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Solution of the Telegrapher's Equation With Boundary Conditions on Only One Characteristic [1]

By George E. Forsythe

Forecasting a certain idealized horizontal, autobarotropic, nonviscous, nondiverging atmospheric flow considered by Rossby leads to an unusual boundary-value problem for the telegrapher's equation, involving boundary values on only one characteristic. It is shown how to find unique solutions periodic in the longitude; these are represented in terms of a Green's function. A procedure for computing the Green's function is set down and is shown to be optimal in a restricted sense. The Green's function is tabulated for 72 longitudes and 14 time-values. An alternative solution by a difference equation is mentioned.

I. Introduction

In one treatment of planetary atmospheric flow as horizontal, autobarotropic, nonviscous, and nondiverging in a plane, Rossby [9][2] considered the idealized case of a constant west-wind component, U, and a south-wind component, v, dependent on the west-to-east distance coordinate, ψ, and time, τ, but independent of the south-to-north distance coordinate, μ. It was shown in [5] that this v satisfies the telegrapher's equation (eq 3) below, where $x=\psi-U\tau$, $t=4\ \beta\tau$. The parameter $\beta=2\Omega\cos\varphi\ (d\varphi/d\mu)$ is here considered constant; Ω is the angular speed of the earth's rotation, and φ is latitude. For this simple atmospheric model, the meteorological forecast problem is one of determining $v(x, t)$ for future times t, given only $v(x, 0)$. But on a plane the specification of $v(x, 0)$ is not sufficient to determine $v(x, t)$ for many $t>0$, because the line $t=0$ is a characteristic of eq 3 (see p. 254 of [11]). Having its initial conditions on only one characteristic is an unusual feature of the present problem that does not seem to have arisen in other physical problems known to the author to lead to the telegrapher's equation.

The author shows that the forecast problem has a unique solution when it is assumed that the world is round, that is, when the solution is assumed to be periodic in x. The problem is stated in section II and solved in section III. In section IV the solution is represented in terms of a Green's function. In section V a procedure is outlined for computing the Green's function by improving the convergence of its Fourier series. In section VI certain auxiliary polynomials, $\sigma_k(x)$, used in section V are discussed and related to the Bernoulli polynomials. In section VII are reported without proof a few results on the approximate solution of the problem by a difference equation, taken from [6]. In section VIII is given a table of values of the Green's function, as computed in the Computation Unit of the Institute for Numerical Analysis.

The present author first reported this work in [7]. Independently of the research reported here, Charney, Eliassen, and Hunt of the Institute for Advanced Study considered the telegrapher's equation while investigating numerical weather prediction in general. Their research was reported in [1] and is written up in [2]. The work of these men includes much of what is reported here, and much more.

II. Statement of the Problem

Let C be the circumference of a unit circle; let us adopt an angle coordinate x for C: $-\pi<x\leq\pi$. Let I be the set of time-instants t: $0\leq t<\infty$. Let R be the closed two-dimensional region consisting of all points (x,t) with x in C and t in I. Let $f(x)$ be a real-valued function that satisfies the follow-

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[2] Figures in brackets indicate the literature references at the end of this paper.

ing hypotheses, but which is otherwise arbitrary:

H_1: *f(x) is sectionally smooth* [3] *on C. Moreover,*

$$f(x)=\tfrac{1}{2}[f(x+0)+f(x-0)], \qquad \text{(all } x). \qquad (1)$$

H_2: *f(x) has the average value zero:*

$$\int_{-\pi}^{\pi} f(x)dx=0. \qquad (2)$$

The problem is to find a real-valued function, $v(x, t)$, defined everywhere on R, with the following four properties:

P_1: *v_t exists* [4] *and is continuous throughout R.*

P_2: *v_x and $v_{tx}=\partial v_t/\partial x$ exist and are continuous everywhere in R except, at most, for a finite number of values of* [5] *x.*

P_3: *Whenever v_{xt} is defined, the following hyperbolic partial differential equation (the telegrapher's equation) is satisfied:*

$$v_{xt}+\tfrac{1}{4}v=0. \qquad (3)$$

P_4: *For $t=0$, $v(x, t)$ reduces to $f(x)$:*

$$v(x, 0)=f(x).$$

III. Solution of the Problem, Uniqueness

One gets a formal solution by separation of variables and use of Fourier series. Assume a solution of eq 3 of form $v(x, t)=X(x)T(t)$. Then $v_{xt}=X'(x)T'(t)$, and eq 3 takes the form

$$\frac{X'(x)}{X(x)}\cdot\frac{T'(t)}{T(t)}=-\tfrac{1}{4}. \qquad (4)$$

The two factors in eq 4 must themselves be constant:

$$\frac{X'(x)}{X(x)}=\lambda, \qquad (5)$$

$$\frac{T'(t)}{T(t)}=-\frac{1}{4\lambda}. \qquad (6)$$

A solution of eq 5 for $-\infty<x<\infty$ is $X(x)=e^{\lambda x}$. For x on the circle C, however, one must have $X(-\pi)=X(\pi)$, or $e^{-\pi\lambda}=e^{\pi\lambda}$. Taking logarithms, one sees that $-\pi\lambda=\pi\lambda+2n\pi i(n=0, \pm 1, \pm 2, \ldots)$. Hence $\lambda=ni(n=0, \pm 1, \pm 2, \ldots)$. Since the value $\lambda=0$ is incompatible with eq 4, there remain the following fundamental solutions of eq 5:

$$X_n(x)=e^{nix}, \qquad (n=\pm 1, \pm 2, \pm 3, \ldots).$$

Corresponding to $X_n(x)$, the solution $T_n(t)$ of eq 6 for $\lambda=ni$ is $T_n(t)=\exp(it/4n)$. Hence for $n=\pm 1, \pm 2, \ldots$ the functions $X_n(x)T_n(t)=\exp[i(nx+t/4n)]$ have properties P_1, P_2, and P_3. By taking linear combinations of the functions X_nT_n and $X_{-n}T_{-n}$, one obtains the equivalent pair of functions $\cos(nx+t/4n)$ and $\sin(nx+t/4n)$. Both of the latter functions have properties P_1, P_2, and P_3.

In order to obtain a solution with enough degrees of freedom to satisfy P_4, consider the series,

$$v(x,t)\sim\sum_{n=1}^{\infty}\left[a_n\cos\left(nx+\frac{t}{4n}\right)+b_n\sin\left(nx+\frac{t}{4n}\right)\right], \qquad (7)$$

where a_n, b_n are undetermined constants. We postpone a discussion of the convergence of the series (eq 7) for $t\neq 0$ and consider it for $t=0$, where $v(x, 0)$ is supposed to equal $f(x)$:

$$v(x,0)\sim\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \qquad (8)$$

If the series in eq 8 actually does converge to $f(x)$ for all x, it is shown on p. 274 of [12] that the coefficients, a_n, b_n, must be the Fourier coefficients of f. Conversely, by p. 25 of [12], the hypothesis H_1 is sufficient to insure that the Fourier series of f actually converges to $f(x)$; it even converges uniformly for x in any interval bounded away from a discontinuity of f. Moreover, the hypothesis H_2 implies that in the Fourier series $a_0=0$. We henceforth stipulate that the series (eq 8) is the Fourier series of f. It is important to note [6] that H_1 implies that a_n and b_n are $O(1/n)$; that is, there exists a constant $M<\infty$ such that

$$|na_n|\leq M, |nb_n|\leq M \qquad \text{(all } n). \qquad (9)$$

1. Proof of Convergence

There remains only a proof that the series in eq 7 actually does converge to a function $v(x, t)$ with the required properties, P_1, P_2, P_3, P_4. It will be useful to have the following representations of $\cos(t/4n)$ and $\sin(t/4n)$. They are proved by Taylor's formula and hold for all values of $t/4n$:

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[3] That is, both $f(x)$ and $f'(x)$ are continuous in C except for a finite number of jump discontinuities.

[4] The subscripts denote partial derivatives.

[5] It is shown on pp. 55 to 57 of [3] that our conditions P_1 and P_2 imply the following: v_{xt} exists and equals v_{tx} for all (x, t) such that x is not one of the excepted values in P_2.

[6] See p. 18 of [12].

$$\cos\frac{t}{4n}=1-\frac{\alpha_n t^2}{32n^2}, \tag{10}$$

where $|\alpha_n|=\left|\alpha\left(\frac{t}{4n}\right)\right|\leq 1$;

$$\sin\frac{t}{4n}=\frac{\beta_n t}{4n}, \tag{11}$$

where $|\beta_n|=\left|\beta\left(\frac{t}{4n}\right)\right|\leq 1$;

$$\sin\frac{t}{4n}=\frac{t}{4n}-\frac{\gamma_n t^3}{384n^3}, \tag{12}$$

where $|\gamma_n|=\left|\gamma\left(\frac{t}{4n}\right)\right|\leq 1$.

