

Calculation of Stress in Electrodeposits from the Curvature of a Plated Strip

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Formulas are derived for the calculation of stress in electrodeposits from the curvature developed in a plated strip. All the formulas are derived from the fundamentals of the theory of elasticity and the relationships among the various formulas are shown. Simplified formulas are derived and their limits of applicability demonstrated. Correction factors for a number of variables and specific formulas for different experimental procedures are presented.

I. Introduction

The methods of determining the stress in an electro deposit by the curvature of a strip that is plated on only one side has been described in a preceding paper [1].¹

The calculation of stress in electrodeposits was first worked out by Stoney [2]. He derived two equations, but only the first one has been used by later investigators, apparently because the second one was written down without making clear either its derivation or meaning. His first equation,

$$S = \frac{Et^2}{6rd}$$

where S is stress in the coating, E is Young's modulus of elasticity of the basis metal, t is the thickness of the basis metal, d is the thickness of the coating, r is the radius of curvature of the bent strip, is really a differential expression in which the thickness of the coating is considered to be infinitesimal compared to the thickness of the basis metal. The equation is therefore approximate, but it can be used without much error in those cases in which the thickness of the coating does not amount to more than a few percent of the thickness of the basis metal. The error involved in using the Stoney equation for thin coatings is usually less than the experimental error, which is ordinarily about 5 to 10 percent.

¹ Figures in brackets indicate the literature references at the end of this paper.

There are occasions however, when it is desirable to measure the stress in thick deposits, which are to be used for electroforming, or to study the variation of stress with thickness. Such a situation could be met by plating on a basis metal of such thickness that the deposit would still contribute only a small proportion of the total thickness. It would nevertheless be an advantage to make measurements of stress, particularly of deposits of low stress, by plating a thick coating on a relatively thin strip of basis metal, because the sensitivity of the measurement is thereby increased. The error involved in using the Stoney equation begins to exceed the experimental error when the thickness of the coating is 5 percent or more of the thickness of the basis metal. For example, in Soderberg's [3] work, in which the thickness of the coatings amounted to as much as 25 percent of the thickness of the basis metal, the error involved in using the Stoney equation would have been more than 50 percent. This was recognized by Soderberg who then made a more satisfactory analysis of the calculations involved than had been made previously.

Although on the practical side, the Stoney equation is sufficiently accurate for most calculations of the stress in electrodeposited coatings, the theoretical side is not so satisfactory. There exists a lack of clarity in the definition of what is meant by the stress in the deposit, and a lack of recognition that the same formula cannot be used

for data obtained by slightly different methods of applying the simple bent strip technique.

Barklie's [4] derivation of the Stoney equation is very similar to that given by Stoney. According to his derivation, his formula yields the approximate stress in the coating on the curved beam. This stress will be referred to as the equilibrium stress. It is smaller than the true stress, because some of the force has been relieved by the bending of the coating. A minor situation that has caused confusion is that Stoney and Barklie both considered that the neutral axis of the bent strip was distant from the surface by about one-third the thickness of the strip, instead of at the center of the strip as would ordinarily be expected. Actually, this confusion does not affect their derivations. The neutral axis is usually defined as that longitudinal axis of a beam, which undergoes no additional strain (no change in length) when the beam is bent. According to this definition the neutral axis lies at the center of a simple beam. Barklie and Stoney considered the neutral axis to be that axis along which the stress was zero after bending. It will be shown later that this axis is not the same as the neutral axis as above defined. Stoney and Barklie did not consider the effect of the difference in Young's modulus of the coating and of the basis metal on the calculation of stress, apparently because for thin coatings this effect is negligible. Soderberg took account of these moduli in his derivation.

Because of the rather confused situation regarding the calculation of stress in electrodeposits from the data obtained by the deflection of a strip, it was considered worth while to give a more rigorous derivation of the involved formulas than has been done previously. An added advantage in having the exact formulas is that in a given case one can determine whether or not the Stoney formula is a sufficiently accurate approximation. The need for clarifying the situation was further shown by the recent appearance of another paper on stress, by Heussner, Balden, and Morse [5], in which another set of formulas, differing slightly from those given by Soderberg, was proposed.

It has not been recognized previously that slight variations in the procedure of measuring stress by the curved-strip method require different methods of calculation. There are three methods of measuring the stress of electrodeposits by the curvature of a strip.

Method I. The deposit is plated on a basis metal, which is so rigidly held that neither contraction nor bending of the plated strip can occur. The constraints are then released and the flat plated strip is allowed to assume its equilibrium curvature. This is the method used by Soderberg, who developed a suitable formula for the calculation.

Method II. The deposit is plated on a strip that is constrained from bending but not from undergoing contraction. The constraints are then released, and the flat plated strip is allowed to assume its equilibrium curvature. Experimentally, this method is more easily realized than method I, because bending can be more readily prevented than can the rather minute longitudinal changes.

Method III. The deposit is plated on a strip that is allowed to bend continuously during plating. This is the method that is most commonly used, but there has been no satisfactory discussion in the literature of the calculations involved for either this method or for method II.

In the following discussion, the formulas for the different methods of measurement will be derived. For thin coatings, all formulas reduce to Stoney's formula. In the summary, the error involved in using Stoney's formula for thick coatings will be considered in more detail, and examples will be given.

1. Symbols

- A_b = coefficient of thermal expansion of basis metal.
- A_c = coefficient of thermal expansion of coating.
- b = width of strip.
- C = diameter of helix.
- c = distance of neutral axis to outside surface of coating in a plated beam whose basis metal and coating have different moduli of elasticity.
- d = thickness of coating.
- D = angular deflection of torque rod in radians.
- D' = angular deflection of torque rod in degrees.
- \bar{D} = angular deflection of geared pointer in degrees.
- E = Young's modulus of elasticity.
- E_b = Young's modulus of elasticity of basis metal.
- E_c = Young's modulus of elasticity of coating.
- F = force.

G = gear ratio.
 h = height of plated (or active) portion of helix.
 I = moment of inertia.
 K = calibration constant of helix with deflection of geared pointer measured in degrees.
 K' = calibration constant of helix with deflection of torque rod measured in degrees.
 L = projected length of helix.
 L' = true length of helix.
 M = bending moment.
 M_b = bending moment of basis metal.
 M_c = bending moment of coating.
 p = pitch of helix.
 $R = E_c/E_b$ = ratio of modulus of coating to modulus of basis metal.
 r = radius of curvature.
 $\Delta(1/r)$ = change in curvature.
 S_b = stress in basis metal.
 S_{bm} = maximum stress in basis metal.
 S_c = stress in coating.
 S_{eq} = stress at equilibrium.
 S_s = stress as calculated by Stoney's formula.
 σ = generalized stress.
 S = true stress.
 ΔT = temperature change.
 t = thickness of basis metal.
 y = distance of fibers from neutral axis.

II. Derivation of Equations for the Stress in an Electrodeposit

In deriving equations for stress, only elementary considerations of beam theory are involved. This presentation is more detailed than would be required for presentation to a specialist in the theory of elasticity. Before beginning the derivations, it is necessary to clearly define what is meant by the stress in a coating. When a coating is plated upon a thin strip of metal that is restrained from bending, the strip is compressed by the tension in the coating, and the latter is thereby also shortened and loses some of its stress. If the constraints are now released and the strip is allowed to curve, the stress in the coating is still further relieved. These losses in stress depend on the dimensions of the basis metal, and hence the final equilibrium stress in the coating is not a constant quantity but depends upon the experimental conditions. To be independent of the mode of measurement, the stress in a coating is defined as the stress that exists in the coating when it is deposited upon a

rigid, incompressible surface, or for practical purposes, upon a basis metal thick enough to undergo no appreciable deformation. This will be referred to as the "true stress."

There are two basic conditions that must be satisfied by the internal, longitudinal fiber stresses of a beam in equilibrium.

$$F = \int \sigma dA = 0, \quad (1)$$

and

$$M = \int \sigma y dA = 0, \quad (2)$$

taken over any cross section of the beam. The first equation states that the sum, F , of the longitudinal forces within the beam is zero, i. e., that the internal compressive forces are equal to the internal tensile forces. The second equation states that at equilibrium the internal bending moment, M , of the beam is zero about any axis. The variable, y , is the distance of the fibers, of stress σ , from the chosen axis, and dA is the element of area of the cross section.

