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Table of Modified Bernoulli Polynomials

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The polynomials tabulated are $b_k(x)$, where $b_k(x)$ is related to the Bernoulli polynomial $B_k(x)$ by the formula

$$b_k(x)=\left[(-1)^k\tfrac{1}{2}(2\pi)^kB_k\left(\frac{x}{2\pi}\right)\right]/k!.$$

Range of parameters: $k=1\,(1)\,11$; $x=\pi y/36$, $y=0(1)36$. The entries are given to 17 decimal places. The functions were computed in connection with the tabulation of a solution to the telegrapher's equation by George E. Forsythe.

I. Definitions

The polynomials $b_k(x)$ tabulated below can be identified with the Bernoulli polynomials[1] $B_k(x)$ by the following relations:

$$b_k(x)=\frac{(-1)^k\frac{1}{2}(2\pi)^kB_k\left(\frac{x}{2\pi}\right)}{k\,!}.$$

For $0<x\leq\pi$, we have

$$b_k(x)=\sum_{n=1}^{\infty}\frac{\sin\,[nx+\frac{1}{2}(k-1)\pi]}{n^k},\ k\geq 1.$$

Explicit expressions for the polynomials tabulated here are as follows:

$$\left.\begin{aligned}
b_1(x)&=-\frac{x}{2}+\frac{\pi}{2};\\
b_2(x)&=\frac{x^2}{4}-\frac{\pi x}{2}+\frac{\pi^2}{6};\\
b_3(x)&=-\frac{x^3}{12}+\frac{\pi x^2}{4}-\frac{\pi^2 x}{6};\\
b_4(x)&=\frac{x^4}{48}-\frac{\pi x^3}{12}+\frac{\pi^2x^2}{12}-\frac{\pi^4}{90};\\
b_5(x)&=-\frac{x^5}{240}+\frac{\pi x^4}{48}-\frac{\pi^2x^3}{36}+\frac{\pi^4x}{90};\\
b_6(x)&=\frac{x^6}{1{,}440}-\frac{\pi x^5}{240}+\frac{\pi^2x^4}{144}-\frac{\pi^4x^2}{180}+\frac{\pi^6}{945};\\
b_7(x)&=-\frac{x^7}{10{,}080}+\frac{\pi x^6}{1{,}440}-\frac{\pi^2x^5}{720}+\frac{\pi^4x^3}{540}-\frac{\pi^6x}{945};\\
b_8(x)&=\frac{x^8}{80{,}640}-\frac{\pi x^7}{10{,}080}+\frac{\pi^2x^6}{4{,}320}-\frac{\pi^4x^4}{2{,}160}+\frac{\pi^6x^2}{1{,}890}-\frac{\pi^8}{9{,}450};\\
b_9(x)&=-\frac{x^9}{725{,}760}+\frac{\pi x^8}{80{,}640}-\frac{\pi^2x^7}{30{,}240}+\frac{\pi^4x^5}{10{,}800}-\frac{\pi^6x^3}{5{,}670}+\frac{\pi^8x}{9{,}450};\\
b_{10}(x)&=\frac{x^{10}}{7{,}257{,}600}-\frac{\pi x^9}{725{,}760}+\frac{\pi^2x^8}{241{,}920}-\frac{\pi^4x^6}{64{,}800}+\frac{\pi^6x^4}{22{,}680}-\frac{\pi^8x^2}{18{,}900}+\frac{\pi^{10}}{93{,}555};\\
b_{11}(x)&=-\frac{x^{11}}{79{,}833{,}600}+\frac{\pi x^{10}}{7{,}257{,}600}-\frac{\pi^2x^9}{2{,}177{,}280}+\frac{\pi^4x^7}{453{,}600}-\frac{\pi^6x^5}{113{,}400}+\frac{\pi^8x^3}{56{,}700}-\frac{\pi^{10}x}{93{,}555}.
\end{aligned}\right\}\quad(1)$$

[1] See H. T. Davis, Tables of the higher mathematical functions, **II**, 181 (Principia Press, Inc., Bloomington, Ind., 1935).

The entries correspond to the argument $x=\pi y/36$, for $y=0(1)36$. The values are given to 17 decimal places, with an error of less than one unit in the last place.

