa,\,m\mu},\qquad(\boldsymbol{r}_k\text{ on }S_k). \tag{34}$$

Since the tangential components of $P_R\boldsymbol{e}_{m\mu}$ are zero on terminal surfaces other than S_k, eq 34 determines $P_R\boldsymbol{e}_{m\mu}$. Hence $D(R)_{l\lambda,\,m\mu}=0$, $l\neq R(m)$; also, since the standard terminal-fields are real, $D(R)$ must be real. The nonvanishing elements of $D(R)$ can be presented in a set of n submatrices $D(R)_{k,\,m}$ ($D(R)_{k,\,m}$ is square and of dimension ν_m). The arrangement of these submatrices within $D(R)$ is in accordance with the scheme of permutation, $m\rightarrow R(m)$, of terminal fields among equivalent waveguides. (In example A, section V, the permutation $1\rightarrow2$, $2\rightarrow1$, $3\rightarrow3$, $4\rightarrow4$ for $n=4$ occurs.) Thinking primarily of the general case $\nu_m>1$, but not excluding $\nu_m=1$, we determine the nature of the submatrices in the following manner. For the moment let two modes (in the same or in equivalent waveguides) be termed equivalent if they have equal wave-admittances, so that equivalent modes are not only "degenerate" but also of the same kind (*TE*, *TM*, or *TEM*). It is clear that the right-hand side of eq 34 can involve only modes equivalent to the μ^{th} mode in the m^{th} waveguide. Hence (assuming suitable ordering of mode indices) the nonvanishing elements of $D(R)_{k,\,m}$ will appear in smaller square

[19] This point of view is helpful in fixing the significance of the waveguide index $m(m=1,2,\cdots n)$, which is to be understood to identify a waveguide lead in a fixed spatial location.

submatrices ("steps") lying on the main diagonal of $D(R)_{k,m}$. Each step relates equivalent modes and is itself a real unitary (=real orthogonal) matrix. For, considering any two equivalent modes μ,μ' (distinct or not), one finds with the aid of eq 4, 5, 27, and 34,

$$\delta_{\mu\mu'}=(\zeta_0\eta_{m\mu})\int_{S_m}\boldsymbol{e}^0_{m\mu}\cdot\boldsymbol{e}^0_{m\mu'}\,dS$$

$$=(\zeta_0\eta_{m\mu})\int_{S_k}(P_R\boldsymbol{e}^0_{m\mu})\cdot(P_R\boldsymbol{e}^0_{m\mu'})\,dS$$

$$=\sum_{\kappa,\kappa'}\delta_{\kappa\kappa'}D(R)_{k\kappa,m\mu}D(R)_{k\kappa',m\mu'}=\sum_{\kappa}D(R)_{k\kappa,m\mu}D(R)_{k\kappa,m\mu'},$$

where $k=R(m)$, and $\eta_{m\mu}$ is the common value of the wave-admittances of the modes involved. Thus each step, hence each $D(R)_{k,m}$, hence $D(R)$ itself, is a real unitary matrix.

Parenthetically it may be noted that even if ν_m is large the dimension of an individual step ordinarily will not exceed two or three. (It can be shown that the highest symmetry degeneracy of wave-guide modes is two-fold.) Of course, $\nu_m=1$ frequently occurs in actual problems; the submatrix $D(R)_{k,m}$ then consists of a single one-dimensional step and is necessarily equal to ±1.

The magnetic fields of the basis $\boldsymbol{h}$ transform under rotation according to the same group-representation as do the electric fields of the basis $\boldsymbol{e}$. Perhaps the most easily visualized proof of this is the following. From eq 34 and 4 one may obtain

$$P_R[\boldsymbol{h}^0_{m\mu}(\boldsymbol{r}_k)\times\boldsymbol{k}_m]=[\sum_{\kappa}\boldsymbol{h}^0_{k\kappa}(\boldsymbol{r}_k)D(R)_{k\kappa,m\mu}]\times\boldsymbol{k}_k,$$
$$(\boldsymbol{r}_k \text{ on } S_k;\ k=R(m))$$

where, again, the wave-admittances drop out. The left-hand side may be replaced by $[P_R\boldsymbol{h}^0_{m\mu}(\boldsymbol{r}_k)]\times\boldsymbol{k}_k$ (cf eq 26); and since the fields are transverse, $\boldsymbol{k}_k$ may be canceled from the resulting equation. Thus one obtains for the tangential component of $P_R\boldsymbol{h}_{m\mu}$ on S_k

$$P_R\boldsymbol{h}^0_{m\mu}(\boldsymbol{r}_k)=\sum_{\kappa}\boldsymbol{h}^0_{k\kappa}(\boldsymbol{r}_k)D(R)_{k\kappa,m\mu},$$
$$(\boldsymbol{r}_k \text{ on } S_k;\ k=R(m)).\qquad(35)$$

Since the tangential components of $P_R\boldsymbol{h}_{m\mu}$ vanish on terminal surfaces other than S_k, eq 35 implies

$$P_R\boldsymbol{h}_{m\mu}(\boldsymbol{r})=\sum_{l\lambda}\boldsymbol{h}_{l\lambda}(\boldsymbol{r})D(R)_{l\lambda,m\mu},\qquad(\boldsymbol{r}\text{ in }V),\qquad(36)$$

where $D(R)_{l\lambda,m\mu}=0$, $l\neq R(m)$, as above. In view of eq 34 and 35 it is clear that to determine $D(R)$ in a concrete case one may consider the transformation of either the electric or the magnetic standard terminal-fields. If the magnetic terminal-fields are considered, it should not be forgotten that they are pseudovectorial.

The transformation of an arbitrary electromagnetic state upon rotation by P_R is obviously determined by eq 33, 36 and is conveniently presented in matrix form. In matrix notation eq 33 and 36 become

$$P_R\boldsymbol{e}=\boldsymbol{e}D(R),\qquad P_R\boldsymbol{h}=\boldsymbol{h}D(R),\qquad(37)$$

where $P_R\boldsymbol{e}$ is interpreted simply as $(P_R\boldsymbol{e}_{11}\cdots P_R\boldsymbol{e}_{m\mu}\cdots P_R\boldsymbol{e}_{n\nu_n})$; $P_R\boldsymbol{h}$, similarly. For $\boldsymbol{E}=\boldsymbol{e}v$ and $\boldsymbol{H}=\boldsymbol{h}i$, then,

$$P_R\boldsymbol{E}=\boldsymbol{e}D(R)v=\boldsymbol{e}v_R,\qquad P_R\boldsymbol{H}=\boldsymbol{h}D(R)i=\boldsymbol{h}i_R,$$