Before considering the application of these general equations to the curvature of a plated strip, it will be helpful to examine first the stresses that exist in a plated strip that has been allowed to assume its equilibrium curvature. In figure 1 is shown the system of stresses that exists in a strip that has been plated according to method I. The distribution of the longitudinal stresses is the same over any cross section, $ABCD$, of the beam. The stresses vary along the direction of the radius of curvature, that is from AB to DC . The stresses do not vary in the direction of the width, b , of the beam, that is from AD to BC , and therefore the system of stresses normal to the cross section can be represented by a two-dimensional graph, DL , as shown in the figure. The magnitude of the tensions in the coating are represented by the length of the vectors directed to the right and the magnitude of the compressive stresses by the vectors directed toward the left. It will be noted that the outer fibers of the basis metal, as well as the coating, are under tension, but that most of the basis metal is under a compressive stress. The neutral axis of the beam is at the midpoint and is the curved surface represented by the dotted line PFU . The method of constructing the stress diagram will be considered later.

The tensile force in the coating is the summation of the stresses, represented by $EKLD$ over the

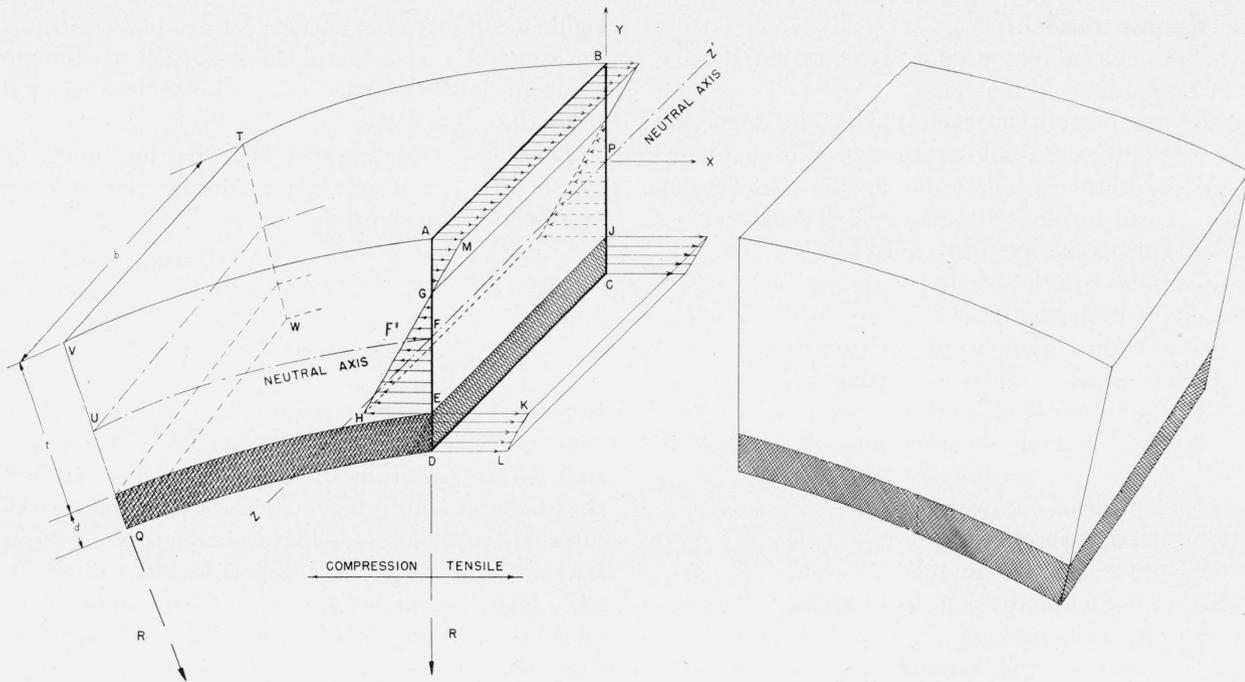


FIGURE 1. Stresses in a plated strip.

The cross-hatched area is the coating.

area of the cross section, $DEJC$, of the coating. The stress can be considered constant over the elemental strips of area, bdy . Tensile force in coating $= b \int_D^E \sigma dy = b(\text{area } EKLD)$. Similarly, the internal forces in the basis metal consists of a compressive force represented by $b(\text{area } EHG)$ and a tensile force represented by $b(\text{area } GAM)$. The first condition of equilibrium as applied to this beam is:

$$\text{Force} = b(\text{area } EKLD + \text{area } GAM) - b(\text{area } EHG) = 0. \quad (3)$$

Tensile forces - compressive forces = 0; or, area $EKLD + \text{area } GAM = \text{area } EHG$.

In figure 1, the internal bending moment of a fiber is represented by the product of a stress, and its distance from the neutral axis, for example, $KE \times EF$. The internal bending moment of the coating is the aggregate of the products of each fiber stress by its lever arm, summed over the area of the cross section of the coating $DEJC$.

$$\text{Bending moment of coating} = b \int_D^E \sigma y dy. \quad (4)$$

The integral represents the moment of the area $EKLD$ with respect to the neutral axis. When the stress, σ , is tensile (arrows to the right in fig. 1), it is considered positive. When the stress is compressive (arrows to the left), it is considered negative. y is positive or negative, depending upon whether the area under consideration is above or below the reference axis. In later summing up these moments, each must be given its proper sign.

$$\text{Bending moment of the coating} = b(\text{moment of area } EKLD). \quad (5)$$

The bending moment of the basis metal strip is represented similarly by the moments of areas EHG and GAM with respect to the neutral axis. The second condition of equilibrium can now be represented graphically as, $b(\text{moment of area } EKLD) + b(\text{moment of area } FEHF') + b(\text{moment of area } FF'G) + b(\text{moment of area } GAM) = 0$. Or by using the absolute values of the moments

$$\text{Moment of area } EKLD = \text{moment of area } GAM + \text{moment of area } FEHF' - \text{moment of area } FF'G. \quad (6)$$

The moments of the areas were considered with respect to the neutral axis for illustrative purposes. However, the sum of the moments, with proper regard to sign, is zero with respect to any axis, for example with respect to an edge, AB , of the beam. If the beam is not at its equilibrium curvature, the net bending moment is not zero and tends to make the beam assume the equilibrium curvature.

Qualitatively, the effect of the stresses in the beam is to give the cross section, $ABCD$, the appearance of having rotated around ZZ' as axis with respect to another cross section, $QVTW$.

The stress in a coating may be calculated from the equilibrium curvature of the beam by different methods. The two general conditions of equilibrium may be applied to the curved beam to determine the equilibrium stress, which is then corrected for the stresses that have been relieved by contraction and curvature. Stoney and Barklie applied this method partially, but this procedure is not very easy to use, because the distribution of stresses in the beam is rather complicated, particularly for methods II and III. A simpler approach is to consider that the plated beam reaches its final equilibrium in stages and to calculate for each stage the relation between the stress in the coating and the geometrical changes of shape that occur. This method will be illustrated graphically for method I by constructing the diagrams of the stresses that exist in the coating and basis metal strip at each stage. Only two-dimensional graphs are necessary to show the stresses, as they are to be interpreted in the sense of the plane graph $DLKHMA$ of figure 1.

1. Equations for Method I

The formulas for method I will now be discussed for the simple case in which the moduli of the basis metal and coating are the same.

In figure 2, A, is shown the stress, S , in the coating. There is no resultant stress in the basis metal, because it has been held rigid and has not been allowed to deform. The stress, S , is thus the true stress in the coating. In figure 2, B, the stresses are shown in the coating and basis metal after the constraints have been partially removed and the beam has been allowed to shorten, but not to curve. As the coating also contracts slightly, its stress is somewhat reduced. It can

be shown that the compressive stress in the basis metal is

$$S \left(\frac{d}{t+d} \right). \quad (7a)$$

The stress in the coating is diminished by this same amount to the new value

$$S \left(\frac{t}{t+d} \right). \quad (7b)$$

The first condition of equilibrium is illustrated here by the equality of the area $MNPR$ and the area $LMGH$, that is, by the equality of the tensile forces and the compressive forces.