The above functions were obtained in the process of computing the function $G(x, z)$ required in the solution of the telegrapher's equation by George E. Forsythe.[2]

II. Method of Computation

The derivatives with respect to y of $b_k(\pi y/36)$, at $y=0$, were computed with the aid of Glaisher's values[3] of π^n; and the leading forward differences at $y=0$ were then computed from the well-known formulas for differences in terms of derivatives. Let $f(y)$ be a polynomial of degree k, and let $\Delta^p f_0$ indicate the p^{th} forward difference of $f(y)$ at $y=0$. Then for integral values of s,

$$f(s)=\sum_{p=0}^{k} {}_sC_p\Delta^p f_0, \tag{2}$$

where

$${}_sC_p=s!/p!(s-p)!.$$

If $\Delta^p f_0$ is not exact, but has the rounding error ϵ_p, then from eq 2 the error in $f(s)$, due to using inexact differences, will be

$$\rho_s=\sum_{p=0}^{k} {}_sC_p\epsilon_p.$$

In order to insure an error no greater than 2×10^{-18} in $f(s)$, the leading differences were computed to a high enough accuracy to satisfy

$$|k_sC_p\epsilon_p|<2\times10^{-18},\ s=36,$$

and the computed differences were tested by generating the last required value (corresponding to $y=36$) from eq 2 and comparing the result with the value of the functions computed from eq 1. This phase of the work was done with the aid of desk calculators.

The leading differences were then key-punched, carefully tested, and used to generate all remaining values on an IBM tabulator, type 405. The last value obtained in the process agreed with precomputed values to within two units in the eighteenth decimal place, and this guaranteed the accuracy of all intermediate values, since each value was based on previously computed values. To eliminate a possible listing error, the values listed on the preliminary manuscript were differenced on an adding machine. Finally the values were summed in groups of nine or ten, and the sums recorded; this was done on the 405 IBM tabulator, from the punched cards. The galley proofs of the printed table were carefully proofread, and then summed in the same groups of nine or ten; from these sums the precomputed sums were subtracted. A zero total insured the agreement of the galley proofs with the tested preliminary manuscript.

The preparation of leading differences at $y=0$, and the checking of the manuscript were performed by Mrs. H. Arens and Misses F. Gordon, B. Harding, S. Marks, and R. Tishman. The computation of the functions on the IBM tabulator was performed by Everett Rea.

[2] See G. E. Forsythe, J. Research NBS **44**, 89 (1950) RP 2059; the polynomials $b_k(x)$ are termed $(-1)^{k-1}\sigma_{k-1}(x)$ in this paper.

[3] J. W. Glaisher, Numerical values of the first twelve powers of π, of their reciprocals, and of certain other related quantities, Proc. London Math. Soc. **VIII**, 140 (1877).