Using eq 10 and 11, one sees that for any fixed t,

$$\left.\begin{aligned}&\sum_{n=1}^{\infty}\left[a_n\cos\left(nx+\frac{t}{4n}\right)+b_n\sin\left(nx+\frac{t}{4n}\right)\right]=\\&\sum_{n=1}^{\infty}\left[(a_n\cos nx+b_n\sin nx)\cos\frac{t}{4n}+\right.\\&\left.(b_n\cos nx-a_n\sin nx)\sin\frac{t}{4n}\right]=\\&\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)-\\&\frac{t^2}{32}\sum_{n=1}^{\infty}\alpha_n\left(\frac{a_n}{n^2}\cos nx+\frac{b_n}{n^2}\sin nx\right)+\\&\frac{t}{4}\sum_{n=1}^{\infty}\beta_n\left(\frac{b_n}{n}\cos nx-\frac{a_n}{n}\sin nx\right)=\\&\Sigma_0+\Sigma_1+\Sigma_2.\end{aligned}\right\} \tag{13}$$

Representation as the sum of three series is permitted because each of the series Σ_0, Σ_1, Σ_2 converges. Σ_0 converges for all x because it is the Fourier series of f; its convergence is uniform in any interval bounded away from a discontinuity of $f(x)$. Fix any positive number t_1, and restrict the consideration to t's such that $0\leq t\leq t_1$. Since a_n and b_n are $O(1/n)$, Σ_1 and Σ_2 are convergent uniformly in x and t. For example, Σ_2 is dominated by $(t_1/4)\sum(|b_n|/n+|a_n|/n)$, a series convergent like $\sum(1/n^2)$. The series of eq 7 is thus convergent for all x, t and defines by its limit a function $v(x,t)$:

$$v(x,t)=\sum_{n=1}^{\infty}\left[a_n\cos\left(nx+\frac{t}{4n}\right)+b_n\sin\left(nx+\frac{t}{4n}\right)\right]. \tag{14}$$

Moreover, the series (eq 14) converges uniformly for x, t such that x is bounded away from a jump of $f(x)$.

Since Σ_1 and Σ_2 converge uniformly, they converge to continuous functions of x and t. Thus the only discontinuities of $v(x,t)$ are those from Σ_0, that is, those of $f(x)$. This is the property of hyperbolic differential equations that discontinuities in their solutions are propagated along characteristics. Let the set of discontinuities of $f(x)$ be denoted by E.

For all x, t one may obtain v_t by termwise differentiation of eq 14, because by eq 9 the resulting series is absolutely and uniformly convergent in both x and t:

$$v_t(x,t)=\tfrac{1}{4}\sum_{n=1}^{\infty}\left[\frac{b_n}{n}\cos\left(nx+\frac{t}{4n}\right)-\frac{a_n}{n}\sin\left(nx+\frac{t}{4n}\right)\right]. \tag{15}$$

Moreover, v_t is continuous for all x, t, so that P_1 holds. Now one may not obtain v_x by termwise differentiation of eq 14, because the resulting series will generally not converge. However, v_x does exist and is a continuous function of x and t for all t and for all x not in E. To see this, one uses eq 10 and 12 to carry the Taylor formula (eq 13) to one higher power of t. It is found that

$$\begin{aligned}v(x,t)=f(x)&+\frac{t}{4}\sum_{n=1}^{\infty}\left(\frac{b_n}{n}\cos nx-\frac{a_n}{n}\sin nx\right)-\\&\frac{t^2}{32}\sum_{n=1}^{\infty}\alpha_n\left(\frac{a_n}{n^2}\cos nx+\frac{b_n}{n^2}\sin nx\right)-\\&\frac{t^3}{384}\sum_{n=1}^{\infty}\gamma_n\left(\frac{b_n}{n^3}\cos nx-\frac{a_n}{n^3}\sin nx\right).\end{aligned} \tag{16}$$

By termwise differentiation of eq 16, it is found that

$$\begin{aligned}v_x(x,t)=f'(x)&-\frac{t}{4}\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)-\\&\frac{t^2}{32}\sum_{n=1}^{\infty}\alpha_n\left(\frac{b_n}{n}\cos nx-\frac{a_n}{n}\sin nx\right)+\\&\frac{t^3}{384}\sum_{n=1}^{\infty}\gamma_n\left(\frac{a_n}{n^2}\cos nx+\frac{b_n}{n^2}\sin nx\right)=\\&f'(x)-\tfrac{1}{4}tf(x)+\Sigma_3+\Sigma_4.\end{aligned}$$

Restricting attention to t with $0\leq t\leq t_1<\infty$, one sees that the expressions Σ_3 and Σ_4 are uniformly convergent with respect to x and t. The series leading to $f(x)$ is uniformly convergent for x in any interval bounded away from a discontinuity of $f(x)$. Hence, for x not in E, termwise differentiation leads to the correct value of v_x. Moreover, v_x is continuous in x and t whenever x is a point of continuity of $f(x)$. To get v_{tx} one may differentiate eq 15 termwise with respect to x:

$$v_{tx}(x,t)=$$
$$-\tfrac{1}{4}\sum_{n=1}^{\infty}\left[a_n\cos\left(nx+\frac{t}{4n}\right)+b_n\sin\left(nx+\frac{t}{4n}\right)\right]. \quad (17)$$

As remarked after eq 14, the series in eq 17 is uniformly convergent for x, t such that x is bounded away from the discontinuities of $f(x)$. Hence v_{tx} is continuous in x and t for all x, t except for x in E. Since v_t, v_x, v_{tx} are all continuous, v_{xt} exists and equals v_{tx} except on the lines corresponding to the discontinuities of $f(x)$. This shows that $v(x, t)$ has property P_2. The eq 17 and 14 show that v satisfies the telegrapher's equation (eq 3). Finally, property P_4 was taken care of by the selection of $\{a_n, b_n\}$. Thus the problem is solved completely.

2. Proof of Uniqueness

It will be shown that $v(x,t)$ is the only function that solves the above problem. Suppose that $v_1(x,t)$ were a second solution. Then the difference, $w(x,t)=v-v_1$, satisfies the same problem with $f(x)\equiv 0$. For each x_1, by property P_1, $w(x_1, t)$ and $w_t(x_1, t)$ are continuous functions of t for $0\leq t<\infty$, while $w(x, 0)\equiv 0$ and $w_x(x, 0)\equiv 0$ are, of course, continuous functions of x. For each t, let the value of $w(x,t)$ be extended as a periodic function of x to all x in the interval $[-2\pi, 2\pi]$. Now the values $w(-\pi, t)$ and $w(x,0)$ are given on two characteristics of eq 3. By pp. 21 to 22 of [10] they are therefore sufficient to determine $w(x,t)$ uniquely for all x, t. On the other hand, the values $w(\pi, t)$ and $w(x, 0)$ are also sufficient to determine $w(x, t)$ for all x, t. Since $w(-\pi, t)=w(\pi, t)$ and $w(x, 0)=w(-x, 0)=0$, it is seen by symmetry that $w(x, t)=w(-x, t)$. Now since the values of x lie on a circle, there is nothing exceptional about the line $x=\pi$. The above argument will also show that, for each value of x_1, $w(x_1+x, t)=w(x_1-x, t)$. It follows that for each fixed t, $w(x, t)=$constant, whence $w(x, t)=h(t)$. By eq 3, $-\tfrac{1}{4}w=w_{tx}=(d/dx)h'(t)=0$. Hence, the constant value of $w(x, t)$ must be everywhere zero. Then $v\equiv v_1$, and the solution $v(x, t)$ given by eq 14 is unique.

The results of section III may be summarized in the following theorem, phrased in the notation of section II.

THEOREM 1. *If the real-valued function $f(x)$ defined on C satisfies hypotheses H_1 and H_2, then there exists a unique function, $v(x, t)$, defined on R and possessing properties P_1, P_2, P_3, and P_4. If eq 8 is the Fourier series of $f(x)$, then $v(x, t)$ is defined explicitly by eq 14.*

It is of mathematical interest[7] to note that Theorem 1 can be extended to general functions $f(x)$ of bounded variation. That is, one may replace H_1 by the weaker hypothesis

H_1': *$f(x)$ is of bounded variation on C. Moreover,*[8]

$$f(x)=\tfrac{1}{2}[f(x+0)+f(x-0)], \qquad \text{(all } x\text{)}.$$

The solution $v(x, t)$ is required to have property P_2', instead of P_2:

P_2': *v_x exists in R except for x in a set E ($E\subset C$) of Lebesgue measure zero; for all t and for all x not in E, v_{tx} exists and is a continuous function of x and*[9] *t.*

The extension of Theorem 1 is stated as follows:

THEOREM 2. *If the real-valued function $f(x)$ defined on C satisfies hypotheses H_1 and H_2', then there exists a unique function $v(x, t)$ defined on R and possessing properties P_1, P_2', P_3, and P_4. If eq 8 is the Fourier series of $f(x)$, then $v(x, t)$ is defined explicitly by eq 14.*

The convergence proof of section III, 2 requires only slight modification to serve as a proof of Theorem 2. For an arbitrary function $f(x)$ of bounded variation, there need be no interval of continuity; one may therefore not expect the series (eq 14) to converge uniformly in any interval. The termwise differentiations of section III, 2 can, however, be justified for almost all x by the fact that the resulting series are Fourier series.

IV. Representation by a Green's Function

The formula (eq 14) for the solution of the problem of section II is directly adapted to numerical computation only when the Fourier coefficients a_n, b_n converge rapidly to zero. But some of the most important cases in meteorology are where $f(x)$ has discontinuities (see footnote 7).

[7] This extension seems to have no meteorological interest. However, it is of much importance in meteorology to deal with functions $f(x)$ with some discontinuities; such discontinuities occur at fronts between air masses.

[8] Same as eq 1.

[9] It follows from P_1 and P'_2 that v_{xt} exists and equals v_{tx} for all t and for all x not in E; see pp. 55 to 57 of [3].