The internal bending moment in the straight beam (see fig. 2, B), which tends to make it curve is equal to the sum of the bending moments of the basis metal and the coating. This is represented by the moments of areas $MNPR$ and $LMGH$ with respect to the neutral axis. Since the forces in the basis metal and coating are equal and opposite, the internal bending moment, M , is equivalent to a couple with the forces applied at the center of the basis metal and of the coating, that is with a distance of $(t+d)/2$ between the lines of application of the two forces, F_2 ,

$$M = F_2 \left(\frac{t+d}{2} \right)$$

$$F_2 = b \text{ (area } MNPR) = S \left(\frac{t}{t+d} \right) db$$

or

$$F_2 = b \text{ (area } LMGH) = S \left(\frac{d}{t+d} \right) tb.$$

Therefore,

$$M = \frac{Stdb}{2}. \quad (7c)$$

This same result could have been obtained directly from the stresses in the beam as shown in figure 2, A. The initial tensile force of the coating is equal to the stress times the area of the cross section or, $F_1 = Sdb$. The force may be considered to act at the midpoint of the coating at a distance of $t/2$ from the neutral axis. The product of the force and the distance gives the same result as before, $Stdb/2$.

When the straight beam is now allowed to bend, a new distribution of the internal stresses occur.

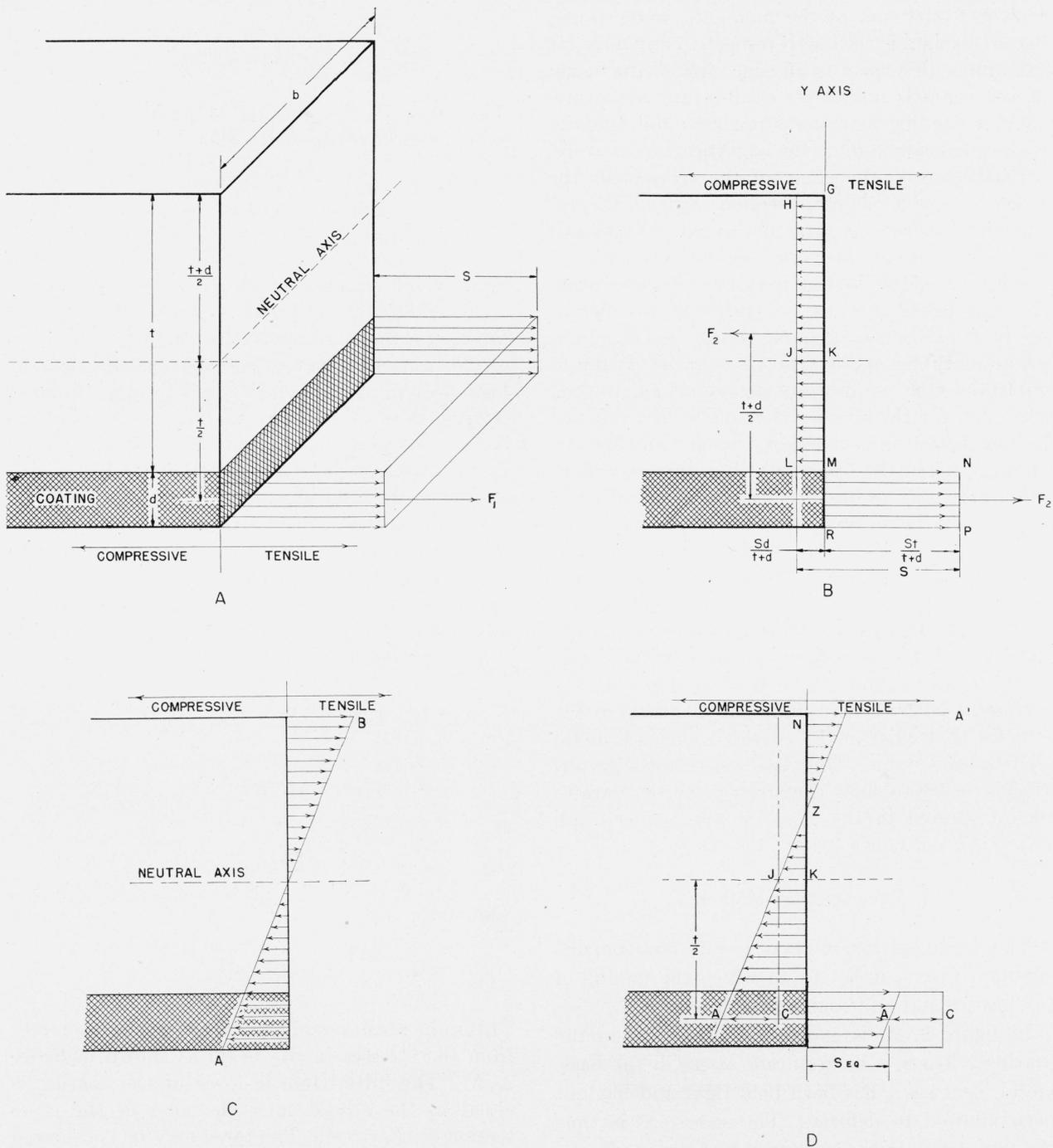


FIGURE 2. *Stresses in a strip held rigid during plating and then released.*

A, Stresses in rigidly held plated strip; B, stresses after contraction only had occurred; C, system of forces superimposed on forces of B as a result of curvature; D, stresses in the free strip at equilibrium; combination of stresses in B and C.

Regardless of what system of stresses may have existed previously in a beam, an additional system of stresses shown in figure 2, C, is produced and superimposed on the initial stresses. Bending of the beam continues until the bending moment of the stresses shown in figure 2, C, is equal and opposite to the bending moment calculated for the stresses in figure 2, B. The algebraic addition of this new system of fiber stresses to those shown in figure 2, B, gives the final system of stresses shown in figure 2, D. It can be shown graphically that both conditions of equilibrium are satisfied by stresses in figure 2, D. In figure 2, C, the tensile force is equal to the compressive force, and thus if the first condition of equilibrium were satisfied by figure 2, B, it will still be satisfied on adding the stresses in 2, B, to those of figure 2, C. The bending moment of the stresses in figure 2, C, is equal and opposite to the bending moment of the stresses calculated for figure 2, B, and thus the sum of the two system of stresses gives a resultant bending moment of zero for figure 2, D.

The equality between the initial bending moment of the flat plated strip, as typified in figure 2, A, or 2, B, and the bending moment of the new stresses produced by curvature typified in figure 2, C, forms the basis for the calculation of the stress in the coating. The bending moment of the stresses of figure 2, C, depends upon the curvature and the dimensions of the beam, and is readily calculated from measurements made directly on the beam.

$$M = \frac{EI}{r} = \frac{E(t+d)^3 b}{12r}, \quad (8)$$

where r is the radius of curvature of the neutral axis.

Here E is taken to be the modulus of both the coating and the basis metal. I is the moment of inertia of the cross section, $ABCD$ (fig. 1), of the beam with respect to the trace, FP , of the neutral axis in the plane of the cross section. In the previous discussion it was shown that the initial bending moment of the beam (fig. 2, B) was numerically equal to the bending moment impressed by curvature (fig. 2, C). Thus, the bending moment in eq 7 can be equated to the bending moment in eq 8:

$$\left. \begin{aligned} M &= \frac{Stdb}{2} = \frac{E(t+d)^3 b}{12r} \\ S &= \frac{E(t+d)^3}{6tdr} \end{aligned} \right\} \quad (9)$$

This is the equation derived by Soderberg.

Some other useful relationships may be made clear by use of the diagrams in figure 2. The stresses in figure 2, C, are proportional to their distance, y , from the neutral axis, and are given by the expression,

$$\sigma = \frac{Ey}{r}, \quad (10)$$

which is the basic relation of simple beam theory. The slope of the stress line, AB , with respect to the y axis is

$$\frac{E}{r}. \quad (11)$$

As the beam bends, the line AB can be visualized as rotating further from the y axis. It will be noted that the compressive stress along the neutral axis, denoted by JK , is the same before bending (fig. 2, B) as after bending (fig. 2, D). This is because there is no strain (or change of length) at the neutral axis as a result of the bending. Certain other features of the stresses in the beam have already been pointed out in the discussion of figure 1.