y	$b_1\left(\frac{\pi y}{36}\right)$	$b_2\left(\frac{\pi y}{36}\right)$	$b_3\left(\frac{\pi y}{36}\right)$
0	1.57079 63267 94896 62	1.64493 40668 48226 44	0.00000 00000 00000 00
1	1.52716 30954 95038 38	1.50976 00868 17874 50	−.13762 18091 82357 85
2	1.48352 98641 95180 14	1.37839 38245 34856 41	−.26361 36063 21136 16
3	1.43989 66328 95321 90	1.25083 52799 99172 19	−.37830 76774 74722 92
4	1.39626 34015 95463 66	1.12708 44532 10821 82	−.48203 63087 01506 12
5	1.35263 01702 95605 42	1.00714 13441 69805 31	−.57513 17860 59873 74
6	1.30899 69389 95747 18	0.89100 59528 76122 65	−.65792 63956 08213 78
7	1.26536 37076 95888 94	.77867 82793 29773 86	−.73075 24234 04914 21
8	1.22173 04763 96030 70	.67015 83235 30758 92	−.79394 21555 08363 02
9	1.17809 72450 96172 46	.56544 60854 79077.84	−.84782 78779 76948 21
10	1.13446 40137 96314 23	.46454 15651 74730 61	−.89274 18768 69057 75
11	1.09083 07824 96455 99	.36744 47626 17717 25	−.92901 64382 43079 64
12	1.04719 75511 96597 75	.27415 56778 08037 74	−.95698 38481 57401 86
13	1.00356 43198 96739 51	.18467 43107 45692 09	−.97697 63926 70412 39
14	0.95993 10885 96881 27	.09900 06614 30680 30	−.98932 63578 40499 23
15	.91629 78572 97023 03	+.01713 47298 63002 36	−.99436 60297 26050 37
16	.87266 46259 97164 79	−.06092 34839 57341 72	−.99242 76943 85453 78
17	.82903 13946 97306 55	−.13517 39800 30351 94	−.98384 36378 77097 45
18	.78539 81633 97448 31	−.20561 67583 56028 30	−.96894 61462 59369 38
19	.74176 49320 97590 07	−.27225 18189 34370 81	−.94806 75055 90657 55
20	.69813 17007 97731 83	−.33507 91617 65379 46	−.92154 00019 29349 94
21	.65449 84694 97873 59	−.39409 87868 49054 25	−.88969 59213 33834 54
22	.61086 52381 98015 35	−.44931 06941 85395 18	−.85286 75498 62499 34
23	.56723 20068 98157 11	−.50071 48837 74402 26	−.81138 71735 73732 33
24	.52359 87755 98298 87	−.54831 13556 16075 48	−.76558 70785 25921 49
25	.47996 55442 98440 63	−.59210 01097 10414 84	−.71579 95507 77454 80
26	.43633 23129 98582 39	−.63208 11460 57420 34	−.66235 68763 86720 27
27	.39269 90816 98724 16	− 66825 44646 57091 99	−.60559 13414 12105 86
28	.34906 58503 98865 92	−.70062 00655 09429 78	−.54583 52319 11999 58
29	.30543 26190 99007 68	−.72917 79486 14433 71	−.48342 08339 44789 40
30	.26179 93877 99149 44	−.75392 81139 72103 78	−.41868 04335 68863 31
31	.21816 61564 99291 20	−.77487 05615 82440 00	−.35194 63168 42609 30
32	.17453 29251 99432 96	−.79200 52914 45442 36	−.28355 07698 24415 36
33	.13089 96938 99574 72	−.80533 23035 61110 86	−.21382 60785 72669 48
34	.08726 64625 99716 48	−.81485 15979 29445 50	−.14310 45291 45759 63
35	.04363 32312 99858 24	−.82056 31745 50446 29	−.07171 84076 02073 81
36	.00000 00000 00000 00	−.82246 70334 24113 22	.00000 00000 00000 00