With such an f the Fourier coefficients are, roughly speaking, of the order $O(1/n)$, and for those f the convergence of eq 14 is hopelessly slow.

It is possible, however, to improve the convergence of eq 14 to such a degree that computation of $v(x, t)$ is reasonably possible. The procedure will be illustrated in section V for one particular choice of $f(x)$:

$$f(x)=\sigma_0(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\begin{cases}\frac{1}{2}(\pi-x) & (0<x\leq\pi)\\ & (x=0)\\ -\frac{1}{2}(\pi+x) & (-\pi\leq x<0).\end{cases} \tag{18}$$

It may be shown by direct computation that the Fourier series of $\sigma_0(x)$ is the series of eq 18. It then follows that the series converges to $\sigma_0(x)$ for all x. The reason for choosing $\sigma_0(x)$ is two-fold: (a) it is of meteorological interest to see how a simple discontinuity in $v(x, 0)$ is propagated, as t increases; (b) for any $f(x)$ that is sectionally smooth, it is possible to represent the corresponding $v(x, t)$ in terms of the solution for the special initial condition $v(x, 0)=\sigma_0(x)$.

The present section is devoted to proving the property (b). Suppose, therefore, that $G(x, t)$ is the solution of the problem of section II with the initial condition $\sigma_0(x)$; then $G(x, 0)=\sigma_0(x)$. Let a sectionally smooth function $f(x)$ be given that satisfies eq 1 and 2. Let $f(x)$ have the jump $J_k=f(x_k+0)-f(x_k-0)$ at the point $x_k (k=1, 2, \ldots, K)$. Let $\sigma_0(x)$ be continued periodically for x in $[-2\pi, 2\pi]$. Then $(J_k/\pi)\sigma_0(x-x_k)$ also has the jump J_k at the point x_k. Now

$$\xi(x)=f(x)-\frac{1}{\pi}\sum_{k=1}^{K}J_k\sigma_0(x-x_k)$$

is a continuous function, since all the jumps have been removed. Moreover, $\xi(x)$ and the functions $(J_k/\pi)\sigma_0(x-x_k)$ all satisfy eq 1 and 2. There is, therefore, a unique solution to the problem of section II for each of these functions. For the function $(J_k/\pi)\sigma_0(x-x_k)$ the solution is $(J_k/\pi)G(x-x_k, t)$. It will be shown below that the solution $y(x, t)$ corresponding to the initial values $\xi(x)$ is given by

$$y(x, t)=\frac{1}{\pi}\int_{-\pi}^{\pi}G(x-u, t)\xi'(u)du. \tag{19}$$

Since the problem of section II is linear in the initial condition $f(x)$, and since

$$f(x)=\xi(x)+\frac{1}{\pi}\sum_{k=1}^{K}J_k\sigma_0(x-x_k), \tag{20}$$

it follows that

$$v(x, t)=\frac{1}{\pi}\sum_{k=1}^{K}J_kG(x-x_k, t)+\frac{1}{\pi}\int_{-\pi}^{\pi}G(x-u, t)\xi'(u)du. \tag{21}$$

The formula (eq 21) is the desired representation of $v(x, t)$ in terms of the solution $G(x, t)$ to the single problem where $f(x)=\sigma_0(x)$. The nature of eq 21 indicates that $G(x, t)$ may be called the *Green's function* of the problem of section II. The representation (eq 21) is not only of theoretical importance, but it can also be used for approximating the solutions for general boundary values, $f(x)$, once the Green's function is tabulated. The practical problem then becomes one of approximating the integral in eq 21 by some numerical process. This latter problem is not treated here.

It remains to prove eq 19. First, it may be observed that, for each x, since σ_0 and ξ are periodic,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sigma_0(x-u)\xi'(u)du=\frac{1}{\pi}\int_{-\pi}^{\pi}\sigma_0(x-u)d\xi(u)=$$
$$-\frac{1}{\pi}\int_{-\pi}^{\pi}\xi(u)d_u\sigma_0(x-u)=\frac{1}{\pi}\int_{-\pi}^{x-0}\xi(u)\sigma_0'(x-u)du$$
$$+\xi(x)+\frac{1}{\pi}\int_{x+0}^{\pi}\xi(u)\sigma_0'(x-u)du=$$
$$\xi(x)-\frac{1}{2\pi}\int_{-\pi}^{\pi}\xi(u)du=\xi(x). \tag{22}$$

In eq 22 we have used two Riemann-Stieltjes integrals. The last step is true because $\xi(x)$ satisfies eq 2. By eq 13,

$$G(x, t)=\sigma_0(x)+\frac{t}{4}\sum_{n=1}^{\infty}\frac{\beta_n}{n^2}\cos nx-\frac{t^2}{32}\sum_{n=1}^{\infty}\frac{\alpha_n}{n^3}\sin nx.$$

Hence

$$\frac{1}{\pi}\int_{-\pi}^{\pi}G(x-u, t)\xi'(u)du=\frac{1}{\pi}\int_{-\pi}^{\pi}\sigma_0(x-u)\xi'(u)du+\frac{t}{4\pi}\int_{-\pi}^{\pi}\left[\sum_{n=1}^{\infty}\frac{\beta_n}{n^2}(\cos nx\cos nu+\sin nx\sin nu)\right]\xi'(u)du-$$
$$\frac{t^2}{32\pi}\int_{-\pi}^{\pi}\left[\sum_{n=1}^{\infty}\frac{\alpha_n}{n^3}(\sin nx\cos nu-\cos nx\sin nu)\right]\xi'(u)du. \tag{23}$$

Since $\xi(u)$ is sectionally smooth, $|\xi'(u)|$ is bounded. Hence the series in eq 23 remain uniformly convergent when multiplied by $\xi'(u)$ and may be integrated termwise. We note that

$$\left.\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}\cos nu\xi'(u)du=\frac{n}{\pi}\int_{-\pi}^{\pi}\sin nu\xi(u)du=nb_n,\\ \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nu\xi'(u)du=-\frac{n}{\pi}\int_{-\pi}^{\pi}\cos nu\xi(u)du=-na_n,\end{aligned}\right\} \tag{24}$$

where a_n and b_n are the Fourier coefficients of ξ.

In view of eq 22 and 24, the termwise integration in eq 23 yields

$$\frac{1}{\pi}\int_{-\pi}^{\pi}G(x-u,t)\xi'(u)du=\xi(x)+$$

$$\frac{t}{4}\sum_{n=1}^{\infty}\frac{\beta_n}{n}(b_n\cos nx-a_n\sin nx)-$$

$$\frac{t^2}{32}\sum_{n=1}^{\infty}\frac{\alpha_n}{n^2}(a_n\cos nx+b_n\sin nx). \tag{25}$$

But, by eq 13 and 14, the right-hand side of eq 25 is $y(x,t)$, the solution corresponding to the initial values $\xi(x)$. This completes the proof of eq 19.

The representations (eq 20 and 21) assume a more symmetric and unified form when the Lebesgue-Stieltjes integral is used. It can be shown that

$$f(x)=\frac{1}{\pi}\int_C\sigma_0(x-u)df(u), \tag{26}$$

and that

$$v(x,t)=\frac{1}{\pi}\int_C G(x-u,t)df(u), \tag{27}$$

where eq 26 and 27 include Lebesgue-Stieltjes integrals over the circle C. Whenever $f(x)$ has a discontinuity (say at x_1), the integral in eq 26 fails to converge as a Riemann-Stieltjes integral for $x=x_1$, because the functions $\sigma_0(x_1-u)$ and $f(u)$ both have a discontinuity for $u=0$. The same holds for eq 27. The integrals (eq 26 and 27) are convergent for all x as Lebesgue-Stieltjes integrals. Moreover, the formula (eq 27) yields the solution of the problem when $f(x)$ is an arbitrary function of bounded variation; the above proof of eq 19 can be modified to serve as a proof of eq 27.

V. Computation of the Green's Function

For the purpose of using the representation (eq 21) and for its own meteorological interest, it was desired to compute the Green's function $G(x, t)$. A tabulation to three decimal places, accurate to approximately 0.001, appeared sufficiently accurate. An x interval of 5 degrees of longitude ($\pi/36$ radian) is convenient in meteorology. It was decided to compute $G(x, t)$ for various times τ up to 96 hours at latitudes ϕ from 32°19′ to[10] 55°. Since the length unit is here the radian of *longitude*, the expression $d\phi/d\mu$ in section I takes the value $\cos\phi$. Then $4\beta=8\Omega\cos^2\phi$, where $\Omega=7.292\times10^{-5}$ radian/second. Now 24 hours corresponds to $\tau=86{,}400$ seconds, or to $t=50.40\cos^2\phi$. At latitude 32°19′, $\cos^2\phi=0.714$, whence $t=36$ at 24 hours. The largest value of t for which $G(x, t)$ was computed corresponds to 96 hours at latitude 32°19′; it is $t=144$.

Let $z=t/4$, for convenience. By eq 14

$$G(x,z)=\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(nx+\frac{z}{n}\right). \tag{28}$$

In summary, a method is required to compute $G(x, z)$ to an accuracy of approximately ±0.001 for[11] $x=-\pi(\pi/36)\pi$ and for various positive values of z up to 36. The present section will present one such procedure, an application of a method for improving the convergence of certain Fourier series, given on pp. 84 to 88 of [10]. *The procedure presented below is not an exact description of the methods actually used in making the table of section VIII.* It is assumed in section V that computing machinery is available capable of dealing with numbers of 10 decimal digits, but no more than 10.

