Checking and Interpolation of Functions Tabulated at Certain Irregular Logarithmic Intervals

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Many functions $f(x)$ behave as polynomials in log x. When tabulated for arguments in geometric progression, $f(x)$ can be checked by ordinary differencing, and interpolation can be performed to a fine extent with existing tables of Lagrangian coefficients. But in practice, $f(x)$ is often known or calculated at some or all of the points 1, 2, 5, 10, 20, 50, 100, 200, 500, and 1000 (same theory for the points 0.001, 0.002, 0.005, 0.01, etc., or 0.01, 0.02, 0.05, 0.1, etc., or any constant multiple of 1, 2, 5, . . .).

The present tables have a twofold use: I. Checking the correctness of $f(x)$ when tabulated at some of the more frequently occurring combinations of points 1, 2, 5, etc. This also includes their use to estimate the least number of tabular entries for interpolation of given accuracy. II. Facilitation of Lagrangian interpolation by a generalization of a scheme given by W. J. Taylor.

I. Introduction

Functions which behave like polynomials in log x are encountered in numerous fields, such as statistics, actuarial studies, economics, biometrics, electronics, nuclear physics, biophysics, physical chemistry, etc. When a function obtained experimentally is suspected to have this form, it is usual to examine this by plotting the values on semilogarithmic graph paper and observing whether they lie on a smooth curve. When it has been decided that the function is approximately a polynomial in log x (or when it is known to be exactly a polynomial in log x), the question arises as to how this information can be used to facilitate checking the table and interpolating in it numerically. This is the question to be discussed here.

II. Arguments in Geometric Progression

If the tabular arguments are in geometric progression, such as 1, 2, 4, 8, . . .; or 1, 10, 100, 1000, . . ., the problems of checking and interpolation are quite simple, because the function is effectively tabulated at a constant interval in log x. Thus an examination of the ordinary differences of the tabular values will reveal any errors, as well as the lowest degree which an approximating polynomial in log x must have in order to yield a certain desired accuracy. Either the differences themselves can be used for interpolation, or if we wish to avoid using differences, any one of several well-known tables of Lagrangian interpolation coefficients may be used for interpolation to a very fine subdivision of the tabular interval. The most extensive of these tables of interpolation coefficients are contained in [1].[1]

III. Divided Differences for Checking

In practice, however, it is often found that the function has been determined at some or all of the points 1, 2, 5, 10, 20, 50, 100, . . ., 1000 (or a constant multiple of those values, e. g., 0.1, 0.2, 0.5, 1.0, . . .). Then it is no longer possible to use the ordinary differences for checking purposes. Instead we use a certain generalization, known as "divided differences," a subject that is treated fully in most textbooks on finite differences (see [2] or [3]). The $(n-1)$th divided difference with respect to log x, of a function $f(x)$ tabulated at the n points $x_1, x_2, \ldots, x_n$, can be written in the form $\sum_{i=1}^{n} A_i f(x_i)$, where the A_i are certain numbers depending only on the x_i, not on the function. It is convenient to tabulate these quantities, once and for all, for the usual sets of values of x_i and for the various values of n, so that the divided difference can be obtained by a single accumulation on a calculating machine. This has been done here for $n=3(1)10$ and x_i ranging from 1, 2, 5, . . ., to 1000, in table 1, A; for $n=3(1)7$ and x_i ranging from 1, 5, 10, . . . to 1000 in table 1, B; and for $n=3(1)7$ and x_i ranging from 1, 2, 10, . . . to 1000 in table 1, C. All entries are given to eight significant figures, correct to about a unit in the last place.