Figure 2, D, shows that the point of zero stress in the beam occurs not at the neutral axis, but at a point, Z , which is distant from the surface of the basis metal by about one-third the thickness of the basis metal when the coating is thin. The point of zero stress can be determined readily with the aid of the diagram.

By eq 11

$$\frac{JK}{KZ} = \frac{E}{r}. \quad (12)$$

JK is equal to the initial compressive stress in the basis metal (fig. 2, B).

$$JK = S \left(\frac{d}{t+d} \right) = \frac{E(t+d)^3}{6rtd} \left(\frac{d}{t+d} \right) = \frac{E(t+d)^2}{6rt}, \quad (13)$$

by substituting for S , using eq 9.

Substituting for JK , as given by eq 13, into eq 12,

$$KZ = \frac{(t+d)^2}{6t} \quad (14)$$

$$NZ = KN - KZ = \frac{t+d}{2} - \frac{(t+d)^2}{6t} = \frac{t}{3} + \frac{d}{6} - \frac{d^2}{6t} \quad (15)$$

When d is small, NZ is approximately equal to $t/3$. Z is the point that Stoney and Barklie called the neutral axis.

It is of interest to calculate the average equilibrium stress in the coating, that is, the stress at the midpoint of the coating after bending has taken place. The equilibrium stress, S_{eq} , illustrated by figure 2, D, differs from the stress in figure 2 B, by the stress AC, which results from curvature.

$$\left. \begin{aligned} S_{eq} &= S \left(\frac{t}{t+d} \right) - AC = \frac{E(t+d)^3}{6rt d} \left(\frac{t}{t+d} \right) - \frac{Et}{2r} \\ S_{eq} &= \frac{E(t^2 - td + d^2)}{6rd} \end{aligned} \right\} \quad (16)$$

The relative difference between the equilibrium stress and the true stress (relative difference = $S_{eq} - S / S$) can be shown to be $-(4d/t) + 10(d/t)^2$, which is fairly large. If d/t is about 5 percent, which is the value in some experiments, the equilibrium stress is about 18 percent smaller than the true stress.

In the measurement of stress it is necessary that the elastic limit of the basis metal be not exceeded. From figure 2, D, it is apparent that the maximum stress in the basis is a compressive stress occurring at the interface with the coating. For method I, this stress can be shown to be given by the formula

$$S_{bm} = \frac{Sd(4t^2 - td + d^2)}{(t+d)^3} \quad (17)$$

If the coating is thinner than the basis metal, this stress is always less than the stress in the coating. By differentiation of eq 17, it can be shown to attain a maximum value of $5S/9$ when the thickness of the coating is one-half the thickness of the basis metal. This simple relation serves as a convenient means of determining whether the elastic limit of the basis metal is likely to be exceeded.

2. Equation for Method II

The stress in the coating will now be calculated for method II, in which it is assumed that the basis metal is prevented from bending but not from contracting during plating, and is allowed to bend subsequently. The stress distribution in

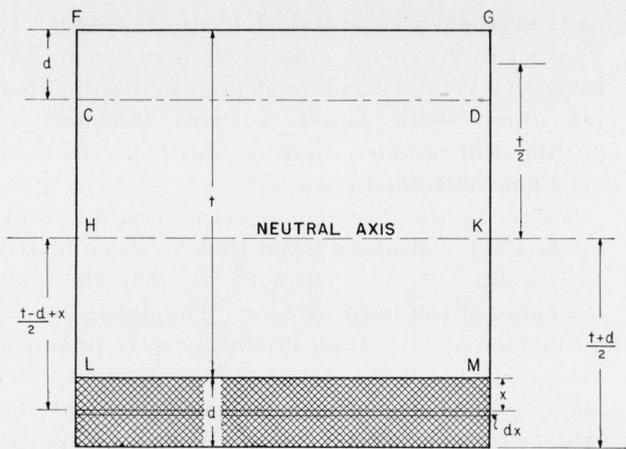
this beam differs from that in method I because, although bending is prevented, contraction of the basis metal can take place progressively as the coating becomes thicker. The calculation is complicated by the situation that the earlier layers of coating are compressed by the tension in the succeeding layers in the same way that the basis metal is compressed. Before bending occurs there thus exists a stress distribution in the coating, as shown in figure 3, B, with the greatest tension in the layer of coating last deposited. It is of interest to make this calculation for method II because it approximates practice more closely than the conditions laid down for method I, inasmuch as it is difficult to prevent the contraction of a thin strip during plating, even if it is bolted down. If it turns out that the calculated value for method II differs from that for method I by less than the experimental error, then the simpler calculation can be used.

Stress S_b in the basis metal produced as a result of the successive additions of coating to a total thickness of d will be calculated first. It has already been noted (eq 7b) that the compressive stress produced in the basis metal of thickness t , by the addition of a coating of thickness d with a tensile stress S , is $S_b = S(d)/(t+d)$. If the coating is added in increments dx , the equation becomes $dS_b = S(dx)/(t+x)$, where x is the thickness of the coating that has been previously laid down. The total compressive stress in the basis metal is

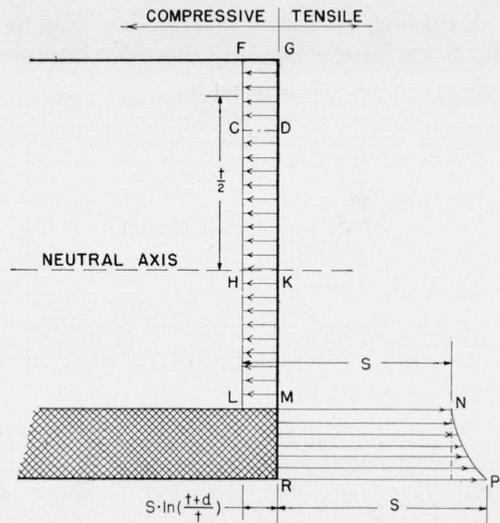
$$S_b = S \int_0^d \frac{dx}{t+x} = S \ln \left(\frac{t+d}{t} \right) = S \ln \left(1 + \frac{d}{t} \right) \quad (18)$$

It will be noted that when d is small, this equation approaches that for method I (eq 7b), since $S \ln (t+d)/t = S(d)/(t+d) + 1/2(d)/(t+d)^2 + \dots$, and the second and higher term may be neglected for thin coatings. An expression for the tension in the coating as a function of the distance, x , from the surface of the basis metal is now required for the purpose of calculating the bending moment of the beam. Qualitatively, it may be observed (fig. 3, B) that the coating has the minimum stress at the interface with the basis metal and that the stress at the outer surface of the coating has the initial value, S .

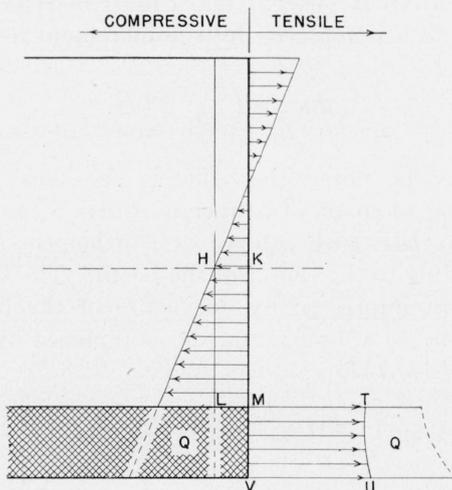
Consider now the stress in an increment of coating dx , at the distance x from the basis metal,



A



B



C

FIGURE 3. Stresses in a strip allowed to contract during plating, but not bend, and then allowed to curve after the plating was completed.

A, Diagram of dimensions involved in computation; B, stresses in strip before bending took place; C, stresses in strip after it curved.

The stress at any point x , in the coating is

$$\left. \begin{aligned} S_c &= S - S \ln \left(\frac{t+d}{t+x} \right) \\ S_c &= S \left[1 + \ln \left(\frac{t+x}{t+d} \right) \right] \end{aligned} \right\} \quad (19b)$$

The moment of the coating with respect to the neutral axis is (see fig. 3, A)

$$M_c = bS \int_0^d \left[1 + \ln \left(\frac{t+x}{t+d} \right) \right] \left(\frac{t-d}{2} + x \right) dx, \quad (20)$$

since the variable lever arm of the fiber stresses is $(t-d)/(2) + x$.