Of the tolerable error 0.001, the amount 0.0005 must be reserved for round-off in the final tabulation to three decimal places. Suppose that 0.0004 is allowed for truncation errors,[12] and 0.0001 for computing errors resulting from round-offs during the calculation with 10-digit numbers. To have a truncation error as low as 0.0004 from use of a partial sum of eq 28 would require about 23,000

[10] The limit $\phi=32°19'$ arose unintentionally.

[11] The notation $x=a(\delta)b$ means $x=a, a+\delta, a+2\delta, a+3\delta, \ldots, b-\delta, b$.

[12] Truncation errors are errors that result from use of approximate mathematical formulas, e. g., use of partial sums of infinite series.

terms for $x=\pi/36$ and $x=35\pi/36$. The convergence must obviously be improved.

1. Representation by Truncated Double Sum

We write

$$G(x,z)=\sum_{n=1}^{\infty}\frac{1}{n}\sin nx\cos\frac{z}{n}+\sum_{n=1}^{\infty}\frac{1}{n}\cos nx\sin\frac{z}{n}=\Sigma_1+\Sigma_2; \tag{29}$$

for simplicity we consider only the first sum Σ_1 in eq 29. It can be shown that the term Σ_2 behaves similarly throughout the analysis. Expanding $\cos(z/n)$ in its Maclaurin series, one has

$$\Sigma_1=\sum_{n=1}^{\infty}\frac{1}{n}\sin nx\sum_{r=0}^{\infty}\frac{(-1)^r z^{2r}}{(2r)!n^{2r}}=\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r z^{2r}\sin nx}{(2r)!n^{2r+1}}. \tag{30}$$

One type of truncation of eq 30 consists in omitting all terms for $n\geq N+1$, $r\geq R$. Let the error caused by this truncation be called ϵ_1. We shall estimate ϵ_1 for $0\leq z\leq Z$:

$$|\epsilon_1|=\left|\sum_{n=N+1}^{\infty}\sum_{r=R}^{\infty}\frac{(-1)^r z^{2r}\sin nx}{(2r)!n^{2r+1}}\right|$$

$$\leq\sum_{r=R}^{\infty}\frac{Z^{2r}}{(2r)!}\sum_{n=N+1}^{\infty}\frac{1}{n^{2r+1}}<\sum_{r=R}^{\infty}\frac{Z^{2r}}{(2r)!}\int_N^{\infty}\frac{d\chi}{\chi^{2r+1}}$$

$$=\sum_{r=R}^{\infty}\frac{(Z/N)^{2r}}{(2r)(2r)!}<\frac{1}{4\sqrt{\pi}}\sum_{r=R}^{\infty}\frac{1}{r^{3/2}}\left(\frac{eZ}{2rN}\right)^{2r}.$$

The last step above uses the Stirling expression for the factorial function, which, according to p. 74 of [8], is a one-sided estimate: $s!>s^s e^{-s}\sqrt{2\pi s}$. Continuing, one finds for $eZ/2RN<1$ that

$$|\epsilon_1|<\frac{1}{4\sqrt{\pi}}\left(\frac{eZ}{2RN}\right)^{2R}\frac{1}{R^{3/2}}\sum_{r=0}^{\infty}\left(\frac{eZ}{2RN}\right)^{2r}$$
$$=\frac{1}{4\sqrt{\pi}R^{3/2}}\left(\frac{eZ}{2RN}\right)^{2R}\left[1-\left(\frac{eZ}{2RN}\right)^2\right]^{-1}=F(N,R). \tag{31}$$

The first estimate in eq 31 seems crude, but it does not affect the values of N or R very much. Thus $F(N,R)$ is an upper bound for the truncation error $|\epsilon_1|$ introduced by omitting terms of type $n\geq N+1$, $r\geq R$ in Σ_1.

One can reasonably tolerate a truncation error $|\epsilon_1|$ of 5×10^{-5}. The corresponding admissible values of N and R come from setting $F(N,R)=5\times10^{-5}$ for $Z=36$. The following pairs of values of N,R were obtained from eq 31 by a numerical calculation followed by a round-off of N to an integral value:

$N=26$	$R=4$	$N=9$	$R=8$	
18	5	7	10	(32)
13	6	4	15	
11	7	3	20	

The selection of the most suitable pair of values N,R from eq 32 will be postponed until we have discussed the summation of the remaining terms of eq 30.

2. Computation of the Double Sum

The terms of Σ_1 for $n=1, 2, \ldots, N$ and all r may be left in the form

$$\Sigma_1'=\sum_{n=1}^{N}\left(\frac{1}{n}\cos\frac{z}{n}\right)\sin nx, \tag{33}$$

and may be computed from this formula. The terms for $n\geq N+1$ and $r=0, 1, \ldots, R-1$ may be written in the form

$$\Sigma_2'=\sum_{r=0}^{R-1}\frac{z^{2r}}{(2r)!}\sigma_{2r}^{(N)}(x), \tag{34}$$

where

$$\sigma_{2r}^{(N)}(x)=(-1)^r\sum_{n=N+1}^{\infty}\frac{\sin nx}{n^{2r+1}}. \tag{35}$$

Once $\sigma_{2r}^{(N)}(x)$ has been tabulated for the one value of N to be selected below, Σ_2' may be computed directly from eq 34. Two methods are needed to get $\sigma_{2r}^{(N)}(x)$, as the calculating machinery is assumed to be limited to ten decimal digits.

The first method is to use the identity

$$\sigma_{2r}^{(N)}(x)=\sigma_{2r}(x)-(-1)^r\sum_{n=1}^{N}\frac{\sin nx}{n^{2r+1}}, \tag{36}$$

where

$$\sigma_{2r}(x)=\sigma_{2r}^{(0)}(x)=(-1)^r\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2r+1}}. \tag{37}$$

In section VI it will be shown that for $0\leq x\leq\pi$, $\sigma_{2r}(x)$ is essentially a Bernoulli polynomial in the variable $x/2\pi$. Hence $\sigma_{2r}(x)$ can readily be com-

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puted or even interpolated from existing tables like [4]. One can then get $\sigma_{2r}^{(N)}(x)$ by carrying out the subtraction indicated in eq 36. The terms $\sigma_{2r}(x)$ and $(\sin x)/1$ of eq 36 are approximately unity; with ten-digit calculating machinery these terms may be carried to ten decimal places. Hence $\sigma_{2r}^{(N)}(x)$, calculated from eq 36, will be good to ten decimal places. In order to get the Green's function to an accuracy of 0.001, it is necessary that all terms in the sums of eq 33 and 34 be substantially correct to five decimal places; hence the terms $z^{2r}\sigma_{2r}^{(N)}(x)/(2r)!$ in eq 34 must be given to five decimal places. It follows that $z^{2r}/(2r)!$ must be less than 10^5; for $z=36$ this means that $2r\leq 4$. Thus the first method of computing $\sigma_{2r}^{(N)}(x)$ is adequate when $2r=0, 2, 4$.

For $2r > 4$, another way of getting $\sigma_{2r}^{(N)}(x)$ is needed. For these larger r, the convergence of the infinite series (eq 35) is good, and we may write

$$\sigma_{2r}^{(N)}(x) \doteq (-1)^r \sum_{n=N+1}^{N_{2r}} \frac{\sin nx}{n^{2r+1}}. \tag{38}$$

The truncation error ϵ_2 in eq 38 may be shown to be at its maximum when $x=\pi/36$ and when $\sin(N_{2r}x)=0$. In this case the error is not greater than the sum of the first 36 omitted terms, which is estimated by

$$|\epsilon_2| < \sum_{n=N_{2r}+1}^{N_{2r}+36} \frac{\sin(n\pi/36)}{n^{2r+1}} < \frac{1}{N_{2r}^{2r+1}} \sum_{n=1}^{35} \sin\frac{n\pi}{36} < \frac{72}{\pi N_{2r}^{2r+1}}.$$

In order to keep $\epsilon_2 z^{2r}/(2r)!$ numerically less than 5×10^{-5} for $z=36$, it is sufficient that

$$\frac{2}{\pi}\left(\frac{36}{N_{2r}}\right)^{2r+1}\frac{1}{(2r)!}=5\times 10^{-5}. \tag{39}$$

Solution of eq 39 gives the following points of truncation of eq 38:

$$\left.\begin{aligned} &N_4=126,\ N_5=78,\ N_6=54,\ N_7=40,\\ &N_8=32,\ N_9=26,\ N_{10}=22,\ \ldots. \end{aligned}\right\} \tag{40}$$

(The values of N_k for odd subscripts k are appropriate to a parallel analysis of the sum Σ_2 of eq 29, but not to the present analysis of Σ_1.) If $\sigma_6^{(N)}(x)$, $\sigma_8^{(N)}(x), \ldots, \sigma_{2R-2}^{(N)}(x)$ are estimated by eq 38, with the values N_{2r} taken from eq 40, the individual summands of eq 34 will each have a truncation error not exceeding 5×10^{-5}.

3. Operational Analysis, Selection of N and R

We now estimate the labor involved in computing the Σ_1 of eq 29, in order to select that pair of values of N and R from eq 32 that makes the computational work a minimum. The resulting computing procedure will be optimal in a limited sense—i. e., optimal among the one-parameter family of truncations considered in section V, 1. Although the resulting procedure will be perfectly feasible for computation—indeed, it differs only moderately from the procedure actually used to get the tables of section VIII—it cannot be said to be optimal among *all* procedures for computing the Σ_1 of eq 29. For it has been based on a certain type of truncation of a certain double series (eq 30), and on a predetermined assignment of truncation errors to several subcalculations (eq 38). Given only the nature of the computational machinery, to describe an absolutely optimal procedure of getting Σ_1 would seem quite beyond the present powers of analysis.