where, by definition, $D(R)v=v_R$, $D(R)i=i_R$. That is to say, if $\boldsymbol{E}$ has the coordinates v relative to $\boldsymbol{e}$ as basis, then $P_R\boldsymbol{E}$ has the coordinates v_R relative to the same basis; similarly for $\boldsymbol{H}$. The interpretation of P_R as a matrix operator, defined with respect to v and i (which represent electromagnetic states) as operands, is immediate:

$$P_Rv\equiv D(R)v=v_R,\qquad P_Ri\equiv D(R)i=i_R.\qquad(38)$$

The matrices $D(R)$ are, of course, basis-dependent. A unitary change of basis induces the similarity transformation

$$D(R)\rightarrow D'(R)=\alpha^{-1}D(R)\alpha,\qquad(39)$$

as is readily verified with aid of eq 15 and (choosing one of several possibilities) eq 37. The essential practical problem in symmetry analysis, briefly stated, is to find a basis—that is, to find a transforming matrix α—such that the representation D (of the symmetry group involved) whose matrices are $D(R)$ will be reduced-out [20] by the transformation 39. Such a basis, in which the matrices $D'(R)$ of the representation D' appear in reduced-out form, will be termed a *symmetry basis* and the corresponding coordinates, *symmetry coordinates*. The principal items to be considered in the remainder of the analysis are the construction of D', the construction of α, and the physical consequences; it is expedient to consider these items in this order.

A reduced-out representation D', in which the irreducible components of D are to appear explicitly, can in principle be written down as soon as the irreducible components of D are determined. Let χ denote the character of D, and let χ^p denote the character of the p^{th} irreducible representation D^p of the symmetry group involved.[21] Then, by a basic theorem, the nonnegative integer c_p that tells how many times D^p must appear in D' is given by [22]

$$c_p=(1/g)\sum_R\overline{\chi^p(R)}\chi(R),\qquad(40)$$

where g is the order of the symmetry group and the

[20] For the concepts and theorems of the theory of group representations that will be needed here it will be convenient to refer to Wigner (Gruppentheorie, cited in footnote 16), especially chapters IX and XII.

[21] Character tables are given, e. g., by Rosenthal and Murphy (footnote 17), and by G. Herzberg, Infrared and Raman spectra of polyatomic molecules (D. Van Nostrand Co., New York, 1945). Irreducible representations have apparently not been tabulated, but many of them are easily found (for one-dimensional representations, $D^p\equiv\chi^p$).

[22] Wigner, p. 95.

summation goes over the group.[23] (The complex conjugate taken in the right-hand side of eq 40, which makes the formula correct in general, is of no significance here because c_p is real in general and χ is real in the present problem.) Let the (distinct) nonequivalent irreducible representations actually contained in D (i. e., those for which $c_p \neq 0$) be numbered from 1 to t. Further, let $D^p(R)_{\pi\rho}$ denote the element in the π^{th} row and the ρ^{th} column of the matrix $D^p(R)$ of the irreducible representation D^p, which must be unitary and is considered known; and let l_p denote the dimension of D^p. For the element in the $(p\pi a)^{\text{th}}$ row and the $(q\rho b)^{\text{th}}$ column of the matrix $D'(R)$ we write

$$D'(R)_{p\pi a,\, q\rho b} = \delta_{pq}\delta_{ab}D^p(R)_{\pi\rho}, \tag{41}$$

where the indices are to have the ranges

$$a=1,2,\cdots,c_p; \qquad \pi=1,2,\cdots,l_p; \qquad p=1,2,\cdots,t;$$

$$b=1,2,\cdots,c_q; \qquad \rho=1,2,\cdots,l_q; \qquad q=1,2,\cdots,t.$$

(Whenever $l_p \neq l_q$, undefined symbols $D^p(R)_{\pi\rho}$ multiplied by zero occur in the right-hand side of eq 41; such "products" are defined to be zero.) The elements of $D'(R)$ furnished by eq 41 may, if desired, be arranged in such a way that $D'(R)$ takes the form

$$D'(R) = \begin{bmatrix} D^1(R) & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & D^1(R) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & D^2(R) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & D^t(R) \end{bmatrix}, \tag{42}$$

in which the matrix $D^1(R)$ appears c_1 times, $D^2(R)$ appears c_2 times, etc. (Any consistent scheme of ordering elements into matrices may be assumed, and it may be noted that an order other than one that leads to eq 42 will be convenient later.)

In the formation of D' by means of eq 41, the choice of the particular irreducible representation D^p among the unitary representations equivalent to D^p is merely a matter of convenience. (This question does not arise when $l_p=1$, of course.) However, an irreducible representation appearing more than once in D' is to be presented in identical form each time it appears, as indeed is insured by eq 41. The arbitrariness in D' implies an arbitrariness in the choice of a symmetry basis; of more interest in this paper, however, is further arbitrariness in a symmetry basis that remains after D' is assumed. This arbitrariness will appear in the course of the construction of the transforming matrix α and will be utilized in a brief discussion at the end of this section.

Let the row-matrices $\boldsymbol{e}'$, $\boldsymbol{h}'$ present the basis fields of a symmetry basis in which the matrices of D' are in fact given by eq 41. In this basis eq 37 become $P_R\boldsymbol{e}'=\boldsymbol{e}'D'(R)$, $\quad P_R\boldsymbol{h}'=\boldsymbol{h}'D'(R)$; where, in accordance with the index notation established in eq 41, $\boldsymbol{e}'=(\boldsymbol{e}'_{p\pi a})$, $\boldsymbol{h}'=(\boldsymbol{h}'_{p\pi a})$. The characteristic transformation equations of basis fields in a symmetry basis follow directly from the form of $D'(R)$:

$$P_R\boldsymbol{e}'_{p\pi a}=\sum_{\rho=1}^{l_p}\boldsymbol{e}'_{p\rho a}D^p(R)_{\rho\pi}, \qquad P_R\boldsymbol{h}'_{p\pi a}=\sum_{\rho=1}^{l_p}\boldsymbol{h}'_{p\rho a}D^p(R)_{\rho\pi}. \tag{43a, b}$$

The l_p members of a set of basis fields (electric or magnetic) identified by fixed p and a transform among themselves according to the irreducible representation D^p of $\{P_R\}$, and they are said to be *partners* and to be of *symmetry species* p (relative to $\{P_R\}$). There are c_p sets of basis fields (electric or magnetic) of the species p, in accordance with the range $1,2,\cdots,c_p$ of the index a. More specifically, $\boldsymbol{e}'_{p\pi a}$ or $\boldsymbol{h}'_{p\pi a}$ is said to belong to the π^{th} row of the irreducible representation D^p; there are, again, c_p such electric or magnetic basis fields. The same terminology may be applied, of course, to other fields or other entities that transform under rotation according to an irreducible representation as in eq 43.