Integration by parts gives the following expression

$$M_c = \frac{Sbtd}{2} \left(1 - \ln \left(\frac{t+d}{t} \right) + \frac{d}{2t} \right). \quad (21)$$

The amount of the basis metal about the neutral axis is

$$M_b = \frac{Sbtd}{2} \ln \left(\frac{t+d}{t} \right), \quad (22)$$

because only the area $FCDG$ (which equals $d \times S \ln(t+d)/t$) contributes to the moment. The moments of areas $HLMK$ and $CHKD$ are equal and opposite, and hence cancel.

before any subsequent metal is deposited. The stress in this last increment is S , because the increment does not produce any finite contraction of the base metal. However, its stress is decreased by the effect of the layers of thickness dx , subsequently deposited upon it. The decrease in stress undergone by this increment of coating will be the same as the increase of stress undergone by the basis metal, namely

$$S \ln \left(\frac{t+d}{t} \right) - S \ln \left(\frac{t+x}{t} \right) = S \ln \left(\frac{t+d}{t+x} \right). \quad (19a)$$

Equating the initial internal bending moment of the beam to the bending moment imposed by the curvature, as given in eq 8

$$\left. \begin{aligned} M_c + M_b &= \frac{E(t+d)^3 b}{12r} \\ \frac{S b t d}{2} \left(1 + \frac{d}{2t}\right) &= \frac{E(t+d)^3 b}{12r} \\ S &= \frac{E(t+d)^3}{6r t d \left(1 + \frac{d}{2t}\right)} = \frac{E(t+d)^3}{3r d (2t+d)} \end{aligned} \right\} \quad (23)$$

On comparing this formula for stress with the one for method I, it will be seen that the relative difference between the two values of S is $1/2 \times d/t$. Since d/t usually is not over 10 percent, the difference between the two methods of calculation of S would be about 5 percent, or within the experimental error. This difference cannot be determined experimentally, because the frequently observed variation of stress with thickness detracts too much from the reproducibility.

The stresses in the coating before and after bending are shown in figure 3, B, and C. It will be noted that the stress in the outer fibers of the coating are higher than in the inner fibers even after bending, in contrast to the coating shown in figure 2, D, for method I. The equilibrium stresses in the coating, represented by area $MTUV$ in figure 3, C, are obtained by subtracting area Q from area $MNPR$ in figure 3, B.

3. Equations for Method III

Method III deals with the bending of the strip as plating proceeds and is the most common method of measurement employed for determining stress in coatings. Before deriving the expression for method III it will be advantageous to consider, qualitatively, the mechanical differences between method I, which is the simplest to visualize, and method III. In method I the mean length of the lever arm of the fiber stresses in the coating is $t/2$. In method III, the length of the lever arm varies from $t/2$ for the first increment of coating laid down to $(t+d)/2$ for the final increment. The longer average lever arm causes the bending moment to be greater in method III. In method I the curvature of the beam is determined by the final thickness of the beam, $t+d$. In method III, the thickness of the

beam varies from t to $t+d$ as the plating proceeds, and on the average is less than in method I. This smaller average cross section of beam also leads to a greater bending of the beam in method III. From both causes, a given thickness of coating will produce more curvature by method III than by method I.

To derive the formula for calculating stress by method III, consider a beam that has been plated with a thickness of coating x , so that the total thickness of the beam is $t+x$. The deposition of an increment of coating of thickness dx produces an increment of internal bending moment, dm , about the neutral axis, which is situated at the center of the beam at a distance, $(t+x)/2$, from the strip dx . The strip increases in curvature from radius r_1 to radius r_2 by the amount $(1/r_2 - 1/r_1 = d(1/r))$. The internal bending moment of the increment dx is represented by a couple that is given by eq 7C, except that t must be replaced by $t+x$ and d is replaced by the increment dx

$$dm = S \left(\frac{t+x}{2} \right) b dx. \quad (24)$$

It may be noted that this is the same as the bending moment of a fiber of stress S , and cross section, $b dx$, with a lever arm of length $(t+x)/2$, extending to the center of the beam. The bending moment impressed by curvature of the beam is given by eq 8, except that $1/r$ is replaced by $d(1/r)$ and M by dM .

$$dM = \frac{E b (t+x)^3}{12} d \left(\frac{1}{r} \right). \quad (25)$$

Equating eq 24 to eq 25

$$S \left(\frac{t+x}{2} \right) b dx = \frac{E b (t+x)^3}{12} d \left(\frac{1}{r} \right). \quad (26)$$

On separating variables and integrating:

$$\left. \begin{aligned} S \int_0^d \frac{dx}{(t+x)^2} &= \frac{E}{6} \int_0^{1/2} d \left(\frac{1}{r} \right)^{1/2} = \frac{E}{6r} \\ S &= \frac{E t (t+d)}{6r d} \end{aligned} \right\} \quad (27)$$

This formula is the simplest of those for the three methods and, it is closer to the simple Stoney formula than the formulas of methods I and II. The relative difference from Stoney's approximation is d/t , which was usually less than 5 percent

for the experiments done with the spiral contractometer [1]. In reviewing the literature, it was found that eq 27 was first written down by Stoney, in the form

$$S = \frac{4E(t^2 + td)z}{d^2} \quad (28)$$

which is the same as eq 27 with r replaced by $L^2/8Z$. This latter expression is the formula that Stoney used for calculating r ; L is the length of the plated strip, and Z is the camber of the curved strip (see eq 50A). This formula will be discussed in more detail later. Equation 28 seems to have been entirely ignored by later investigators, probably because Stoney did not give its derivation or indicate its application. It was also passed over by the authors until the above derivation had been arrived at and the similarity of eq 27 with eq 28 was discovered. The diagram for the stresses involved in method III is shown in figure 4. The distinguishing feature is that the stress in the outer fiber of the coating is considerably higher than in the inner fibers. The stress is equal to S , the initial stress, since the last increment of coating does not undergo bending or contraction.

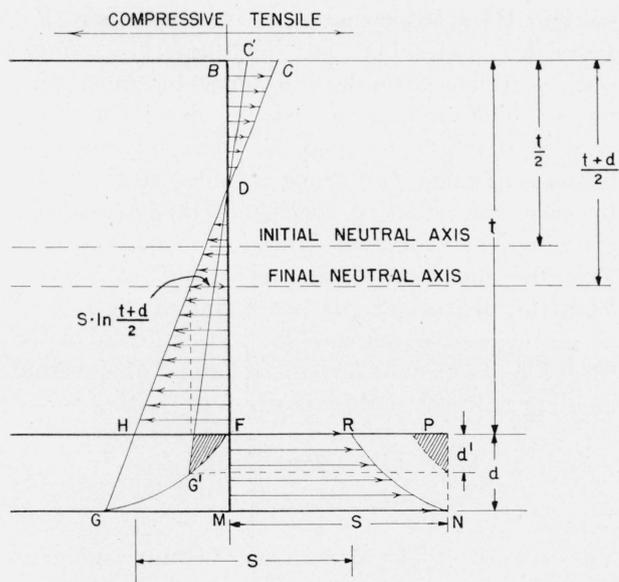


FIGURE 4. Stress in a plated strip allowed to continuously curve during plating.

The stresses in the coating are not as readily calculated as in methods I and II, because each increment of coating is deposited in a curved condition, and the stress relief resulting from the

subsequent curvature is not given by the simple relation, Ey/r (eq 10). In addition, the compression of the first increments of coating by the tension in the latter increments produces a stress relief, given by eq 19A. The stress in the coating at a distance x from the interface with the basis metal is given by the equation

$$Sx = S \left[1 + \ln \frac{t+x}{t+d} - \frac{3(t-d+2x)(d-x)}{(t+d)(t+x)} \right]. \quad (29)$$

In figure 4, area GHF represents the compressive force that is generated in the coating as a result of the bending of the beam and the tension of the outer layers of the coating. This area is subtracted from the initial tensile force of the coating (area $FMNP$) to give the final stresses $FMNR$ in the coating. Area GHF may be considered to have been swept out by the end of the line GDC when it starts from its initial position FDB and goes through the intermediate position $C'DG'$, as the coating increases in thickness.