It is customary and quite realistic to estimate the cost of a computation by the number of multiplications required.[13] We shall consider the multiplications required to get Σ_1 for one value of z and for one value of x. In eq 33 one may ignore the desk computation required to get $(1/n)\cos(z/n)$ and $\sin nx$; there are then essentially N multiplications involved in eq 33. In using eq 36, one may ignore the work of getting $\sigma_{2r}(x)$, which is chargeable to basic table development, and count $3N$ multiplications needed in all to get $\sigma_0^{(N)}(x)$, $\sigma_2^{(N)}(x)$, $\sigma_4^{(N)}(x)$. To get $\sigma_6^{(N)}(x)$ from eq 38, in view of eq 40, requires $54-N$ multiplications. To get $\sigma_8^{(N)}, \sigma_{10}^{(N)}, \ldots, \sigma_{2R-2}^{(N)}$ from eq 40 requires in all approximately $(R-4)(30-N)$ multiplications, where 30 is a rough average of the higher values of N_{2r} in eq 40. To put Σ_2' together by use of eq 34 involves R multiplications.

Summarizing, we find that getting Σ_1 by the outlined procedure requires for each value of z and each value of x the number of multiplications

$$\begin{aligned} W_1(N,R)&=-66+7N+31R-RN=\\ &151-(7-R)(31-N). \end{aligned} \tag{41}$$

Minimizing $W_1(N,R)$ over the pairs given in eq 32 selects the pair $N=18$, $R=5$, for which $W_1(18,5)=125$. Assuming that the computation of Σ_2 in

[13] This method is expecially useful with respect to computations on International-Business-Machines equipment.

eq 29 involves the same considerations, we may therefore propose $N=18$, $R=5$ as being the optimal values to use in getting $G(x, z)$ by eq 33 and 34 for one value of x and one value of z. The number of multiplications will be $2W_1(N, R)$.

Getting $G(x, z)$ for 72 values of x and one value of z involves no change of N, R. Since sin nx [cos nx] is an odd [even] function of x, the number of multiplications in getting $G(x, z)$ for all x will be $72W_1(N, R)$. However, getting $G(x, z)$ for several values of z changes the analysis, because the functions $\sigma_{2r}^{(N)}(x)$, once computed, serve for each new z without change. To get Σ_1 for 13 values of z and one value of x, for example, will require, in addition to the multiplications in eq 41, only $12N$ multiplications from eq 33 and $12R$ from eq 34. The total number of multiplications will then be

$$W_{13}(N, R) = -66+19N+43R-NR = 751-(19-R)(43-N).$$

The minimum of $W_{13}(N, R)$ is 361, and occurs for $N=13$, $R=6$. The optimal choice of N, R has changed, though not greatly. Since we expect to use 13 values of z, we adopt the values $N=13$, $R=6$.

4. Summary of the Computation Method

With the above choice of N and R, the computation of Σ_1 may proceed as follows:

(a) Compute

$$\Sigma_1' = \sum_{n=1}^{13} \left(\frac{1}{n} \cos \frac{z}{n}\right) \sin nx.$$

(b) For $r=0, 1, 2$, compute

$$\sigma_{2r}^{(13)}(x) = \sigma_{2r}(x) - (-1)^r \sum_{n=1}^{13} \frac{\sin nx}{n^{2r+1}}, \qquad (42)$$

where $\sigma_{2r}(x)$ is computed from section VI.

(c) Compute

$$\left.\begin{aligned} \sigma_6^{(13)}(x) &= -\sum_{n=14}^{54} \frac{\sin nx}{n^7}, \\ \sigma_8^{(13)}(x) &= \sum_{n=14}^{32} \frac{\sin nx}{n^9}, \\ \sigma_{10}^{(13)}(x) &= -\sum_{n=14}^{22} \frac{\sin nx}{n^{11}}. \end{aligned}\right\} \qquad (43)$$

(d) Compute

$$\Sigma_2' = \sum_{r=0}^{5} \frac{z^{2r}}{(2r)!} \sigma_{2r}^{(13)}(x).$$

(e) Compute $\Sigma_1 = \Sigma_1' + \Sigma_2'$.

The number of multiplications involved in getting Σ_1 for one x-value and for one z-value is: (a) 13; (b) 39; (c) 69; (d) 6; (e) 0. When getting Σ_1 for one x-value and 13 z-values, one adds 156 multiplications to (a) and 72 to (d). The total for 13 z-values is 355 multiplications per x-value. (The slight discrepancy with the number 361 in subsection 3 is due to the rough estimate previously made for step (c).) For all x-values (essentially 36), one gets a total of 12,780 multiplications to get Σ_1.

To get Σ_2 in eq 29, one follows analogous steps involving $\sigma_{2r+1}(x)$, $\sigma_{2r+1}^{(13)}(x)$, etc. There will be approximately 12,750 more multiplications, making a total of about 25,500 multiplications to get $G(x, z)$ for the 72 x-values and 11 z-values.

The total truncation error in getting Σ_1 is bounded by 2×10^{-4}. This is divided into four truncation errors of 5×10^{-5}, one for each of the three steps in (c), and one for the terms left out of (e). The truncation error for Σ_2 is also bounded by 2×10^{-4}, making a total truncation error of 4×10^{-4}. The final round-off of the final answer to three decimal places may introduce an error of 5×10^{-4}. The third source of error is the accumulation of round-offs from adding five-decimal-place terms. Each term is accurate to 5×10^{-6}; with an assumed rectangular distribution these terms have a dispersion near 3×10^{-6}. Each value of $G(x, z)$ is obtained from the addition of about 270 such terms. The dispersion σ of the sum is therefore about $\sqrt{270}\times5\times10^{-6}$, or about 8×10^{-5}. One may expect the accumulated error to exceed $2.5\ \sigma=2\times10^{-4}$ in only 1.3 percent of the cases. The sum of the three errors is effectively bounded by 11×10^{-4}, or slightly more than 0.001.

VI. The Polynomials $\{\sigma_k(x)\}$

In section V we made use of certain functions $\sigma_k(x)$ defined as follows:

$$\left.\begin{aligned} \sigma_{2r}(x) &= (-1)^r \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2r+1}} \qquad (r=0, 1, 2, \ldots); \\ \sigma_{2r+1}(x) &= (-1)^{r+1} \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2r+2}} \qquad (r=0, 1, 2, \ldots). \end{aligned}\right\} \qquad (44)$$

The function $\sigma_0(x)$ was used in section IV; see eq. 18. For $k>0$ the series $\sigma_k(x)$ in eq. 44 are absolutely convergent; hence they represent continuous functions. Since $\sigma_{2r}(x)$ is odd and $\sigma_{2r+1}(x)$ is even, it is necessary to sum the series (eq. 44) only for $0\leq x\leq \pi$.

As stated in eq. 18,

$$\sigma_0(x)=\frac{\pi}{2}-\frac{1}{2}x, \qquad (0<x\leq\pi). \tag{45}$$

Now

$$\sigma_1(x)=-\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\int_0^x \sigma_0(\xi)d\xi-\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Hence

$$\sigma_1(x)=-\frac{\pi^2}{6}+\frac{\pi}{2}x-\frac{1}{4}x^2, \qquad (0\leq x\leq \pi).$$

Similarly, for $0\leq x\leq\pi$, one finds

$$\left.\begin{aligned}
\sigma_2(x)&=-\frac{\pi^2}{6}x+\frac{\pi}{4}x^2-\frac{1}{12}x^3;\\
\sigma_3(x)&=\frac{\pi^4}{90}-\frac{\pi^2}{12}x^2+\frac{\pi}{12}x^3-\frac{1}{48}x^4;\\
\sigma_4(x)&=\frac{\pi^4}{90}x-\frac{\pi^2}{36}x^3+\frac{\pi}{48}x^4-\frac{1}{240}x^5.
\end{aligned}\right\} \tag{46}$$

Use has been made of the formulas $\sum_{n=1}^{\infty} n^{-2}=\pi^2/6$; $\sum_{n=1}^{\infty} n^{-4}=\pi^4/90$.

The functions $\sigma_k(x)$ are therefore all polynomials. Their use in improving the convergence of Fourier series is pointed out on pp. 84 to 88 of [10]. Although they may be easily tabulated from eq 46, they may also be adapted from existing tables because they are essentially Bernoulli polynomials. Let $\{B_n(x)\}$ be the Bernoulli polynomials given on p. 181 of [4].