Let $\boldsymbol{e}'_{p\pi a}$ and $\boldsymbol{h}'_{p\pi a}$, which are to obey eq 43, be given by

$$\boldsymbol{e}'_{p\pi a}=\sum_{m\mu}\boldsymbol{e}_{m\mu}\alpha_{m\mu,\, p\pi a}\equiv\boldsymbol{e}\alpha_{p\pi a},$$

$$\boldsymbol{h}'_{p\pi a}=\sum_{m\mu}\boldsymbol{h}_{m\mu}\alpha_{m\mu,\, p\pi a}\equiv\boldsymbol{h}\alpha_{p\pi a},$$

where $\alpha_{p\pi a}$ denotes the $(p\pi a)^{\text{th}}$ column of the unitary transforming matrix α (cf eq 15 and the parenthetical remark following eq 16). (Note that the column matrix $\alpha_{p\pi a}$ presents the coordinates, relative to the primitive basis, of $\boldsymbol{e}'_{p\pi a}$ and $\boldsymbol{h}'_{p\pi a}$.) From either eq 43a or 43b we may derive an equation similar to eq 43 but applying to the $\alpha_{p\pi a}$. Namely, if in eq 43a we replace $\boldsymbol{e}'_{p\pi a}$ by $\boldsymbol{e}\alpha_{p\pi a}$ (and $\boldsymbol{e}'_{p\rho a}$ by $\boldsymbol{e}\alpha_{p\rho a}$) and then invoke the matrix interpretation of P_R (eq 37, 38) and abstract $\boldsymbol{e}$, we obtain

$$P_R\alpha_{p\pi a}=\sum_{\rho=1}^{l_p}\alpha_{p\rho a}D^p(R)_{\rho\pi}. \tag{44}$$

Thus $\alpha_{p\pi a}$, as well as $\boldsymbol{e}'_{p\pi a}$ and $\boldsymbol{h}'_{p\pi a}$, may be said to belong to the π^{th} row of the irreducible representation D^p. Conversely, it may be seen that if the columns of a unitary matrix obey eq 44, then the corresponding basis fields will obey eq 43. (Equation 44 is, moreover, equivalent to eq 39 and may be derived from that equation by equating the $(p\pi a)^{\text{th}}$ columns of the matrix products on the two sides of $D(R)\alpha=\alpha D'(R)-D'(R)$ being given by eq 41.)

A systematic procedure for the construction of α may be based upon eq 44, assuming that the matrices of the irreducible representations involved are known. (In a concrete problem various expedients, including judicious guessing, may often be employed to advantage.) We shall give the con-

[23] Only finite symmetry groups will be considered explicitly.

struction [24] almost in the form of a recipe, deferring the justification to the next paragraph. Define the $N\times 1$ matrix $v^{\binom{p}{\pi}}$ as a function of an arbitrary ($N\times 1$ matrix) v by means of the equation

$$v^{\binom{p}{\pi}}=(l_p/g)\sum_R \overline{D^p(R)}_{\pi\pi}P_R v, \tag{45a}$$

which may be written

$$v^{\binom{p}{\pi}}=[(l_p/g)\sum_R \overline{D^p(R)}_{\pi\pi}D(R)]v=G^{\binom{p}{\pi\pi}}v, \tag{45b}$$

thereby defining the $N\times N$ "generating" matrix $G^{\binom{p}{\pi\pi}}$. Corresponding to any set of N linearly independent v's, eq 45 yields a set of exactly c_p linearly independent $v^{\binom{p}{\pi}}$'s; each $v^{\binom{p}{\pi}}$, hence any linear combination of $v^{\binom{p}{\pi}}$'s, belongs to the π^{th} row of D^p. Select a set of c_p such combinations that are mutually orthogonal and normalized to unity (in the Hermitian sense): these may be taken as $\alpha_{p\pi 1}$, $\alpha_{p\pi 2}$, $\cdots$, $\alpha_{p\pi c_p}$. Define the matrix $G^{\binom{p}{\rho\pi}}$ and construct the l_p-1 partners of $\alpha_{p\pi a}$ by means of

$$\alpha_{p\rho a}=(l_p/g)\sum_R \overline{D^p(R)}_{\rho\pi}P_R\alpha_{p\pi a}, \tag{46a}$$

$$=[(l_p/g)\sum_R \overline{D^p(R)}_{\rho\pi}D(R)]\alpha_{p\pi a}\equiv G^{\binom{p}{\rho\pi}}\alpha_{p\pi a}, \tag{46b}$$

for $\rho=1, \cdots, \pi-1, \pi+1, \cdots, l_p$ and $a=1, 2, \cdots, c_p$. The $\alpha_{p\rho a}$'s so obtained will automatically satisfy $\alpha^*_{p\rho a}\alpha_{p\sigma b}=\delta_{\rho\sigma}\delta_{ab}$. The procedure is to be carried out for each symmetry species ($p=1, 2, \cdots, t$); the orthogonality $\alpha^*_{p\pi a}\alpha_{q\rho b}=0$, $p\neq q$, will automatically hold. Thus one obtains altogther a complete set of column matrices that satisfy eq 44 and are properly orthonormalized to serve as columns of α.

For the most part, the above construction represents an immediate application of theorems given by Wigner.[25] Beyond this it is only necessary to show that (a) $G^{\binom{p}{\pi\pi}}$ is in fact of rank c_p and (b) $\alpha_{p\rho a}$ and $\alpha_{p\rho b}$, as given by eq 46, are orthogonal when $\alpha_{p\pi a}$ and $\alpha_{p\pi b}$ are. As for (a): since the representations D and D' are unitary and equivalent, there is no question of the existence of a unitary transformation connecting them. Hence the equation defining $G^{\binom{p}{\pi\pi}}$ may be transformed formally into

$$\alpha^{-1}G^{\binom{p}{\pi\pi}}\alpha=(l_p/g)\sum_R \overline{D^p(R)}_{\pi\pi}D'(R).$$

Considering $D'(R)$ in the form 42 (say) and applying the orthogonality theorem given below (eq 50) one finds that $\alpha^{-1}G^{\binom{p}{\pi\pi}}\alpha$ is a diagonal matrix having exactly c_p nonvanishing diagonal elements (each equal to 1), so that $\alpha^{-1}G^{\binom{p}{\pi\pi}}\alpha$, hence $G^{\binom{p}{\pi\pi}}$, is of rank c_p. As for (b): from the relation

$$(G^{\binom{p}{\rho\pi}})^*=(l_p/g)\sum_R D^p(R)_{\rho\pi}D(R)^*=$$

$$(l_p/g)\sum_R \overline{D^p(R^{-1})}_{\pi\rho}D(R^{-1})=G^{\binom{p}{\pi\rho}},$$

and the observation that, for a set of partners, eq 46 holds for all π,ρ (in their proper range),[26] one obtains

$$\alpha^*_{p\pi a}\alpha_{p\pi a}=\alpha^*_{p\pi a}(G^{\binom{p}{\rho\pi}})^*\alpha_{p\rho b}=\alpha^*_{p\pi a}G^{\binom{p}{\pi\rho}}\alpha_{p\rho b}=\alpha^*_{p\pi a}\alpha_{p\pi b},$$

which establishes (b).