The explicit expression for A_i is

$$A_i = 1/\underset{j}{\pi'}\,(\log_{10}x_i - \log_{10}x_j), \tag{1}$$

where $\underset{j}{\pi'}$ indicates the product over all $j \neq i$. It can be shown that if $f(x)$ is a polynomial of degree $n-2$ in log x, then the $(n-1)$th divided difference is identically zero. If the $(n-1)$th divided difference is sufficiently small, then $f(x)$ considered as a polynomial of degree $n-2$ in log x, is in all likelihood, free from error. (However the user is cautioned that a sufficiently small value of $\sum_{i=1}^{n} A_i f(x_i)$ is only a strong indication, rather than conclusive evidence, of the correctness of the n entries, considered as values of a polynomial of degree $n-2$.) Now the tabular entries may be entirely correct and the user might still want to employ the most economical Lagrangian formula. Again, this information is conveyed in the sufficiently small $(n-1)$th divided difference, which establishes the adequacy of an $(n-1)$ point Lagrangian interpolation formula. In brief, if the function is checked

[1] Figures in brackets indicate the literature references at the end of this paper.

TABLE 1. *Coefficients A_i for checking and interpolation, based upon the points x_i*

A

x_i	A_i	Three-point	Four-point	Five-point	Six-point
$x_1=1$	$A_1=$	4.75260 47	−4.75260 47	3.65295 55	−2.15010 00
$x_2=2$	$A_2=$	−8.34781 13	11.94301 8	−11.94301 8	8.54329 79
$x_3=5$	$A_3=$	3.59520 66	−11.94301 8	19.83692 3	−19.83692 3
$x_4=10$	$A_4=$	------------	4.75260 47	−15.78781 1	22.58725 1
$x_5=20$	$A_5=$	------------	------------	4.24095 00	−10.65726 0
$x_6=50$	$A_6=$	------------	------------	------------	1.51373 42

A—continued

x_i	A_i	Seven-point	Eight-point	Nine-point	Ten-point
$x_1=1$	$A_1=$	1.07505 00	−0.46720 383	0.17310 449	−0.05770 1497
$x_2=2$	$A_2=$	−5.02851 60	2.51425 80	−1.04850 75	0.38848 430
$x_3=5$	$A_3=$	15.24709 1	−9.51717 89	4.75858 94	−2.06802 58
$x_4=10$	$A_4=$	−22.58725 1	17.36105 3	−10.21857 5	5.10928 77
$x_5=20$	$A_5=$	15.24709 1	−15.24709 1	10.90682 8	−6.41967 08
$x_6=50$	$A_6=$	−5.02851 60	8.35218 43	−8.35218 43	6.41967 08
$x_7=100$	$A_7=$	1.07505 00	−3.57123 89	5.10928 77	−5.10928 77
$x_8=200$	$A_8=$	------------	0.57521 751	−1.44548 80	2.06802 58
$x_9=500$	$A_9=$	------------	------------	0.11694 543	−0.38848 430
$x_{10}=1000$	$A_{10}=$	------------	------------	------------	0.05770 1497

B

x_i	A_i	Three-point	Four-point	Five-point	Six-point	Seven-point
$x_1=1$	$A_1=$	1.43067 66	−0.84208 465	0.42104 232	−0.15600 111	0.05200 0371
$x_2=5$	$A_2=$	−4.75260 47	4.75260 47	−3.65295 55	1.82647 77	−0.79376 529
$x_3=10$	$A_3=$	3.32192 81	−4.75260 47	4.75260 47	−2.79734 47	1.39867 23
$x_4=50$	$A_4=$	------------	0.84208 465	−2.79734 47	2.79734 47	−2.15010 00
$x_5=100$	$A_5=$	------------	------------	1.27665 32	−1.82647 77	1.82647 77
$x_6=500$	$A_6=$	------------	------------	------------	0.15600 111	−0.51822 448
$x_7=1000$	$A_7=$	------------	------------	------------	------------	0.18493 938

C

x_i	A_i	Three-point	Four-point	Five-point	Six-point	Seven-point
$x_1=1$	$A_1=$	3.32192 81	−2.55330 63	1.27665 32	−0.55481 813	0.18493 938
$x_2=2$	$A_2=$	−4.75260 47	4.75260 47	−2.79734 47	1.39867 23	−0.51822 448
$x_3=10$	$A_3=$	1.43067 66	−4.75260 47	4.75260 47	−3.65295 55	1.82647 77
$x_4=20$	$A_4=$	------------	2.55330 63	−3.65295 55	3.65295 55	−2.15010 00
$x_5=100$	$A_5=$	------------	------------	0.42104 232	−1.39867 23	1.39867 23
$x_6=200$	$A_6=$	------------	------------	------------	0.55481 813	−0.79376 529
$x_7=1000$	$A_7=$	------------	------------	------------	------------	0.05200 0371

by the A_i's for a certain n, we use a different set of A_i's corresponding to $n-1$, for interpolation.