LEMMA.[14] *For $0\leq x\leq\pi$, and $k=0, 1, 2, \ldots,$*

$$\sigma_k(x)=-\frac{(2\pi)^{k+1}}{2(k+1)!}B_{k+1}\left(\frac{x}{2\pi}\right).$$

Proof: Define the Bernoulli number B_n by the relation $B_n=B_n(0)$. These are the Bernoulli numbers used on p. 21 of [8]; they are $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=1/6$, $B_3=0$, $B_4=-1/30$, $B_5=0$, $B_6=1/42, \ldots,$ Davis uses other notations in [4]. Now fix x in the interval $0\leq x\leq\pi$. For each $k=1, 2, 3, \ldots,$

$$\sigma_k(x)=\int_0^x \sigma_{k-1}(\xi)d\xi-\sin\frac{k\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^{k+1}}=$$

$$\int_0^x \sigma_{k-1}(\xi)d\xi-\frac{(2\pi)^{k+1}B_{k+1}}{2(k+1)!}.$$

The last step is by eq 9 on p. 21 of [8], which is correct except for sign. Hence, letting $x=2\pi t$,

$$2(2\pi)^{-k-1}\sigma_k(2\pi t)=$$

$$\int_0^t 2(2\pi)^{-k}\sigma_{k-1}(2\pi\eta)d\eta-\frac{B_{k+1}}{(k+1)!}\ (k=1,2,3,\ \ldots\ ). \tag{47}$$

Now define $2(2\pi)^0\,\sigma_{-1}(2\pi t)=-B_0/0!\,(=-1)$. Use eq 47 formally to get $2(2\pi)^{-1}\sigma_0(2\pi t)$:

$$2(2\pi)^{-1}\sigma_0(2\pi t)=-\frac{B_0}{0!}\frac{t}{1!}-\frac{B_1}{1!}(=-t+\tfrac{1}{2}). \tag{48}$$

Note that eq 48 agrees with eq 45 for $\sigma_0(x)$. Hence eq 48 is a correct formula, although it was only derived formally.

We now apply formula (eq 47) repeatedly, getting always correct expressions:

$$2(2\pi)^{-2}\sigma_1(2\pi t)=-\frac{B_0}{0!}\frac{t^2}{2!}-\frac{B_1}{1!}\frac{t}{1!}-\frac{B_2}{2!};$$

$$2(2\pi)^{-3}\sigma_2(2\pi t)=-\frac{B_0}{0!}\frac{t^3}{3!}-\frac{B_1}{1!}\frac{t^2}{2!}-\frac{B_2}{2!}\frac{t}{1!}-\frac{B_3}{3!};$$

$$\cdots$$

$$2(2\pi)^{-k-1}\sigma_k(2\pi t)=-\sum_{j=0}^{k}\frac{B_j}{j!}\frac{t^{k+1-j}}{(k+1-j)!}.$$

Hence

$$2(k+1)!(2\pi)^{-k-1}\sigma_k(2\pi t)=-\sum_{j=0}^{k}\frac{(k+1)!B_jt^{k+1-j}}{j!(k+1-j)!}$$

$$=-\sum_{j=0}^{k}\binom{k+1}{j}B_jt^{k+1-j}. \tag{49}$$

But it follows from the top of p. 188 of [4] that

$$B_{k+1}(t)=\sum_{j=0}^{k+1}\binom{k+1}{j}B_jt^{k+1-j}. \tag{50}$$

Comparing eq 49 and 50, we see that

$$B_{k+1}(t)=-2(k+1)!(2\pi)^{-k-1}\sigma_k(2\pi t).$$

Let $2\pi t=x$, and the lemma is proved.

VII. Solution by a Difference Equation

Our first approximate solution of the problem stated in section II consisted of the approximate evaluation of the integral (eq 21) by means of numerical integration formulas, using the approxi-

[14] After this paper was completed, Professor D. H. Lehmer called the author's attention to the statement of this lemma on p. 65 of N. E. Nörlund's Vorlesungen über Differenzenrechnung (Julius Springer, Berlin, 1924).

mate values of $G(x, z)$ tabulated in section VIII below. A second approximate solution of the problem consists in solving with appropriate boundary conditions a difference equation that is closely related to the differential equation (eq 3). The latter method is considered in detail in [6], where proofs may be found; only a summary is given in the present section.

For any positive integer $2N$, let $h=\pi/2N$; let $k>0$ be arbitrary. A net is formed of all points (x, t) of form $(\mu h, \nu k)$, where μ and ν are integers satisfying the conditions

$$\mu+\nu\equiv 0 \pmod 2),\ |\mu|\leq 2N,\ \nu\geq 0. \tag{51}$$

Where necessary we extend the net and the functional values periodically in x with period 2π. The differential equation (eq 3) is approximated by the difference equation,

$$\begin{aligned} v(x+h, t+k)-v(x-h, t+k)= \\ v(x+h, t-k)-v(x-h, t-k)-hkv(x, t). \end{aligned} \tag{52}$$

The boundary conditions of the difference-equation problem are prescribed values of $v(x, t)$ on the two rows $t=0$, $t=k$. Assume that for $t=k$,

$$\sum_x v(x,t)=0, \tag{53}$$

where the sum is extended over all points of the second row of the net. The boundary conditions and eq 52 then determine the value of $v(x, t)$ on the row $t=2k$ up to an additive constant. The additive constant and hence $v(x, 2k)$ are determined uniquely by requiring that eq 53 hold also for $t=2k$. Continuing row after row, one thus determines $v(x, t)$ over the whole net. Let the function so determined be denoted by $v^{(N)}(x, t)$; it depends on N, on k, and on the initial values prescribed for the first two rows. The problem of [6] is to see whether $v^{(N)}(x, t)\rightarrow v(x, t)$ as $N\rightarrow\infty$.

Let the initial values $v(x, 0)$ on the first row be defined by the relation $v(x, 0)=f(x)$, where $f(x)$ is the function of eq 1. Let k be fixed. Then it is possible to choose the initial values $v(x, k)$ on the second row of the net in such a manner that, as $N\rightarrow\infty$, $v^{(N)}(x, t)\rightarrow v(x, t)$ for each t of the net and for each x that is an abscissa of continuity of $f(x)$. If k is allowed to vary with N in such a manner that $k\rightarrow 0$ as $N\rightarrow\infty$, then $v^{(N)}(x, t)\rightarrow v(x, t)$ for each $t\geq 0$ and for each x that is an abscissa of continuity of $f(x)$. In neither case may one, in general, expect the convergence to be uniform in x or t.

The method referred to for choosing the values $v(x, k)$ on the second row is not an economical one, and in a practical computation one would prefer a cheaper though approximate method. Two things are shown in [6] about the effects of an approximation of the values of $v(x, k)$: First, they may introduce ultimate instability into the solution. Even though the solution $v(x, t)$ of eq 3 be identically zero, it is possible that for fixed N and x,

$$\overline{\lim_{t\rightarrow\infty}}\, v^{(N)}(x, t)=+\infty.$$

Second, the approximation does not prevent convergence of $v^{(N)}(x, t)$ to $v(x, t)$, provided that the error of the approximation of $v(x, k)$ vanishes as $N\rightarrow\infty$. One reasonable way of causing the error to vanish is to let $k\rightarrow 0$.

These results show that the difference-equation method is a feasible method of solving the problem of this paper.

VIII. Table of the Green's Function

In this section is tabulated the Green's function $G(x, z)$, as computed in the Computation Unit of the Institute for Numerical Analysis. The value of the time parameter z corresponding to h hours at latitude ϕ is

$$z=0.52502\ h \cos^2 \phi. \tag{51}$$

(Except for the last digit of the constant, formula (eq 51) can be verified from the introduction to section V.) Meteorological considerations suggested that h should be chosen in convenient multiples of 12 hours, and that ϕ should be 35°, 45°, or 55°. The latitude 32°19′ resulted from a numerical error by the author. A limited number of pairs of values of h and ϕ were selected for the computation; these pairs are shown in table 1, together with the corresponding values of z determined from eq 51.

For each of the 13 values of z given in table 1 (and for $z=0$) and for $x=-\pi(\pi/36)\pi$, the Green's function $G(x, z)$ is presented in table 2 to 5 decimal places. Since $G(x, z)$ has a discontinuity of the first kind at $x=0$, the values $G(-0, z)$, $G(0, z)$, and $G(+0, z)$ are all given. In every case, $G(0, z)=\frac{1}{2}[G(-0, z)+G(0, z)]$ and $G(+0, z)-G(-0, z)=\pi$. The computational procedure followed that of section V in general outline, with certain deviations. It was decided to use $N=18$, $R=6$. The auxiliary functions $\sigma_{2r}^{(N)}(x)$ and

$\sigma_{2r+1}^{(N)}(x)$ were computed from formulas like eq 36; formulas like eq 38 were not used. This necessitated carrying considerably more than 10 digits, and so the polynomials $\sigma_k(x)$ were first computed to 17 decimal places. Choice of formula (eq 36) was based on the value of getting these tables of Bernoulli polynomials as a byproduct of general interest.

To be definite,[15] let us write $G(x, z)=G_1(x, z)+G_2(x, z)$, where

$$G_1(x, z)=\sum_{n=1}^{18}\frac{1}{n}\sin\left(nx+\frac{z}{n}\right), \quad (52)$$

and

$$G_2(x, z)=\sum_{n=19}^{\infty}\frac{1}{n}\sin\left(nx+\frac{z}{n}\right).$$

Let

$$(\partial^k/\partial z^k)G_2(x, z)=G^{(k)}(x, z), \text{ for } k=0, 1, 2, \ldots.$$

Then

$$G^{(k)}(x, 0)=\sum_{n=19}^{\infty}\frac{1}{n^{k+1}}\sin\left(nx+k\frac{\pi}{2}\right)=g_k(x)-h_k(x),$$

where

$$g_k(x)=\sum_{n=1}^{\infty}\frac{1}{n^{k+1}}\sin\left(nx+k\frac{\pi}{2}\right),$$

$$h_k(x)=\sum_{n=1}^{18}\frac{1}{n^{k+1}}\sin\left(nx+k\frac{\pi}{2}\right).$$

By section VI, the functions $g_k(x)$ are polynomials. They were generated on an International-Business-Machines tabulator to 17 decimal places. The values of $h_k(x)$ were computed and subtracted from $g_k(x)$ to yield $G^{(k)}(x, 0)$. For any z,

$$G_2(x, z)=\sum_{k=0}^{K}\frac{z^k}{k!}G^{(k)}(x, 0)+R_k(x, z), \quad (53)$$

where there exists a z_1 $(0<z_1<z)$ such that

$$|R_k(x,z)|\leq\frac{z^{K+1}}{(K+1)!}G^{(K+1)}(x, z_1).$$

If $0\leq z\leq 36$, $z^{11}/11!\leq .33\times 10^{10}$. And $|G^{(11)}(x, z_1)|\leq \sum_{n=19}^{\infty} n^{-12}<1.5\times 10^{-15}$. Hence, for $0\leq z\leq 36$, $|R_{11}(x, z)|<0.5\times 10^{-5}$.