III. Calculation of Stress When the Young's Moduli of the Coating and Basis Metal are Different

The treatment up to this point has dealt with measurements in which the moduli of the basis metal and coating were the same. The moduli of the common metals range from 2.4×10^6 lb/in.² for lead to 30×10^6 lb/in.² for steel. However, by suitable choice of basis metal, the modulus of the coating need not differ from that of the basis metal by more than a factor of 2. The effect of differences in moduli is negligible when the thickness of the coating amounts to only a few percent of the thickness of the basis metal, but may be significant with thicker coatings.

When the basis metal and the coating have different moduli, the neutral axis no longer passes through the center of mass of the cross section, at $(t+d)/2$, and the moment of inertia of the cross section does not enter simply into eq 8. For method I, Soderberg has derived an expression for the stress of a coating plated on a basis metal of different modulus, but it is too lengthy to be convenient for calculation. In the following derivation, which is similar to that given previously for method I, a simpler expression than that given by Soderberg will be obtained. Also, a simple

empirical formula can be given that will simplify calculations still further.

In the following discussion the ratio of the modulus of the coating to that of the basis metal will be denoted by R ($R = E_c/E_b$). When the moduli of the coating and basis metal differ, the initial stresses (fig. 2,C) in the coating and basis metal are given by the expressions $S(t)/(t+Rd)$ and $S(d)/(t+Rd)$, instead of by eq 7a and 7b. The bending moment of the coating is equivalent to a couple and is calculated in the same manner as was done for eq 7c, except that the equal and opposite forces in the coating and basis metal are now given by the expression $F = S(t)/(t+Rd)db$ or $F = S(d)/(t+Rd)tb$. The distance between the points of application of the forces is $(t+d)/2$, as before. The initial bending moment of the coating is

$$M = \left(\frac{Std}{t+Rd} \right) \left(\frac{t+d}{2} \right). \quad (30)$$

The expression for the opposing bending moment impressed by curvature is similar to that in eq 8, except that the moment of inertia, I , of the cross section must be replaced by a more complicated expression. For purposes of calculation, the beam consisting of materials of two different moduli is replaced by an equivalent T-beam, in which the widths of the basis metal and the coating are proportional to their moduli. Thus, the basis metal strip is considered to have a width b , and the coating a width equal to Rb . The neutral axis of this equivalent beam (and also the neutral axis of the original beam) is located in the basis metal at a distance, c , from the outside surface of the coating. This distance is given by the expression

$$c = \frac{t^2 + 2td + Rd^2}{2(t+Rd)}. \quad (31)$$

The moment of inertia of the cross section of the T-beam with respect to the neutral axis is given by the expression

$$I = \frac{b[R(t+d)^4 - t^4 - Rd^4](R-1)}{12(t+Rd)}. \quad (32)$$

To simplify later calculations, the Rd^4 quantity in the $(t^4 - Rd^4)$ term in eq 32 will be considered negligible and dropped. That this may be done is shown by the fact that in the extreme case when the thickness of the coating is one-half that of the

basis metal, and the ratio of their moduli is 2, the error thus introduced is less than 1 percent. On substituting I from eq 32 into eq 8, replacing E with E_b , and setting this equal to the right-hand side of eq 30, an expression for S is obtained

$$S = \frac{E_c(t+d)^3}{6rdt} - \frac{(E_c - E_b)t^3}{6rd(t+d)}, \quad (33)$$

or

$$S = \frac{E_b[R(t+d)^4 - (R-1)t^4]}{6rdt(t+d)}.$$

It will be noticed that the first member of the formula is of the same form as eq 9. The second member of the formula is of a similar form except that t and $(t+d)$ are interchanged, and the expression is multiplied by the difference $E_c - E_b$, of the moduli. The following semi-empirical formula was found to agree within a few percent with eq 33.

$$S = \frac{E_b(t+Rd)^3}{6rdt}, \quad (34)$$

and is much simpler to use.

The derivation of the expression for the stress in an electrodeposited coating, measured by method III is somewhat more involved than that given for method I, but fortunately a simple semi-empirical formula will serve for most purposes. The derivation follows along the same lines as that given for eq 27. The increment of bending moment dm , which is added to the beam by each increment of coating of thickness dx , is given by an expression similar to eq 24 except that the distance of the increment of coating from the neutral axis is not equal to $(t+x)/2$ as in the case of equal moduli, but is given by eq 31. The equation for the change of internal bending moment with deposition of coating is

$$dm = S \left[\frac{t^2 + 2tx + Rx^2}{2(t+Rx)} \right] bdx. \quad (35)$$

The increment of the opposed bending moment that is developed by the bending of the beam is given by an expression similar to eq 8, except that M and $1/r$ are replaced by the corresponding differentials and I is given by eq 32.

$$dm = \frac{E_b b [R(t+x)^4 - t^4(R-1)]}{12(t+Rx)} d \left(\frac{1}{r} \right). \quad (36)$$

Equations 35 and 36 yield the following equation for the stress

$$S \int_0^d \frac{t^2 + 2tx + Rx^2}{R(t+x)^4 - (R-1)t^4} dx = \frac{E}{6r} \quad (37)$$

By making the change of variable to $z = (t+x)/t$, breaking the fraction into partial fractions and integrating, the following expression is obtained for the stress

$$S \left[\left(\frac{1+H^2}{2H} \right) \tanh^{-1} \left(\frac{H-Q}{1-HQ} \right) + \left(\frac{1-H^2}{2H} \right) \tan^{-1} \left(\frac{H-Q}{1+HQ} \right) - H^2 \tanh^{-1} \left(\frac{H^2-Q^2}{1-H^2Q^2} \right) \right] = \frac{E_b t}{6r}, \quad (38)$$

where $H^4 = (E_c - E_b)/E_c$, and $Q = Ht/(t+d)$.

It is of interest to note that eq 38 can be considerably simplified. When the absolute value of x is less than 1, the $\tan^{-1} x$ and $\tanh^{-1} x$ may be represented by a series of the form, $x + ax^3 + bx^5 \dots$. The first term approximation is accurate enough for present purposes. On replacing the arc tangents by their arguments and simplifying, the following equation is obtained

$$S = \frac{E_b t(t+d)}{6rd} - \frac{(E_c - E_b)t^3}{6rd(t+d)} \quad (39)$$

This equation parallels eq 33, because the first term is the same as the expression for materials of the same modulus (eq 27), and the second term is the same as that of eq 33.

Over a range of thickness of coating of $d=0$ to $t/2$, good agreement with eq 38 is obtained by the simple semiempirical formula

$$S = \frac{E_b t(t + R^{5/4}d)}{6rd} \quad (40)$$

However, over a more limited range the formula

$$S = \frac{E_b t(t + Rd)}{6rd}, \quad (41)$$

is quite satisfactory.

IV. Calculation of the Effect of Temperature Changes on the Measurement of Stress

The plated strip of metal constitutes a bimetallic element, such as is used in thermoregulators, and

will curve as a result of temperature changes if the two metals differ in coefficient of expansion, which is usually the case. The measurements made with the spiral contractometer are made at the temperature of the plating solution, and the only interest in the effect of temperature changes is to determine the precision of temperature regulation required to keep the error from this cause below 1 percent. After the contractometer is removed from a warm plating solution, an appreciable movement of the pointer occurs, thus showing that the effect of the temperature is not negligible. The effect of temperature change may be serious if the curvature of the plated strip is measured after it is removed from a warm plating solution, as is usually done in applying methods I and II.