y	$b_4\left(\frac{\pi y}{36}\right)$	$b_5\left(\frac{\pi y}{36}\right)$	$b_6\left(\frac{\pi y}{36}\right)$	$b_7\left(\frac{\pi y}{36}\right)$
0	−1.08232 32337 11138 19	0.00000 00000 00000 00	1.01734 30619 84449 14	0.00000 00000 0000 00
1	−1.07623 25654 18722 86	.09427 20984 43369 98	1.01322 57895 62271 90	−.08866 01183 78126 93
2	−1.05864 20001 20314 86	.18750 35275 10128 93	1.00091 98346 01091 12	−.17660 29769 11157 54
3	−1.03055 19491 12506 09	.27873 45939 21055 58	0.98055 85220 35610 13	−.26311 91866 24851 45
4	−0.99293 38262 63001 78	.36709 16415 00405 36	.95235 51533 59016 81	−.34751 45375 76905 39
5	−.94673 00480 10620 39	.45178 45206 72866 91	.91659 56088 83335 46	−.42911 66537 35111 02
6	−.89285 40333 65293 69	.53210 40579 60518 64	.87363 11708 27827 89	−.50728 10119 16383 73
7	−.83219 02039 08066 70	.60741 95254 79785 38	.82387 15672 35443 63	−.58139 63440 57545 18
8	− 76559 39837 91097 75	.67717 61104 38394 83	.76777 82367 17319 46	−.65088 94420 88742 24
9	−.69389 17997 37658 42	.74089 23846 32334 25	.70585 78140 25327 98	−.71522 93846 80385 05
10	−.61788 10810 42133 58	.79815 77739 42806 92	.63865 58364 52675 55	−.77393 12051 34486 87
11	−.53833 02595 70021 37	.84863 00278 33188 80	.56675 06710 62549 30	− 82655 90196 91288 49
12	−.45597 87697 57933 22	.89203 26888 45985 03	.49074 76627 44813 36	−.87272 86355 22049 73
13	−.37153 70486 13593 83	.92815 25620 99786 55	.41127 35031 00754 39	−.91210 96576 78890 98
14	−.28568 65357 15841 16	.95683 71847 86226 63	.32897 08201 55876 12	−.94442 71142 72567 20
15	−.19907 96732 14626 49	.97799 22956 66937 46	.24449 29889 00743 28	−.96946 26191 49057 28
16	−.11233 99058 31014 34	.99157 93045 70506 72	.15849 91626 59874 63	−.98705 50913 35851 36
17	−.02606 16808 57182 53	.99761 27618 89434 14	+.07164 95252 88685 17	−.99710 10505 28818 88
18	+.05918 95518 43577 87	.99615 78280 77088 06	−.01539 92358 01522 42	−.99955 45078 90539 91
19	.14287 73398 36862 48	.98732 77431 44662 04	− 10199 82357 90517 10	−.99442 64714 30982 65
20	.22449 42281 17153 66	.97128 12961 58131 37	−.18751 04777 68049 65	−.98178 40852 41409 66
21	.30356 17591 07820 50	.94822 02947 35209 68	−.27131 47172 24713 67	−.96174 94218 52395 53
22	.37963 04726 61118 78	.91838 70345 42305 50	−.35280 91057 14757 32	−.93449 79469 86838 77
23	.45227 99060 58191 05	.88206 17687 91478 82	−.43141 46136 90845 86	−.90025 66759 78850 52
24	.52111 85940 09066 54	.83956 01777 37397 68	−.50657 82325 10775 09	−.85930 20411 29402 81
25	.58578 40686 52661 22	.79123 08381 74294 71	−.57777 59556 16135 46	−.81195 74892 69619 13
26	.64594 28595 56777 79	.73745 26929 32923 71	−.64451 55388 82927 11	−.75859 08288 02589 85
27	.70129 04937 18105 65	.67863 25203 77516 25	−.70633 90401 44125 56	−.69961 13454 94595 36
28	.75155 14955 62220 96	.61520 24039 02738 17	−.76282 51378 84198 38	−.63546 67062 86619 57
29	.79647 93869 43586 56	.54761 72014 30646 22	−.81359 12291 05572 51	−.56663 96703 97036 32
30	.83585 66871 45552 03	.47635 20149 07644 58	−.85829 53063 67052 47	−.49364 46269 86351 71
31	.86949 49128 80353 69	.40189 96598 01441 48	−.89663 76139 94189 35	−.41702 39786 54884 66
32	.89723 45782 89114 56	.32476 81345 98005 69	−.92836 20834 61600 57	−.33734 43900 44268 79
33	.91894 51949 41844 38	.24547 80902 98523 17	−.95325 75479 47240 51	−.25519 29208 13657 95
34	.93452 52718 37439 63	.16456 02999 16353 60	−.97115 87360 58621 85	−.17117 30622 61518 39
35	.94390 23154 03683 51	.08255 31279 73986 93	−.98194 70447 30987 80	−.08590 06968 63890 09
36	.94703 28294 97245 92	.00000 00000 00000 00	−.98555 10912 97435 10	.00000 00000 00000 00