IV. Interpolation

Once we have ascertained, by use of table 1, that a function is adequately represented by a polynomial of degree $n-1$ in $\log x$ (remembering that $n-1$ is here the $n-2$ of III.), we can interpolate by use of a rearrangement of Lagrange's formula, which was suggested by a paper of W. J. Taylor [4]. The original method and notation of Taylor are not described in this article because he developed them only for the special case of equally spaced arguments, and here they would be superfluous. The interpolation formula is

$$f(x) \sim \left(\sum_{i=1}^{n} a_i f(x_i) \right) \Big/ \sum_{i=1}^{n} a_i, \tag{2}$$

where

$$a_i = A_i / (\log x - \log x_i) \tag{3}$$

and A_i is given by (1).

V. Logarithms to Other Bases

The coefficients A_i have been computed from common (base 10) logarithms, according to (1). It would have been possible, instead, to have the A_i's calculated for any other logarithmic base. Furthermore, it is permissible when computing the a_i's from (3), to use logarithms to a base different from that underlying the A_i's. To illustrate this point, natural logarithms (to the base $e=2.71828\ 183$) have been used in the two examples in section VIII. Extensive tables of $\log_e x$ are given in [5].

VI. Conversion to Arguments in Geometric Progression

If a large number of interpolations are required for a function given for any one set of values x_i, it may be convenient to prepare, by the method in section IV, an auxiliary table giving the function at a new set of values y_i, where the y_i are now in geometric progression. Then we can use ordinary interpolation formulae and coefficients, as suggested in section II. This point was called to the writer's attention by Churchill Eisenhart and Julius Lieblein, of the Statistical Engineering Laboratory.

VII. Use of A_i for Other Arguments

If the given set of values x_i does not begin with 1, the table of coefficients A_i may sometimes still be used by a suitable change of the independent variable x. Notice that a polynomial in $\log (ax)$ or $\log (b/x)$ is still a polynomial in $\log x$. Thus, for instance, if a function is given at the points $x=10, 20, 50$ and 100, we may consider it as a function of $x'=100/x$, given at the points $x'=1, 2, 5$ and 10, so that table 1,A, may be used. Again, if a function is given at the points $x=20, 100, 200$ and 1000, it may, instead, be considered as a function of $x'=1000/x$ given at the points $x'=1, 5, 10$ and 50, so that table 1,B, may be used. Schedules A, B, and C list such transformations of the independent variable, giving both the sets of x_i for which the change of variable to $1/x$ is applicable, and the new arguments x_i', which are proportional to $1/x_i$.

Transformation schedules

SCHEDULE A.

x_i	x_i'	x_i	x_i'	x_i	x_i'	x_i	x_i'	x_i	x_i'
2	5	2	50	2	500	5	20	5	200
5	2	5	20	5	200	10	10	10	100
10	1	10	10	10	100	20	5	20	50
		20	5	20	50	50	2	50	20
		50	2	50	20	100	1	100	10
		100	1	100	10			200	5
				200	5			500	2
				500	2			1000	1
				1000	1				

SCHEDULE B.

x_i	x_i'	x_i	x_i'	x_i	x_i'	x_i	x_i'
2	50	2	500	5	10	5	100
10	10	10	100	10	5	10	50
20	5	20	50	50	1	50	10
100	1	100	10			100	5
		200	5			500	1
		1000	1				

SCHEDULE C.

x_i	x_i'	x_i	x_i'	x_i	x_i'	x_i	x_i'
5	20	5	200	2	10	2	100
10	10	10	100	10	2	10	20
50	2	50	20	20	1	20	10
100	1	100	10			100	2
		500	2			200	1
		1000	1				

VIII. Illustrations of Use of Tables

The following two illustrative examples show how to use the tables of A_i.