Structural symmetry of a waveguide junction places certain restrictions on the form of the matrices Z, Y, and S (defined in eq 12, 18): some matrix elements may be forced to vanish, and the number of independent elements may be reduced materially,[27] as is well known in a variety of cases. One form of the governing equations may be found in the following way. The relation $v=Zi$ holds, of course, for any given electromagnetic state in the junction. If the fields are subjected to a symmetry operation, then $v\rightarrow v_R$, $i\rightarrow i_R$ (eq 38) and $v_R=Zi_R$ must hold, for v_R,i_R are coordinates of a possible state in the given invariant structure (cf footnote 19). Now $v_R=Zi_R$ is the same as $D(R)v=ZD(R)i$, v may be replaced by Zi, and i is arbitrary. Hence

$$D(R)Z=ZD(R), \qquad \text{every } R \text{ of } \{P_R\},$$

and we say that Z commutes with the representation D of the symmetry group. It immediately follows that $Z^{-1}(=Y)$ commutes with D and that $(Z-\zeta_0)$ and $(Z+\zeta_0)^{-1}$, hence $(Z-\zeta_0)(Z+\zeta_0)^{-1}(=S)$, commute with D. (In fact any rational matrix function of Z commutes with D.) Thus, if M denotes Z, Y, or S,

$$D(R)M=MD(R), \qquad \text{every } R \text{ of } \{P_R\}. \tag{47}$$

This equation contains the conditions imposed by symmetry upon the several matrices characterizing the junction. To find the consequences of eq 47 we consider it presented in a symmetry basis.

Upon transformation to a symmetry basis, M and D undergo one and the same similarity transformation (eq 15, 19, 39), and eq 47 becomes $D'(R)M'=M'D'(R)$. This may be written $M'=D'(R)^*M'D'(R)$, and for the element $M'_{p\pi a,q\rho b}$ of M' one obtains, using eq 41,

$$M'_{p\pi a,q\rho b}=\sum_{\sigma,\tau}\overline{D^p(R)}_{\sigma\pi}D^q(R)_{\tau\rho}M'_{p\sigma a,q\tau b}. \tag{48}$$

[24] The method to be given is essentially equivalent to one given (for the construction of molecular symmetry coordinates) by J. R. Nielson and L. H. Berryman, J. Chem. Phys. **17**, 659 (1949).

[25] Chap XII, eq 1, 3, 3a, 6, and 8. Compare eq 44, 45a, 46a with eq 1, 6, 3a, respectively. The reinterpretation of Wigner's equations with his functional P_R and f replaced by our matrix P_R and v is immediate.

[26] Wigner, Chap XII, eq 3a.

[27] Results more detailed than those to be obtained in the text are obtained in appendix, 2.

The g equations of this type (one for each R of $\{P_R\}$) may be added to obtain

$$gM'_{p\pi a,\,q\rho b}=\sum_{\sigma,\tau}\left[\sum_R \overline{D^p(R)}_{\sigma\pi}D^q(R)_{\tau\rho}\right]M'_{p\sigma a,\,q\tau b}. \qquad (49)$$

Now according to the orthogonality theorem for unitary irreducible representations,[28]

$$\sum_R \overline{D^p(R)}_{\sigma\pi}D^q(R)_{\tau\rho}=\delta_{pq}\,\delta_{\sigma\tau}\,\delta_{\pi\rho}(g/l_p), \qquad (50)$$

so that eq 49 becomes

$$\left.\begin{aligned} &M'_{p\pi a,\,q\rho b}=0,\text{ unless } q=p \text{ and } \rho=\pi,\\ &M'_{p\pi a,\,p\pi b}=(1/l_p)\sum_\sigma M'_{p\sigma a,\,p\sigma b}\equiv m^p_{ab}, \end{aligned}\right\} \qquad (51)$$

where m^p_{ab} is a constant independent of π. M' is determined by the values of the $c_1^2+c_2^2+\cdots+c_t^2$ constants m^p_{ab}, which are arbitrary so far as symmetry of the junction is concerned. A convenient step-matrix form for M' is obtained by arranging elements in dictionary-like order according to the values of the indices ρ, π, a in the sequence $p\pi a$. (The sequence $pa\pi$ yields $D'(R)$ in the form 42.) If $t=2$, for example, and $l_1=2$, $c_1=2$, $l_2=2$, $c_2=1$, then M' will have the particular step form

	$s\pi a$	$s\pi b$	$s\rho a$	$s\rho b$	$t\pi a$	$t\rho a$
$s\pi a$	m^s_{aa}	m^s_{ab}	0	0	0	0
$s\pi b$	m^s_{ba}	m^s_{bb}	0	0	0	0
$s\rho a$	0	0	m^s_{aa}	m^s_{ab}	0	0
$s\rho b$	0	0	m^s_{ba}	m^s_{bb}	0	0
$t\pi a$	0	0	0	0	m^t_{aa}	0
$t\rho a$	0	0	0	0	0	m^t_{aa}

The general M' will have, for each p, l_p identical steps of dimension c_p. (If $c_p=1$, every p, the steps are elements and M' is diagonal.)

Additional restrictions on M' corresponding to the conditions discussed in section II, eq 20, 21, are readily imposed. To fulfill I or II, M must satisfy one of the conditions $M^*=-M$, $M^*=M$, $M^*=M^{-1}$; and M' must satisfy the same condition. The reciprocity condition (III) requires $\tilde{M}=M$, and M' must satisfy

$$\tilde{M}'=(\tilde{\alpha}\alpha)\,M'\,(\tilde{\alpha}\alpha)^{-1}, \qquad (52)$$

—or simply $\tilde{M}'=M'$ when α is real.

The form of M may obviously be found from that of M' by calculating $M=\alpha M'\alpha^{-1}$.

[28] Wigner, p. 91.