y	$b_8\left(\frac{\pi y}{36}\right)$	$b_9\left(\frac{\pi y}{36}\right)$	$b_{10}\left(\frac{\pi y}{36}\right)$	$b_{11}\left(\frac{\pi y}{36}\right)$
0	−1.00407 73561 97944 34	0.00000 00000 00000 00	1.00099 45751 27818 09	0.00000 00000 00000 00
1	−1.00020 62153 51232 07	.08750 96417 14427 19	0.99717 37896 56244 93	−.08724 20849 15461 46
2	−0.98862 41183 39830 38	.17434 45534 55891 85	.98574 08937 74000 80	−.17381 81741 69565 97
3	−.96942 46955 17581 81	.25983 54624 82064 52	.96678 40307 41818 64	−.25906 74069 98368 28
4	−.94276 28715 46864 46	.34332 39509 48970 98	.94044 93141 55890 82	−.34233 91509 11500 30
5	−.90885 32457 09580 54	.42416 77383 99322 75	.90693 96491 58770 37	−.42299 80124 76811 84
6	−.86796 79193 58689 85	.50174 57983 79693 22	.86651 30995 52858 63	−.50042 87253 17202 57
7	−.82043 38251 65595 70	.57546 32633 37467 70	.81948 08173 26696 62	−.57404 08763 83469 48
8	−.76662 96111 36988 79	.64475 60766 44693 00	.76620 45548 99386 57	−.64327 34331 76355 18
9	−.70698 21306 93052 87	.70909 53551 45393 31	.70709 37837 80161 14	−.70759 90365 13834 10
10	−.64196 25884 17234 17	.76799 14300 25604 20	.64260 24463 41200 03	−.76652 80256 41369 42
11	−.57208 23894 06074 93	.82099 75380 61304 78	.57322 53700 25071 42	−.81961 21649 29885 51
12	−.49788 87384 65909 54	.86771 31394 08601 51	.49949 43755 66457 88	−.86644 80440 66116 99
13	−.41996 00337 21519 95	.90778 68420 62932 66	.42197 41127 13918 42	−.90668 01264 81524 85
14	−.33890 10975 20145 52	.94091 89169 29723 50	.34125 76585 04148 24	−.94000 34237 58934 79
15	−.25533 82858 32540 31	.96686 33911 17825 76	.25796 19143 81331 71	−.96616 57768 71396 93
16	−.16991 45156 71069 43	.98542 97105 89223 23	.17272 28393 80549 04	−.98496 97283 17548 04
17	−.08328 42483 63134 06	.99648 39666 73877 13	+.08619 05572 19600 66	−.99627 39724 95154 66
18	+.00389 15351 63486 90	.99994 96841 87220 09	−.00097 56245 19138 94	−.99999 43749 73823 70
19	.09095 03326 08492 42	.99580 81719 69687 69	−.08811 22448 73325 83	−.99610 45546 64497 92
20	.17723 30592 46927 69	.98409 84396 02799 10	−.17455 67316 08839 28	−.98463 60262 13837 68
21	.26208 88964 85582 69	.96491 66868 44666 08	−.25965 24173 59067 21	−.96567 79032 53575 48
22	.34488 00205 98140 12	.93841 53749 69419 97	−.34275 34928 90093 01	−.93937 61663 87180 84
23	.42498 61838 90993 39	.90480 18916 89901 78	−.42322 98614 98015 34	−.90593 25029 78573 66
24	.50180 91222 33357 64	.86433 68236 91058 31	−.50047 18596 67370 71	−.86560 27289 01174 44
25	.57477 67645 66997 90	.81733 18530 02830 49	−.57389 48104 40872 38	−.81869 48053 92398 03
26	.64334 72216 92600 79	.76414 72954 95905 76	−.64294 33774 57623 54	−.76556 64670 22019 97
27	.70701 25333 11517 61	.70518 93016 91536 99	−.70709 56892 96800 09	−.70662 24795 07023 72
28	.76530 21539 83308 55	.64090 67418 37703 84	−.76586 72056 03721 43	−.64231 15486 76045 08
29	.81778 61603 41219 56	.57178 77988 18210 97	−.81881 42984 62430 70	−.57312 29042 89954 89
30	.86407 81635 89424 93	.49835 62939 28878 37	−.86553 75246 00580 38	−.49958 25846 49169 25
31	.90383 79129 87570 90	.42116 77718 75785 23	−.90568 45662 55760 22	−.42224 94499 51771 44
32	.93677 35777 09856 73	.34080 53725 24577 65	−.93895 28208 84602 95	−.34171 09541 99414 02
33	.96264 36961 47592 04	.25787 55179 57143 85	−.96509 16223 44254 46	−.25857 87070 71300 69
34	.98125 87834 05870 50	.17300 34442 72497 43	−.98390 40787 07290 08	−.17348 38585 92503 96
35	.99248 25894 26702 14	.08682 86083 02490 12	−.99524 85144 73092 57	−.08707 23406 04748 99
36	.99623 30018 52647 90	.00000 00000 00000 00	−.99903 95075 98271 57	.00000 00000 00000 00

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