Example 1. *Given*

x_i	$f(x_i)$	x_i	$f(x_i)$
1	0. 52	50	6333. 56
2	1. 11	100	15752. 58
5	50. 30	200	34648. 66
10	343. 74	500	85417. 78
20	1447. 15	1000	155833. 72

To calculate f(18). In the absence of knowledge of an explicit formula for $f(x)$, we find that its seventh divided difference, employing the coefficients A_i in table 1, A, for the eight points from 1 to 200, is less than 0.05 in absolute value. This indicates that the function behaves, up to the last place, like a sixth degree polynomial in $\log x$ (any base). Hence only seven points are needed for the interpolation. The work may be arranged in the following way: (Note that an extra place is carried in some columns to reduce the error that would arise in rounding before the final answer is reached.)

x_i	$\log_e 18-\log_e x_i$	$a_i\equiv A_i/(\log_e 18-\log_e x_i)$	$a_i f(x_i)$
1	2. 89037 176	0. 371942	0. 193
2	2. 19722 458	−2. 288576	−2. 540
5	1. 28093 385	11. 903106	598. 726
10	0. 58778 667	−38. 427634	−13209. 115
20	−0. 10536 051	−144. 713527	−209422. 181
50	−1. 02165 125	4. 921950	31173. 466
100	−1. 71479 843	−0. 626925	−9875. 686
		$\sum_{i=1}^{7} a_i=-168.859664$	$\sum_{i=1}^{7} a_i f(x_i)=-200737.137$
			$\sum a_i f(x_i)/\sum a_i=1188.78$

The answer found here, 1188.78, is correct to its last significant figure. The function $f(x)$ was chosen to be $(\log_e x)^6+3(\log_e x)^5+\frac{\pi}{6}$.

Example 2. Given

x_i	x_i'	$f(x_i)$ or $\bar{f}(x_i')$
20	50	15. 45981
100	10	40. 07131
200	5	54. 18217
1000	1	95. 09949

To calculate f(160). To use the present tables of coefficients A_i it is necessary to change the variable to $x'=1000/x$. (See Transformation schedule B.) If $f(x)$ is called $\bar{f}(x')$, then $f(160)=\bar{f}(6.25)$. As in the previous example, in the absence of knowledge of an explicit formula for $f(x)$, we find that the absolute value of its third divided difference, employing the coefficients A_i in table 1, B, for four points, is less than 1½ units in the fifth decimal place. This indicates that the function behaves, up to the last place, like a second degree polynomial in $\log x$. Hence only three points are needed for the interpolation, which is carried out as before:

x_i'	$\log_e 6.25-\log_e x_i'$	$a_i\equiv A_i/(\log_e 6.25-\log_e x_i')$	$a_i\bar{f}(x_i')$
1	1. 83258 146	0. 78068 92	74. 24314
5	0. 22314 355	−21. 29841 84	−1153. 99453
10	−0. 47000 363	−7. 06787 75	−283. 21911
		$\sum_{i=1}^{3} a_i=-27.58560\ 67$	$\sum_{i=1}^{3} a_i\bar{f}(x_i')=-1362.97050$
			$\sum a_i\bar{f}(x_i')/\sum a_i=49.40876$

This answer, 49.40876, is also correct to the last figure given, the function $f(x)$ having been chosen to be $2.20(\log_e x)^2-1.43\log_e x$.

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IX. References

[1] Tables of Lagrangian interpolation coefficients. (Columbia University Press, New York, 1944.)

[2] L. M. Milne-Thomson, Calculus of finite differences, ch. I, p. 1 (Macmillan and Co., London, 1933).

[3] E. T. Whittaker and G. Robinson, Calculus of observations, ch. II, p. 20 (Blackie and Son, London, 1944).

[4] W. J. Taylor, A method of Lagrangian curvilinear interpolation, J. Research NBS **35**, 151 (1945) RP1667.

[5] Table of natural logarithms, vol. I to IV, Mathematical Tables NBS, MT7, MT9, MT10, MT12.

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