It is of interest to consider briefly the special case in which M is a *normal*[29] matrix. Either of the conditions I and II is sufficient to insure that M be normal.[30] When M is normal, M can be reduced to diagonal form by transformation to a suitably chosen symmetry basis. This statement is easily established by considering the transformation needed to diagonalize M'. If M is normal, so also is M'; from the form of M' (eq 51) it is evident that M' can be diagonalized by transformation with a unitary matrix β of the form

$$\beta_{p\pi a,\,q\rho b}=0,\text{ unless } p=q \text{ and } \pi=\rho,$$

$$\beta_{p\pi a,\,p\pi b}=\beta^p_{ab},$$

where β^p is a suitable unitary matrix of dimension c_p. (The argument is trivial if $c_p=1$, every p.) The change of basis corresponding to this transformation may be denoted $\boldsymbol{e}'\rightarrow\boldsymbol{e}''$, where $\boldsymbol{e}''=\boldsymbol{e}'\beta=\boldsymbol{e}(\alpha\beta)$, and we have

$$M'\rightarrow M''=\beta^{-1}M'\beta=(\alpha\beta)^{-1}M(\alpha\beta),$$

where the diagonal matrix M'' presents the eigenvalues of M, and the columns of the combined transforming matrix $\alpha\beta$ are eigenvectors of M. Let us put $\alpha\beta=\gamma$ and examine the $(p\pi a)^{th}$ column of γ. From the form of β,

$$\gamma_{p\pi a}=\sum_{b=1}^{c_p}\beta^p_{ab}\alpha_{p\pi b}; \qquad (53)$$

that is, the eigenvector $\gamma_{p\pi a}$ is a linear combination of the columns of α that belong to the π^{th} row of the irreducible representation D^p. Equation 53 is in fact an expression of the indeterminacy that appeared in the construction of α (following eq 45). Hence γ is, as it were, a possible α, and the basis $\boldsymbol{e}''$ is a symmetry basis.

The degeneracies (of the eigenvalues of M) that arise in consequence of symmetry may be made wholly explicit by a consideration of the formulas at hand. The eigenvector $\gamma_{p\pi a}$ of M belongs to the eigenvalue $M''_{p\pi a,\,p\pi a}$ presented in M''; according to eq 51 (applied to M'') $M''_{p\pi a,\,p\pi a}=\lambda^p_a$, say, where λ^p_a is independent of π. Hence the l_p eigenvectors $\gamma_{p\pi a}$ for $\pi=1,2,\ldots,l_p$ all belong to the eigenvalue λ^p_a. These eigenvectors are certainly linearly independent (being columns of a unitary matrix), and so the eigenvalue λ^p_a is at least l_p-fold degenerate. This degeneracy is necessitated by the symmetry of the waveguide junction and may be termed *symmetry degeneracy.* Degeneracy higher than that necessitated by structural symmetry, when it occurs, may be termed *accidental,* as is customary in mathematically similar circumstances.

[29] A matrix that satisfies $MM^*=M^*M$ is said to be a normal matrix; $MM^*=M^*M$ is a necessary and sufficient condition that M be reducible to diagonal form by a similarity transformation with a unitary matrix. See e. g., Murnaghan, Theory of group representations, p. 26.

[30] Although M may be and indeed usually will be symmetric (reciprocity), this condition is neither necessary nor sufficient to insure that M be normal.

V. Examples

In order to illustrate some of the text material by means of examples, we shall consider the waveguide junctions shown in figures 3, 5, and 6. The parameters σ, μ, ϵ in the interior of the junctions may be the most general consistent with the assumed symmetry in the respective cases. Although the figures indicate specific external geometries, it will be recognized that the exact manner in which the waveguides join is of no importance here, provided that this also is consistent with the assumed symmetries. Lowest-mode operation is assumed in all waveguides ($\nu_m=1$, $m=1,2,\ldots,n$); the heavy arrows in the figures, considered as localized vectors, suffice to characterize the electric terminal-fields for the lowest mode in rectangular waveguide. In applying the notation of the text, unneeded indices will be dropped; for example, $\boldsymbol{e}^0_{m\mu}$, $\boldsymbol{e}_{m\mu}$ will be written simply as $\boldsymbol{e}^0_m$, $\boldsymbol{e}_m$ (since $\nu_m=1$).

Example A. The junction shown in figure 3 has the single symmetry plane and the external geometry of a junction widely known as a "magic T". Let the indicated symmetry plane be $x=0$ of Oxyz. The symmetry group P_R consists of the identity P_I and the reflection P_S in the plane; the corresponding matrices $R=(R_{xy})$ are

$$I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\qquad S=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

(Det $(S)=-1$; P_S is a particular improper rotation.) An inspection of the figure makes it clear that

$$P_S\boldsymbol{e}^0_1=\boldsymbol{e}^0_2,\qquad P_S\boldsymbol{e}^0_2=\boldsymbol{e}^0_1,\qquad P_S\boldsymbol{e}^0_3=-\boldsymbol{e}^0_3,\qquad P_S\boldsymbol{e}^0_4=\boldsymbol{e}^0_4,$$

(cf eq 34); hence

$$P_S\boldsymbol{e}_1=\boldsymbol{e}_2,\qquad P_S\boldsymbol{e}_2=\boldsymbol{e}_1,\qquad P_S\boldsymbol{e}_3=-\boldsymbol{e}_3,\qquad P_S\boldsymbol{e}_4=\boldsymbol{e}_4,$$

(cf eq 33), and so the $N\times N$ $(=4\times 4)$ representation D is

$$D(I)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\qquad D(S)=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}.$$

($D(I)$ is always the N-dimensional unit matrix, of course.) D is real and unitary, as it must be. The character χ of D is $\chi(I)=4$, $\chi(S)=0$.

$\{P_R\}$ has two irreducible representations; both are one-dimensional:

$$D^p:\quad D^p(I)=(1),\qquad D^p(S)=(1).$$

$$D^q:\quad D^q(I)=(1),\qquad D^q(S)=(-1).$$

(The 1×1 matrices are unitary.) Since D^p and D^q are one-dimensional, $\chi^p\equiv D^p$ and $\chi^q\equiv D^q$. From eq 40,

$$c_p=(1/2)(1\cdot 4+1\cdot 0)=2,\qquad c_q=(1/2)(1\cdot 4+(-1)\cdot 0)=2,$$

and a reduced-out representation D' (cf eq 41, 42) is given by

$$D'(I)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\qquad D'(S)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}.$$