In the following discussion of thermal effects, a different approach is employed than that used by Soderberg, and the equations developed are not the same. After a strip has come to an equilibrium curvature under the tension of the coating, the effect of a subsequent temperature change is to alter the curvature. In order to calculate the stress in the deposit, it is first necessary to correct the final observed curvature, $1/r_f$, for the increment of curvature, $\Delta(1/r)$, resulting from the temperature change. This is readily done as follows: The equations for stress, as developed for the three different methods may be written in the form, $S = Q(1/r)$, where $1/r$ is the curvature resulting from the stress in the coating alone. The equation is corrected by subtracting the curvature superimposed upon it by thermal stresses and takes the form

$$S = Q \left[\frac{1}{r_f} - \Delta \left(\frac{1}{r} \right) \right] = Q \left(\frac{1}{r} \right) \quad (42)$$

The value of $\Delta(1/r)$ is calculated in the manner used for method I, from the known thermal stresses set up in the beam. The additional internal bending moment, produced in the curved beam by the thermal stresses alone, is equated to the opposing bending moment impressed by the corresponding increment of curvature. Since the longitudinal forces, F , set up by the thermal effect are equal and opposite in the coating and basis metal, their bending moment is equivalent to a couple with the distance, $(t+d)/2$, between their points of application. Equating the bending

moment of the couple to the bending moment impressed by curvature gives

$$F \frac{(t+d)}{(2)} = EI \Delta \frac{(1)}{(r)} = \frac{E(t+d)^3 b}{12} \Delta \frac{(1)}{(r)}. \quad (43)$$

$$\Delta \frac{(1)}{(r)} = \frac{6F}{Eb(t+d)^2}. \quad (44)$$

It is now necessary to have an expression for the force, F , in the coating or base metal. It should be remembered that F is the force which existed before the strip was allowed to undergo the additional increment of curvature, $\Delta(1/r)$ (just as in method I, where the bending moments were calculated from the forces in the uncurved beam). A condition of equilibrium for the bimetallic strip (assuming that it is held at constant curvature) is that the change in length, or the strain of each metal strip must be equal.

$$A_c \Delta T + \frac{F}{dbE} = A_b \Delta T - \frac{F}{tbE}. \quad (45)$$

A_c and A_b are the coefficients of expansion, and the temperature change is ΔT . The strains produced by thermal expansion are $A \Delta T$. The other two terms represent the strains produced by the force F .

From eq 45, $F = [Etdb(A_b - A_c)\Delta T]/(t+d)$, and substituting in eq 44,

$$\Delta \left(\frac{1}{r} \right) = \frac{6td(A_b - A_c)\Delta T}{(t+d)^3}. \quad (46)$$

Applying this correction for $\Delta 1/r$ to eq 42, the following equations for stress are developed for methods I and III, respectively:

$$S = \frac{E(t+d)^3}{6tdr} - E\Delta T(A_b - A_c). \quad (47)$$

$$S = \frac{Et(t+d)}{6dr} - E\Delta T(A_b - A_c) \left(\frac{t}{t+d} \right)^2. \quad (48)$$

It will be noted that the formulae for stress are the same as the original formula except for the subtraction of a term involving the difference $A_b - A_c$, of the coefficients of expansion. The correction term in eq 47 agrees with the expression given by Heussner [5] but not with the one given by Soderberg [3]. For metals that differ in coefficient of thermal expansion by 5×10^{-6} , for example copper and steel, the correction

amounts to about $150 \text{ lb/in.}^2/^\circ \text{C}$. When measuring low stresses, such as a few thousand pounds per square inch, it would be important to control the temperature of plating within 1 or 2 deg, but a control of 5 deg C would probably be sufficient when measuring stresses above $20,000 \text{ lb/in.}^2$.

If the moduli of the coating and the basis metal are different, the appropriate correction term is subtracted from the expressions previously given for the stress in eq 33, 34, or 38 to 41. The correction for method I can be shown to have the same form as that given in eq 47, namely $E_c \Delta T(A_b - A_c)$ (except that E is replaced by E_c). The correction term for method III can be evaluated by the method given, but for practical purposes the same correction term as that given in eq 48 (except that E is replaced by E_c) may be used.

V. Stress Remaining in Coating After Removal of Basis Metal

Some observers have attempted to note the trend of stress with thickness by observing the direction of bending of the deposit after the basis metal was dissolved. However, in order to draw any conclusions, it is first necessary to know, as a basis of reference, the direction of bending when the stress is uniform throughout the deposit. The three diagrams (fig. 2, D, 3, C, and 4), show the stress existing in the coating after bending has taken place. In method I it is obvious that the coating would straighten out if the basis metal were removed, since the coating was uniformly stressed throughout when it was in its initial straight position. When the basis metal is removed, the coating will simply contract slightly and return to its original straight condition. In method II, since the outer fibers of the coating are under slightly higher stress than in method I, the coating should not quite straighten out, but should remain slightly curved. This slight curvature would be difficult to demonstrate in practice, since the tendency for the stress in many coatings to decrease with thickness usually more than compensates for the excess outer stress. The stresses in the coating in method III differ from the two preceding cases in that the outer fibers of the coating have a considerably higher stress than the inner fibers. This indicates that the curved coating should curve still further after the basis metal

is removed. The percentage difference in stress between the inner and outer fibers is approximately equal to $3(d/t)$, which amounts to about 15 percent for $d/t=0.05$, and hence the additional curvature should be detectable if the coatings are moderately thick. By comparison, the percentage difference in stress between the inner and outer fibers in method I is $6(d/t)^2$, which is much smaller, but yet sufficient to bend the coating back nearly straight.

The expected increase in curvature was not observed when the basis metal (copper) was dissolved from a deposit obtained from a Watts nickel bath, probably because the stress in the outer layers decreased as the coating became thicker. However, the curvature of a deposit from a chloride nickel bath increased considerably when the basis metal was dissolved.

It has seemed paradoxical to some platers that a sheet of plated metal, which has been deposited on a rigid backing, does not curl when it is stripped from the basis metal, whereas a thin plated sheet will develop considerable curvature. The explanation of this paradox is that curving will take place only when there are differences of stress between the two faces of a strip. If the stress in an electro-deposited coating is uniform, it is relieved only by contraction of the coating when it is stripped from the basis metal.

VI. Comparison of Formulas

It is interesting to compare the Stoney formula with the other formulas that have been derived for computing stress measured by the three differ-

ent methods. This comparison will be made by expanding the formulas and comparing only the approximations that involve the first two terms.

In the last column of table 1, the approximation formulas are expressed in terms of the Stoney formula, S_s . The first term of each formula consists of the Stoney formula, and the second term consists of a correction, involving the ratio, d/t , of the thickness of the coating to that of the basis metal. The correction diminishes as the thickness of the coating decreases.

A meaning can be attached to the two terms of the formulas for method I, (eq 9 and 34). The S_s term is the approximate stress in the coating, calculated without regard for the thickness or physical properties of the coating. The second term, $S_s(3Rd)/(t) = E_c T/2r$, can be shown to be the average stress that the coating loses in bending from its initial straight position to a curvature of $1/r$. The midpoint of the coating is approximately at a distance $t/2$ from the neutral axis of the beam. The compressive stress that is set up as a result of curvature is $E_f/2r$ (by eq 10), which is the second term of the formula. This compressive stress diminishes the tensile stress of the coating by this amount.

If carried further, the expansions of the formulas for methods I and II would yield terms containing higher powers of d/t , and therefore their contribution to the calculation would be small. The terms in d^2/t^2 represent the correction involved in the shift of the neutral axis from the center of the unplated beam to the center of the plated beam.

In table 2 are shown the errors involved in

TABLE 1. First two terms in the expansion of the formulas for stress

Stoney's equation.....		$S_s = \frac{Et^2}{6rd}$		$= S_s$
Method	Equation			
I.....	9	$S = \frac{E(t+d)^3}{6tdr}$	$= \frac{E(t^3+3t^2d)}{6tdr} + \dots$	$= S_s \left(1 + 3\frac{d}{t}\right) + \dots$
		$S_{s0} = \frac{E(t^2-td)}{6rd} + \dots$		$= S_s \left(1 - \frac{d}{t}\right) + \dots$
			34	$S = \frac{E_b(t+Pd)^3}{6rdt}$
II.....	23	$S = \frac{E(t+d)^3}{3dr(2t+d)}$	$= \frac{E(t^3+3t^2d)}{3dr(at+d)} + \dots$	$= S_s \left(1 + \frac{5}{2}\frac{d}{t}\right) + \dots$
III.....	27	$S = \frac{Et(t+d)}{6rd}$	$=$	$S_s \left(1 + \frac{d}{t}\right) + \dots$
		41	$S = \frac{E_f(t+Pd)}{6rd}$	$=$

using the approximation formulas for coatings of various relative thickness (compared with the

TABLE 2. Errors involved in using approximation formulas for calculating stress in coating—Continued

TABLE 2 Errors involved in using approximation formulas for calculating stress in coatings