[15] This description of the computational procedure was furnished by Gertrude Blanch of the Computation Unit, Institute for Numerical Analysis

Once the values of $G^{(k)}(x, 0)$ were obtained it was possible to generate, very easily, the function $G(x, z)$ for any values of z in the range $0<z\leq 36$. To summarize,

(a) $G_1(x, z)$ was computed from eq (52).
(b) The functions $G^{(k)}(x, 0)$ were computed for $k=0,1,2, \ldots, 10$.
(c) $G_2(x, z)$ was obtained from eq (53).
(d) $G_1(x, z)$ and $G_2(x, z)$ were added, to yield $G(x, z)$.

The subsidiary computations in (a) and (b) were carried to nine decimal places, those in (c) to at least seven decimal places in the partial products.

Table 2 is believed to be accurate to ± 0.00002 for all x and all z. The cosine component of $G(x, z)$ was given a final check by use of the following formula:

$$\sum_{k=-36}^{35}G(k\pi/36, z)=\sum_{\nu=1}^{\infty}\frac{1}{\nu}\sin\frac{z}{72\nu}=p(z).$$

The check showed a deviation between the sum and $p(z)$, which was never greater than 0.00005. The sine component of $G(x, z)$ was given a partial check by the formula

$$\sum_{k=-36}^{35}\sin\frac{k\pi}{2}G(k\pi/36,z)=2\cos\frac{z}{18}+2\sum_{\nu=1}^{\infty}\left[\frac{1}{4\nu+1}\cos\left(\frac{z/18}{4\nu+1}\right)-\frac{1}{4\nu-1}\cos\left(\frac{z/18}{4\nu-1}\right)\right].$$

Comparable agreement was found. Finally, the table differences with respect to x are very satisfactory.

TABLE 1. *Values of the time parameter, z, corresponding to hours, h, at latitude, φ*

h	φ			
	32°19′	35°	45°	55°
Hours				
12	------	------	3.150	-------
24	9.000	8.455	6.300	4.145
48	18.000	16.909	12.600	8.291
72	27.000	------	18.900	-------
96	36.000	------	25.200	-------

TABLE 2. *Table of Green's Function G(x, z).*

x	z						
	0.00000	3.15012	4.14543	6.30024	8.29086	8.45505	9.00000
Degrees							
−180	−0.00000	0.32695	1.08321	−0.16762	−1.34038	−1.26672	−0.85722
−175	−.04363	.39530	1.08444	−.30315	−1.29132	−1.19935	−.74063
−170	−.08727	.45756	1.07647	−.43082	−1.22533	−1.11601	−.61286
−165	−.13090	.51370	1.05993	−.54955	−1.14399	−1.01848	−.47630
−160	−.17453	.56372	1.03552	−.65844	−1.04899	−0.90869	−.33337
−155	−.21817	.60768	1.00392	−.75679	−0.94217	−.78862	−.18646
−150	−.26180	.64565	0.96581	−.84405	−.82542	−.66032	−.03791
−145	−.30543	.67775	.92189	−.91985	−.70067	−.52583	.11005
−140	−.34907	.70413	.87284	−.98398	−.56985	−.38717	.25533
−135	−.39270	.72496	.81933	−1.03636	−.43487	−.24630	.39595
−130	−.43633	.74042	.76202	−1.07704	−.29757	−.10512	.53015
−125	−.47997	.75072	.70154	−1.10620	−.15975	.03460	.65632
−120	−.52360	.75611	.63851	−1.12412	−.02308	.17118	.77306
−115	−.56723	.75679	.57351	−1.13116	.11085	.30309	.87916
−110	−.61087	.75304	.50712	−1.12779	.24058	.42893	.97363
−105	−.65450	.74510	.43985	−1.11454	.36479	.54746	1.05568
−100	−.69813	.73324	.37221	−1.09199	.48229	.65759	1.12472
−95	−.74176	.71771	.30468	−1.06078	.59203	.75838	1.18034
−90	−.78540	.69880	.23769	−1.02160	.69310	.84906	1.22233
−85	−.82903	.67677	.17165	−0.97516	.78475	.92902	1.25066
−80	−.87266	.65189	.10695	−.92219	.86637	.99779	1.26545
−75	−.91630	.62442	.04392	−.86344	.93749	1.05505	1.26700
−70	−0.95993	.59462	−.01712	−.79967	.99777	1.10063	1.25573
−65	−1.00356	.56277	−.07590	−.73162	1.04701	1.13450	1.23218
−60	−1.04720	.52910	−.13215	−.66004	1.08513	1.15674	1.19701
−55	−1.09083	.49386	−.18566	−.58566	1.11218	1.16755	1.15098
−50	−1.13446	.45730	−.23624	−.50920	1.12832	1.16724	1.09494
−45	−1.17810	.41965	−.28373	−.43136	1.13378	1.15623	1.02978
−40	−1.22173	.38113	−.32798	−.35278	1.12892	1.13502	0.95646
−35	−1.26536	.34197	−.36889	−.27411	1.11416	1.10417	.87600
−30	−1.30900	.30236	−.40638	−.19596	1.09002	1.06431	.78941
−25	−1.35263	.26250	−.44037	−.11887	1.05705	1.01616	.69774
−20	−1.39626	.22259	−.47084	−.04338	1.01588	0.96044	.60203
−15	−1.43990	.18280	−.49776	.03002	0.96717	.89794	.50333
−10	−1.48353	.14330	−.52114	.10089	.91164	.82944	.40264
−5	−1.52716	.10426	−.54100	.16881	.85000	.75578	.30096
−0	−1.57080	.06582	−.55736	.23344	.78302	.67778	.19926
0	0.00000	1.63662	1.01344	1.80424	2.35382	2.24858	1.77006
+0	1.57080	3.20741	2.58423	3.37503	3.92461	3.81938	3.34085
5	1.52716	2.36366	1.53355	1.93219	1.95959	1.81396	1.21684
10	1.48353	1.62904	0.66827	0.87990	0.63440	0.47195	−0.16296
15	1.43990	0.99385	−.03330	.14729	−.20075	−.36465	−.98574
20	1.39626	.44908	−.59093	−.32689	−.66889	−.82505	−1.40257
25	1.35263	−.01369	−1.02264	−.59533	−.87018	−1.01399	−1.53397
30	1.30900	−.40226	−1.34473	−.70310	−.88521	−1.01530	−1.47484
35	1.26536	−.72390	−1.57202	−.68848	−.77804	−0.89523	−1.29869
40	1.22173	−.98534	−1.71788	−.58382	−.59873	−.70520	−1.06130
45	1.17810	−1.19282	−1.79437	−.41615	−.38563	−.48429	−0.80387
50	1.13446	−1.35212	−1.81235	−.20791	−.16734	−.26137	−.55574
55	1.09083	−1.46858	−1.78153	.02253	.03555	−.05690	−.33671
60	1.04720	−1.54714	−1.71063	.26034	.20903	.11545	−.15898
65	1.00356	−1.59233	−1.60740	.49376	.34437	.24746	−.02884
70	0.95993	−1.60833	−1.47871	.71367	.43706	.33521	.05197
75	.91630	−1.59898	−1.33066	.91321	.48585	.37809	.08507
80	.87266	−1.56780	−1.16858	1.08748	.49200	.37799	.07448