For the symmetry basis $\boldsymbol{e}'$, $\boldsymbol{h}'$ we write

$$\boldsymbol{e}'=(\boldsymbol{e}'_{pa}\,\boldsymbol{e}'_{pb}\,\boldsymbol{e}'_{qa}\,\boldsymbol{e}'_{qb}),\qquad (\boldsymbol{h}'=\boldsymbol{h}'_{pa}\,\boldsymbol{h}'_{pb}\,\boldsymbol{h}'_{qa}\,\boldsymbol{h}'_{qb});$$

the basis fields must transforms according to eq 43. Thus, for example,

$$P_I\boldsymbol{e}'_{pa}=D^p(I)\boldsymbol{e}'_{pa}=(1)\boldsymbol{e}'_{pa},\qquad P_S\boldsymbol{e}'_{pa}=D^p(S)\boldsymbol{e}'_{pa}=(1)\boldsymbol{e}'_{pa},$$

$$P_I\boldsymbol{e}'_{qa}=D^q(I)\boldsymbol{e}'_{qa}=(1)\boldsymbol{e}'_{qa},\qquad P_S\boldsymbol{e}'_{qa}=D^q(S)\boldsymbol{e}'_{qa}=(-1)\boldsymbol{e}'_{qa}.$$

The same equations hold with a replaced by b, and $\boldsymbol{h}'$ transforms in the same way as does $\boldsymbol{e}'$. Fields of species p and q may, respectively, be termed symmetric and antisymmetric (or even and odd) with respect to reflection in the plane $x=0$. Now $\boldsymbol{e}_4$ is already symmetric and $\boldsymbol{e}_3$ is already antisymmetric; one may choose $\boldsymbol{e}'_{pa}=\boldsymbol{e}_4$, $\boldsymbol{e}'_{qa}=\boldsymbol{e}_3$. Suitable linear combinations to form one further electric basis-field of each species may be found immediately by inspection in this problem; a simple choice is $\boldsymbol{e}'_{pb}=(\boldsymbol{e}_1+\boldsymbol{e}_2)/\sqrt{2}$, $\boldsymbol{e}'_{qb}=(\boldsymbol{e}_1-\boldsymbol{e}_2)/\sqrt{2}$. The transformation $\boldsymbol{e}'=\boldsymbol{e}\alpha$ so determined is

$$(\boldsymbol{e}'_{pa}\,\boldsymbol{e}'_{pb}\,\boldsymbol{e}'_{qa}\,\boldsymbol{e}'_{qb})=(\boldsymbol{e}_1\,\boldsymbol{e}_2\,\boldsymbol{e}_3\,\boldsymbol{e}_4)\begin{pmatrix}0&1/\sqrt{2}&0&1/\sqrt{2}\\0&1/\sqrt{2}&0&-1/\sqrt{2}\\0&0&1&0\\1&0&0&0\end{pmatrix}$$

and we see that α is unitary, as is required, and also that α is real, as is convenient (but not always possible). (One may verify $D'(S)=\alpha^{-1}D(S)\alpha$ for the above matrices.) From eq 51, M' must be of the form

$$M'=\begin{pmatrix}m^p_{aa}&m^p_{ab}&0&0\\m^p_{ba}&m^p_{bb}&0&0\\0&0&m^q_{aa}&m^q_{ab}\\0&0&m^q_{ba}&m^q_{bb}\end{pmatrix}.$$

Upon calculating $M=\alpha M'\alpha^{-1}$ one finds, in particular, $M_{34}=M_{43}=0$, as may well be expected even under the general conditions permitted here. Indeed, these zeros can be predicted as soon as it is recognized that $\boldsymbol{e}_3$ and $\boldsymbol{e}_4$ are of distinct symmetry species.

The fields $\boldsymbol{e}_4(=\boldsymbol{e}'_{pa})$ and $\boldsymbol{h}_4(=\boldsymbol{h}'_{pa})$ are both of species p (even); they satisfy $\boldsymbol{e}_4(x,y,z)=P_S\boldsymbol{e}_4(x,y,z)$, $\boldsymbol{h}_4(x,y,z)=P_S\boldsymbol{h}_4(x,y,z)$. The x,y,z components of these two equations are

$$e_{4x}(x,y,z)=-e_{4x}(-x,y,z),\qquad h_{4x}(x,y,z)=h_{4x}(-x,y,z),$$

$$e_{4y}(x,y,z)=e_{4y}(-x,y,z),\qquad h_{4y}(x,y,z)=-h_{4y}(-x,y,z),$$

$$e_{4z}(x,y,z)=e_{4z}(-x,y,z),\qquad h_{4z}(x,y,z)=-h_{4z}(-x,y,z),$$

where for $\boldsymbol{e}_4$ eq 24b is used, and for $\boldsymbol{h}_4$ eq 24c and 23 are used (cf also eq 25). The even symmetry with respect to the plane $x=0$ forces $e_{4x}\equiv 0$ and $h_{4y}\equiv h_{4z}\equiv 0$ on the plane. (For $\boldsymbol{e}_3$ and $\boldsymbol{h}_3$, which are odd, $e_{3y}\equiv e_{3z}\equiv 0$ and $h_{4x}\equiv 0$ on $x=0$.) Despite the differences in behavior among their components, the tensor entities $\boldsymbol{e}_4$, $\boldsymbol{h}_4$ are unambiguously classified as of the same species by eq 43.

Example B. For the junction shown in figure 5, the symmetry group $\{P_R\}$ is of order $g=3$ and consists of the identity P_I, a counter-clockwise rotation P_{C_3} of 120° around the indicated axis, and the inverse $P_{C_3}^{-1}$ of P_{C_3}. It is evident from the figure that $P_{C_3}\boldsymbol{e}_1=\boldsymbol{e}_2$, $P_{C_3}\boldsymbol{e}_2=\boldsymbol{e}_3$, etc., and one finds $\chi(I)=3$, $\chi(C_3)=\chi(C_3^{-1})=0$. The characters of the irreducible representations of the symmetry group (as well as the representations themselves) are furnished in the table

	P_I	P_{c_3}	$P_{c_3}^{-1}$
χ^p:	1	1	1
χ^q:	1	w	$\overline{w}$
χ^r:	1	$\overline{w}$	w,

where $w\equiv\exp(2\pi j/3)$. From eq 40, $c_p=c_q=c_r=1$: in this case there will be exactly one (electric or magnetic) basis field of each of the three possible symmetry species. One may easily verify that the fields

$$\boldsymbol{e}'_p=(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3)/\sqrt{3},$$

$$\boldsymbol{e}'_q=(\boldsymbol{e}_1+\overline{w}\boldsymbol{e}_2+w\boldsymbol{e}_3)/\sqrt{3},$$

$$\boldsymbol{e}'_r=(\boldsymbol{e}_1+w\boldsymbol{e}_2+\overline{w}\boldsymbol{e}_3)/\sqrt{3},$$

obey eq 43 and in fact constitute a symmetry basis. We list α, $\tilde{\alpha}\alpha$, and the necessary form of M'.

$$\alpha=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & w & w\\ 1 & w & w\end{bmatrix},\qquad \tilde{\alpha}\alpha=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix},$$

$$M'=\begin{bmatrix}m^p & 0 & 0\\ 0 & m^q & 0\\ 0 & 0 & m^r\end{bmatrix}.$$

It happens that M' is necessarily diagonal, so that M is certainly a normal matrix. In this example the reciprocity condition (eq 52) forces the degeneracy $m^q=m^r$. This degeneracy is technically accidental; it might well be termed reciprocity degeneracy.

Example C. The discussion of the waveguide Wheatstone bridge (fig. 6) is limited mainly to a sketch of results. The symmetry group of this junction is identical with that of a regular tetrahedron and is of order $g=24$. Of the five nonequivalent irreducible representations of this group, the two three-dimensional ones (D^p and D^q, say) are con-

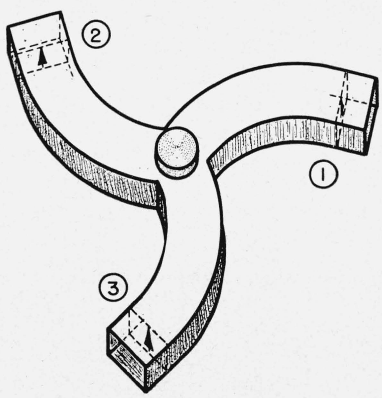

FIGURE 5. *Three-arm junction possessing a threefold axis but no plane of symmetry*

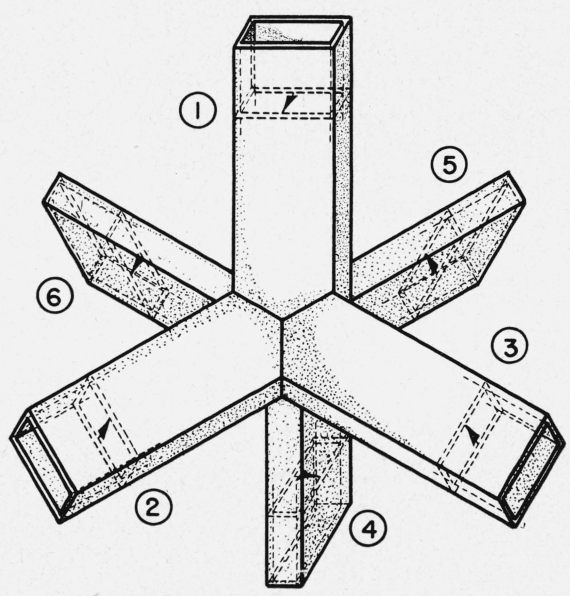

FIGURE 6. *Waveguide Wheatstone bridge.*

tained in D'. There are accordingly three fields (partners) of species p and three of species q in the symmetry basis; it can be shown that a symmetry basis is furnished by [31]

$$\begin{bmatrix} \boldsymbol{e}'_{p\pi} \\ \boldsymbol{e}'_{p\rho} \\ \boldsymbol{e}'_{p\sigma} \\ \boldsymbol{e}'_{q\pi} \\ \boldsymbol{e}'_{q\rho} \\ \boldsymbol{e}'_{q\sigma} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \\ \boldsymbol{e}_5 \\ \boldsymbol{e}_6 \end{bmatrix}.$$

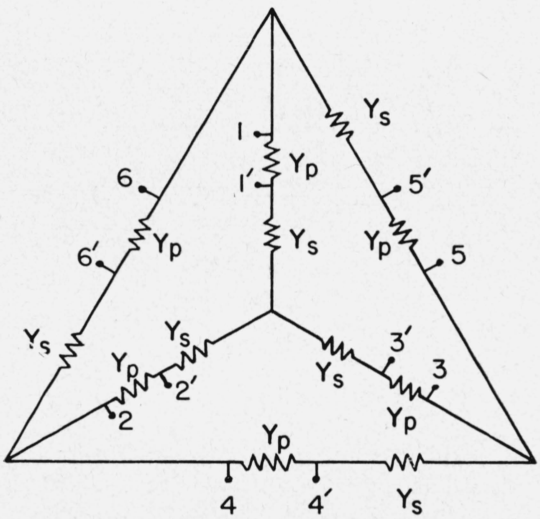

FIGURE 7. *Equivalent network.*

Sign convention: m' is the positive terminal for v_m at the terminal-pair mm'; positive i_m is into the network at m'.

(Note that this matrix equation is the transpose $\tilde{\boldsymbol{e}}'=\tilde{\alpha}\tilde{\boldsymbol{e}}$ of the equation $\boldsymbol{e}'=\boldsymbol{e}\alpha$.) Here, instead of considering impedance, admittance, and scattering matrices in common, we shall consider the admittance matrices particularly. From eq 51, Y' must be of the form

$$Y' = \begin{bmatrix} y^p & 0 & 0 & 0 & 0 & 0 \\ 0 & y^p & 0 & 0 & 0 & 0 \\ 0 & 0 & y^p & 0 & 0 & 0 \\ 0 & 0 & 0 & y^q & 0 & 0 \\ 0 & 0 & 0 & 0 & y^q & 0 \\ 0 & 0 & 0 & 0 & 0 & y^q \end{bmatrix}.$$

The eigen-admittances y^p, y^q are in general complex; the condition $\mathrm{Re}(W)\gtreqless 0$ (mentioned on p. 271) requires $\mathrm{Re}(y^p)\gtreqless 0$, $\mathrm{Re}(y^q)\gtreqless 0$. A calculation of $Y=\alpha Y'\alpha^{-1}$ yields

$$Y = \begin{bmatrix} \lambda & \mu & \mu & 0 & -\mu & \mu \\ \mu & \lambda & \mu & \mu & 0 & -\mu \\ \mu & \mu & \lambda & -\mu & \mu & 0 \\ 0 & \mu & -\mu & \lambda & -\mu & -\mu \\ -\mu & 0 & \mu & -\mu & \lambda & -\mu \\ \mu & -\mu & 0 & -\mu & -\mu & \lambda \end{bmatrix},$$

where $\lambda\equiv(y^p+y^q)/2$, $\mu\equiv(y^p-y^q)/4$. The matrix Y is both normal and symmetric in virtue of the symmetry of the junction.

The six-terminal-pair network shown in figure 7 is a Wheatstone-bridge scheme of connections modified by the presence of the "parallel" and "series" elements Y_p and Y_s. If $Y_p=\lambda+2\mu=y^p$ and $Y^s=-4\mu=y^q-y^p$, then (with the scheme of signs noted under the figure) the admittance matrix of the network is the same as that of the junction, and the network is an "equivalent" network for the junction.