TABLE 2. *Table of Green's Function G(x, z).*

x	z						
	0. 00000	3. 15012	4. 14543	6. 30024	8. 29086	8. 45505	9. 00000
Degrees							
85	.82903	−1.51800	−0.99717	1.23323	.45862	.33860	.02585
90	.78540	−1.45251	−.82049	1.34855	.39011	.26486	−.05417
95	.74176	−1.37403	−.64207	1.43272	.29171	.16247	−.15846
100	.69813	−1.28496	−.46492	1.48592	.16915	.03753	−.27984
105	.65450	−1.18751	−.29158	1.50910	.02832	−.10376	−.41132
110	.61087	−1.08367	−.12420	1.50380	−.12493	−.25537	−.54637
115	.56723	−0.97523	.03545	1.47204	−.28498	−.41155	−.67907
120	.52360	−.86377	.18594	1.41614	−.44662	−.56703	−.80424
125	.47997	−.75074	.32616	1.33870	−.60507	−.71708	−.91745
130	.43633	−.63740	.45522	1.24244	−.75611	−.85757	−1.01507
135	.39270	−.52487	.57252	1.13016	−.89609	−.98499	−1.09431
140	.34907	−.41413	.67764	1.00467	−1.02197	−1.09649	−1.15312
145	.30543	−.30603	.77037	0.86874	−1.13129	−1.18985	−1.19019
150	.26180	−.20133	.85064	.72505	−1.22217	−1.26346	−1.20491
155	.21817	−.10064	.91855	.57617	−1.29329	−1.31627	−1.19725
160	.17453	−.00450	.97429	.42452	−1.34386	−1.34779	−1.16777
165	.13090	.08665	1.01819	.27234	−1.37355	−1.35800	−1.11746
170	.08727	.17245	1.05065	.12170	−1.38248	−1.34730	−1.04773
175	.04363	.25262	1.07214	−.02551	−1.37114	−1.31650	−0.96032
180	.00000	.32695	1.08321	−.16762	−1.34038	−1.26672	−.85722
	12.60048	16.91010	18.00000	18.90072	25.20096	27.00000	36.00000
−180	0.21759	1.38333	0.88710	−0.22251	−0.23708	−0.52214	0.75323
−175	.20848	1.28719	.68046	−.44845	−.13949	−.34513	.66966
−170	.18520	1.17912	.47714	−.65407	−.04531	−.19161	.59996
−165	.14768	1.06395	.28213	−.83538	.03742	−.06711	.55216
−160	.09639	0.94615	.09946	−.98956	.10193	.02510	.53133
−155	.03224	.82961	−.06745	−1.11498	.14294	.08392	.53944
−150	−.04345	.71765	−.21599	−1.21108	.15684	.11027	.57545
−145	−.12901	.61292	−.34473	−1.27829	.14173	.10689	.63557
−140	−.22253	.51745	−.45323	−1.31787	.09736	.07795	.71381
−135	−.32192	.43256	−.54152	−1.33178	.02503	.02877	.80257
−130	−.42494	.35899	−.61014	−1.32256	−.07262	−.03461	.89326
−125	−.52932	.29691	−.66035	−1.29314	−.19180	−.10578	.97703
−120	−.63276	.24596	−.69412	−1.24669	−.32784	−.17834	1.04492
−115	−.73306	.20536	−.71352	−1.18651	−.47538	−.24602	1.09064
−110	−.82808	.17397	−.72062	−1.11592	−.62883	−.30340	1.10661
−105	−.91587	.15038	−.71767	−1.03810	−.78241	−.34564	1.08871
−100	−.99463	.13297	−.70728	−0.95602	−.93051	−.36894	1.03429
−95	−1.06278	.12001	−.69185	−.87240	−1.06785	−.37053	0.94265
−90	−1.11899	.10975	−.67337	−.78959	−1.18968	−.34876	.81504
−85	−1.16216	.10043	−.65362	−.70959	−1.29191	−.30309	.65453
−80	−1.19145	.09041	−.63445	−.63398	−1.37124	−.23406	.46575
−75	−1.20628	.07817	−.61745	−.56396	−1.42518	−.14319	.25459
−70	−1.20633	.06238	−.60358	−.50029	−1.45213	−.03286	.02788
−65	−1.19154	.04192	−.59343	−.44339	−1.45134	.09382	−.20697
−60	−1.16210	.01592	−.58753	−.39331	−1.42289	.23321	−.44192
−55	−1.11840	−.01623	−.58625	−.34978	−1.36773	.38109	−.67081
−50	−1.06108	−.05488	−.58940	−.31226	−1.28738	.53326	−.88534
−45	−0.99094	−.10011	−.59636	−.28000	−1.18407	.68527	−1.07970
−40	−.90897	− 15176	−.60651	−.25205	−1.06049	.83281	−1.24859
−35	−.81632	−.20939	−.61926	−.22736	−0.91976	.97176	−1.38784

Table 2. *Table of Green's Function G(x, z).*

x	z						
	12.60048	16.91010	18.00000	18.90072	25.20096	27.00000	36.00000
Degrees							
−30	−.71423	−.27239	− 63367	− 20477	−.76527	1.09836	−1.49453
−25	−.60408	−.33990	−.64843	−.18315	−.60056	1.20927	−1.56697
−20	−.48731	−.41091	−.66232	−.16133	−.42926	1.30167	−1.60479
−15	−.36540	−.48428	−.67437	−.13823	−.25493	1.37335	−1.60877
−10	−.23989	−.55874	−.68350	−.11286	−.08102	1.42266	−1.58080
−5	−.11230	−.63295	−.68839	−.08436	.08924	1.44862	−1.52370
−0	.01585	−.70554	−.68784	−.05201	.25293	1.45084	−1.44108
0	1.58665	.86526	.88296	1.51879	1.82373	3.02164	0.12972
+0	3.15744	2.43605	2.45375	3.08958	3.39452	4.59243	1.70052
5	0.67142	−0.81525	−0.84342	−0.27249	−0.37260	0.54092	−2.54939
10	−.51209	−2.00417	−1.87971	−1.21362	−.63490	.27468	−1.76069
15	−.85078	−2.07925	−1.75697	−0.97035	.26636	1.10072	−0.49519
20	−.68020	−1.65134	−1.16076	−.29000	1.20809	1.82328	−.00356
25	−.24015	−1.09098	−0.48829	.41213	1.76793	2.06387	−.20922
30	.30379	−0.60220	.05701	.93791	1.89206	1.84177	−.68549
35	.84267	−.27771	.39671	1.22420	1.69102	1.32871	−1.06815
40	1.31002	−.13946	.52770	1.28534	1.32178	0.71692	−1.17421
45	1.67048	−.16770	.48736	1.17420	0.92635	.15777	−0.98652
50	1.91086	−.32146	.33030	0.95687	.60563	−.25779	−.59209
55	2.03312	−.55228	.11399	.69716	.41414	−.49626	−.11820
60	2.04891	−.81299	−.11013	.44820	.36577	−.56829	.31487
65	1.97550	−1.06293	−.30155	.24868	.44447	−.51226	.62341
70	1.83275	−1.27046	−.43250	.12229	.61561	−.37956	.76541
75	1.64092	−1.41383	−.48723	.07901	.83661	−.22221	.74192
80	1.41914	−1.48076	−.46049	.11749	1.06504	−.08486	.58453
85	1.18449	−1.46731	−.35588	.22780	1.26435	.00000	.34287
90	0.95137	−1.37647	−.18386	.39425	1.40727	.01370	.07206
95	.73134	−1.21644	.04076	.59799	1.47711	−.04915	−.17737
100	.53301	−0.99910	.30108	.81915	1.46763	−.18262	−.36632
105	.36223	−.73849	.57953	1.03865	1.38174	−.37234	−.47053
110	.22235	−.44954	.85891	1.23947	1.22959	−.59864	−.48116
115	.11450	−.14704	1.12358	1.40751	1.02640	−.83951	−.40310
120	.03798	.15526	1.36058	1.53214	0.79018	−1.07324	−.25141
125	−.00939	.44516	1.55988	1.60630	.53965	−1.28065	−.04972
130	−.03091	.71247	1.71412	1.62645	.29276	−1.44613	.17798
135	−.03066	.94918	1.81868	1.59230	.06507	−1.55895	.40643
140	−.01322	1.14954	1.87196	1.50636	−.13095	−1.61332	.61405
145	.01665	1.31003	1.87498	1.37347	−.28656	−1.60827	.78435
150	.05419	1.42912	1.83068	1.20024	−.39683	−1.54718	.90692
155	.09488	1.50714	1.74337	0.99449	−.46051	−1.43694	.97765
160	.13453	1.54593	1.61879	.76479	−.47971	−1.28710	.99822
165	.16945	1.54859	1.46378	.51991	−.45930	−1.10889	.97514
170	.19650	1.51918	1.28553	.26848	−.40628	−0.91426	.91844
175	.21318	1.46239	1.09103	.01857	−.32913	−.71502	.84021
180	.21759	1.38333	0.88710	−.22251	−.23708	−.52214	.75323

IX. References

[1] J. Charney, A. Eliassen, G. Hunt, Report on a program for numerical weather predicting, paper presented at a meeting of the American Meteorological Society, New York, N. Y., January 28, 1949.

[2] J. Charney, manuscript in preparation for Journal of Meteorology on the work reported in [1].

[3] R. Courant, Differential and integral calculus (translated by E. J. McShane) **2** (Blackie and Sons, London and Glasgow, 1936).

[4] Harold T. Davis, Tables of the higher mathematical functions, **2** (Principia Press, Bloomington, Ind., 1935).

[5] G. E. Forsythe, J. Meteorology **4,** 67 (1947).

[6] G. E. Forsythe, Manuscript in preparation on the approximate solution by difference equations of the telegrapher's equation with boundary values on only one characteristic.

[7] G. E. Forsythe, A solution of the telegrapher's equation with initial conditions on only one characteristic. Preliminary Report. Paper presented at a meeting of American Mathematical Society, Columbus, Ohio, December 30, 1948.

[8] Erwin Madelung, Die Mathematischen Hilfsmittel des Physikers, American ed. (Dover Publications, New York, N. Y., 1943).

[9] C. G. Rossby and collaborators, J. Marine Res. **2,** 38 (1939).

[10] J. D. Tamarkin and Willy Feller, Partial differential equations (mimeographed lecture notes, Brown University, Providence, R. I., 1941).

[11] A. G. Webster, Partial differential equations of mathematical physics, 2d corrected ed. (Hafner Publishing Co., New York, N. Y., 1947).

[12] A. Zygmund, Trigonometrical Series (Monografje Matematyczne, Warsaw–Lwow, 1935).

Los Angeles, March 8, 1949.