The particular values of y^p and y^q for a given structure depend upon the (common) distance of the terminal surfaces from the center of the junction. If the structure is nondissipative, it is always possible, and sometimes convenient, to assume terminal surfaces so located that $y^p=0$ or, alternatively, $y^q\rightarrow\infty$. In the first alternative the equivalent network is simplified by the absence of Y_p; in the second, by the virtual absence of Y_s (Y_s being replaceable by a perfect conductor). (This second case is the one presented in the reference in footnote 4.)

The author is greatly indebted to K. F. Herzfeld for valuable discussions and guidance in the course of this work. The writer is also very grateful to C. C. J. Roothaan for helpful discussions, and to H. Lyons for his wholehearted support of this work.

VI. Appendix

1. Transformation of Equation 9

The integral of $(1/2)(\boldsymbol{E}^1\times\overline{\boldsymbol{H}}^2)\cdot\boldsymbol{k}$ taken over the complete boundary of V (assuming an infinite domain) is equal to

$$[\boldsymbol{E}^1, \boldsymbol{H}^2]+(1/2)\int_{S_\infty}(\boldsymbol{E}^1\times\overline{\boldsymbol{H}}^2)\cdot\boldsymbol{k}\,dS=-(1/2)\int_V\nabla\cdot(\boldsymbol{E}^1\times\overline{\boldsymbol{H}}^2)dV,$$

where in the left-hand side the definition of the bracket (eq 9) and the boundary condition on the surfaces S'_m (p. 268) are used, and the right-hand side is given by the divergence theorem of vector analysis. From the above equation the desired express on,

$$\left.\begin{aligned} [\boldsymbol{E}^1, \boldsymbol{H}^2]=(\eta/2)\int_{S_\infty}\boldsymbol{E}^1\cdot\overline{\boldsymbol{E}}^2\,dS+(1/2)\int_V\boldsymbol{E}^1\cdot\sigma\cdot\overline{\boldsymbol{E}}^2\,dV \\ +(j\omega/2)\int_V(\overline{\boldsymbol{H}}^2\cdot\mu\cdot\boldsymbol{H}^1-\boldsymbol{E}^1\cdot\epsilon\cdot\overline{\boldsymbol{E}}^2)dV, \end{aligned}\right\} \quad (54)$$

may be obtained with the aid of further vector identity, Maxwell's equations (eq 1), and the following relations hold-

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[31] This transformation represents an adaption of information contained in the tables on p. 638 of the reference in footnote 4.

ing (for both electromagnetic fields) on S_∞: $\boldsymbol{H}^2 = \eta \boldsymbol{E}^2 \times \boldsymbol{k}$ and $\boldsymbol{k}\cdot\boldsymbol{E}^2 = 0$. where $\eta \equiv \sqrt{\epsilon/\mu}$ and vacuum values of ϵ, μ may be assumed. Equation 54 applies also in the case of a finite domain provided merely that the S_∞-integral is omitted.

2. Additional Symmetry Properties

The complex conjugate of the element $Y'_{p\pi a, q\rho b}$ of the admittance matrix Y' is, from eq 12, equal to $[\boldsymbol{e}'_{p\pi a}, \mathfrak{H}(\boldsymbol{e}_{q\rho b})]$, so that, making appropriate substitutions in eq 54 and taking the complex conjugate, one obtains

$$\left.\begin{aligned} Y'_{p\pi a, q\rho b} &= (\eta/2)\int_{S_\infty} \bar{\boldsymbol{e}}'_{p\pi a}\cdot\boldsymbol{e}'_{q\rho b}dS + (1/2)\int_V \bar{\boldsymbol{e}}'_{p\pi a}\cdot\sigma\cdot\boldsymbol{e}'_{q\rho b}dV \\ &+ (j\omega/2)\int_V \bar{\boldsymbol{e}}'_{p\pi a}\cdot\epsilon\cdot\boldsymbol{e}'_{q\rho b}dV - (j\omega/2)\int_V \mathfrak{H}(\boldsymbol{e}'_{q\rho b})\cdot\mu\cdot\overline{\mathfrak{H}(\boldsymbol{e}'_{p\pi a})}dV. \end{aligned}\right\} \quad (55)$$

We wish to show here that an equation of the form 51, which holds for the elements of Y', holds also for each of the four integrals making up the right-hand side of eq 55. It will suffice to consider the third and the fourth of these integrals. Since the structure is invariant with respect to P_R, and since P_R is unitary in the sense of eq 27, one may write

$$\int_V \bar{\boldsymbol{e}}'_{p\pi a}\cdot\epsilon\cdot\boldsymbol{e}'_{q\rho b}dV = \int_V (P_R\bar{\boldsymbol{e}}'_{p\pi a})\cdot\epsilon\cdot(P_R\boldsymbol{e}'_{q\rho b})dV, \quad (56)$$

$$\int_V \mathfrak{H}(\boldsymbol{e}'_{q\rho b})\cdot\mu\cdot\overline{\mathfrak{H}(\boldsymbol{e}'_{p\pi a})}dV = \int_V [P_R\mathfrak{H}(\boldsymbol{e}'_{q\rho b})]\cdot\mu\cdot[\overline{P_R\mathfrak{H}(\boldsymbol{e}'_{p\pi a})}]dV. \quad (57)$$

The right-hand side of eq 56 may immediately be expanded with the aid of eq 43a:

$$\int_V \bar{\boldsymbol{e}}'_{p\pi a}\cdot\epsilon\cdot\boldsymbol{e}'_{q\rho b}dV = \sum_{\sigma,\tau}\overline{D^p(R)_{\sigma\pi}}D^q(R)_{\tau\rho}\int_V \bar{\boldsymbol{e}}'_{p\sigma a}\cdot\epsilon\cdot\boldsymbol{e}'_{q\tau b}dV. \quad (58)$$

The right-hand side of eq 57 may be expanded similarly as soon as it is observed that $P_R\mathfrak{H}(\boldsymbol{e}'_{p\pi a}) = \mathfrak{H}(P_R\boldsymbol{e}'_{p\pi a})$ (as is directly implied by eq 32) and that $\mathfrak{H}$ is a linear function of its argument. Equation 58 is of the form 48, and it is thus clear that an equation of this form holds for each of the integrals involved. The derivation in the text leading to eq 51 obviously applies. An equation of the type 55 may of course be written for $Z'_{p\pi a, q\rho b}$, and a similar argument leads to the same results for the integrals making up $Z'_{p\pi a, q\rho b}$.

WASHINGTON, June 7, 1950.