Method of measurement	Formula	Error in using approximation formula for d/t equal to—			
		0.05	0.1	0.2	0.5
Moduli of coating and basis metal are equal					
*Method I. Cathode completely rigid.	$\frac{E(t+d)^3}{6rd}$	Per-cent	Per-cent	Per-cent	Per-cent
Stoney.....	$\frac{Et^2}{6rd}$	-14	-25	-42	-70
Two-term approximation...	$\frac{Et^2}{6rd} = \left(1 + \frac{3d}{t}\right)$	-1	-2	-7	-26
Method II. Cathode restricted from bending.	$\frac{E(t+d)^3}{3rd(2t+d)}$	---	---	---	---
Stoney.....	$\frac{Et^2}{6rd}$	-11	-21	-36	-63
Two-term approximation...	$\frac{Et^2}{6rd} \left(1 + \frac{5d}{2t}\right)$	0	-1	-4	-17
*Method III. Cathode bends during plating.	$\frac{Et(t+d)}{6rd}$	---	---	---	---
Stoney.....	$\frac{Et^2}{6rd}$	-5	-9	-17	-33
Moduli of coating and basis metal differ by $\frac{E_c}{E_b} = R = 2$					
Method I. Exact formula...	$\begin{cases} \frac{E_c(t+d)^3}{6rd} \\ \frac{(E_c - E_b)t^3}{6rd(t+d)} \end{cases}$	---	---	---	---
Stoney.....	$\frac{Et^2}{6rd}$	-26	-43	-62	-84
*Semiempirical.....	$\frac{E_b(t+Rd)^3}{6rd}$	-2	-1	+5	+32
Two-term approximation...	$\frac{Et^2}{6rd} \left(1 + 3R\frac{d}{t}\right)$	-5	-9	-16	-34
Method III, Exact formula...	See eq 38.....	---	---	---	---
Stoney.....	$\frac{Et^2}{6rd}$	-11	-19	-33	-52
*Two-term approximation...	$\frac{Et(t+Rd)}{6rd}$	-2	-3	-6	-4
Empirical.....	$\frac{Et}{6rd} (t + R^{3/4}d)$	0	0	-1	+4
Simplified.....	$\begin{cases} \frac{E_c t}{6rd} (t+d) \\ \frac{(E_c - E_b)t^3}{6rd(t+d)} \end{cases}$	+3	+4	+5	12

Method of measurement	Formula	Error in using approximation formula for d/t equal to—			
		0.05	0.1	0.2	0.5
Moduli of coating and basis metal differ by $\frac{E_c}{E_b} = R = 4$					
Method III. Exact formula.	See eq 38.....	Per-cent	Per-cent	Per-cent	Per-cent
Stoney.....	$\frac{Et^2}{6rd}$	---	---	-51	---
Two-term approximation...	$\frac{Et(t+Rd)}{6rd}$	---	---	-12	---
Empirical.....	$\frac{Et}{6rd} (t + R^{3/4}d)$	---	---	+4	---
Simplified.....	$\begin{cases} \frac{E_c t}{6rd} (t+d) \\ \frac{(E_c - E_b)t^3}{6rd(t+d)} \end{cases}$	---	---	-13	---

basis metal). It will be noted that if the moduli of the basis metal and coating are the same, the Stoney formula does not cause a serious error if used for method III, provided that d/t is less than 0.1, but that the error is appreciable for method I. However, if the moduli differ appreciably, the Stoney formula would not be a safe one to use for either method. The formulas that are the most convenient to use for calculation have been designated by an asterisk.

VII. Calculation of the Radius of Curvature

The value for the curvature $1/r$ of the strip must be substituted into the formulas for the stress in the coating. The curvature of the bent strip is usually measured indirectly, although it could be determined directly by comparison with a set of curvature gages, as was done by Soderberg [3]. Several different methods of measuring and calculating the curvature have been used, and there is a chance of some confusion. In at least one instance in the literature, an incorrect formula for calculating the curvature resulted through an error in selecting the proper deflection of the bent strip. Three methods of calculating the curvature, or change in curvature, of the plated strip have been used. These involve (a) measuring the camber of the curved strip, (b) measuring the

deflection of the end of the strip, and (c) measuring the angular deflection of the end of a helix.

In figure 5, AOA' represents the strip that has been curved by plating. Initially the strip occupied the position of the straight line ABC . The other lines in the figure are auxiliary lines for aiding the discussion. The method of calculating the curvature that was used by Stoney, involves measuring the sagitta, Z , of the arc AOA' . The curved strip is set on a flat surface with the convex side up, and the camber of the arc measured with a microscope or special micrometer.

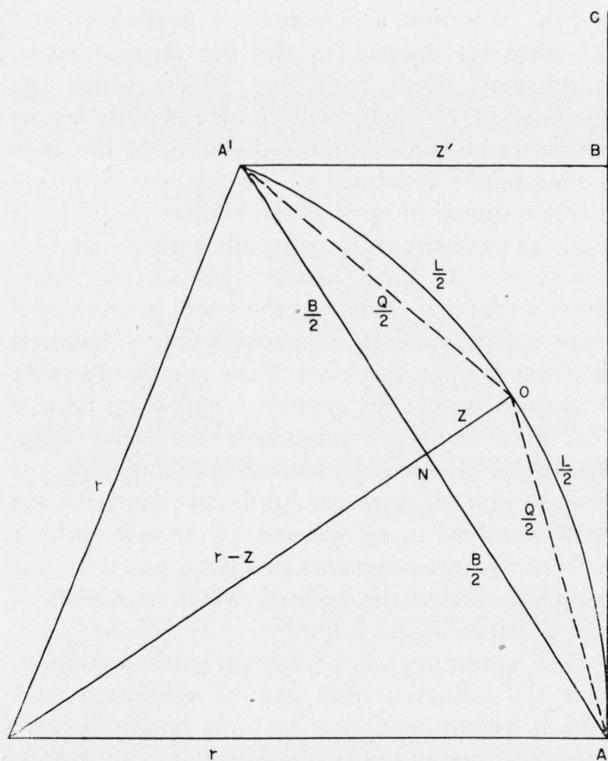


FIGURE 5. Diagram showing the relation between the radius of curvature of a plated strip and the deflection of the end or center.

The curvature $1/r$ is calculated from the following formula:

$$\frac{1}{r} = \frac{2Z}{B^2 + Z^2} \quad (49)$$

Another formula is

$$\frac{1}{r} = \frac{8Z}{Q^2} \quad (50)$$

The broken line $Q = AOA'$ may be replaced by the arc length $AOA' = L$, which is the length of the curved strip, without making an error of more

than about 2 percent, providing that the sagitta, Z , is not more than 10 percent of the length of the strip. The formula then becomes

$$\frac{1}{r} = \frac{8Z}{L^2} \quad (50a)$$

In some experiments the strip is anchored at A , and the strip then curves away from its initial position ABC . The deflection Z' is measured and used to calculate the curvature of the strip. The formula for calculating the curvature is

$$\frac{1}{r} = \frac{2Z'}{B^2} \quad (51)$$

The arc length L , which is the length of the strip, may be substituted for the chord without much error

$$\frac{1}{r} = \frac{2Z'}{L^2} \quad (52)$$

By comparing eq 52 with eq 50a, it will be noted that the deflection Z' is approximately four times greater than the sagitta Z . The error involved in the approximation of eq 52 is less than 2 percent if the deflection Z' is not greater than 20 percent of the length of the strip. For equal curvatures, the error of this approximation is about twice as large as that of eq 50a.

The main application of the spiral contractometer is for making measurements by method III, as the spiral continuously curves during the plating. However, by anchoring the needle, and releasing it at the conclusion of the plating, the experiment may be conducted according to method II.

Since the spiral is initially curved, the measurement consists not in determining the curvature $1/r$, as was done for an initially straight strip, but in determining the change in curvature of the spiral $\Delta(1/r) = (1/r_2) - (1/r_1)$. This expression is to be substituted for $1/r$ in the formulas that are applied to measurements made with the spiral contractometer. The change in curvature is computed from the angular deflection D , of the spiral as follows. When the curvature of an arc of fixed length is increased, the angle subtended by the arc increases. The change in the angle subtended by the arc is equivalent in the case of the helix of the contractometer to the angular deflection of the torque rod. The arc has an initial curvature of L/r_1 radians and after further bending has a curv-

ature L/r_2 radians. The angular deflection (or change in angle) D , (see fig. 6) is:

$$D = FOB - COB = \frac{L}{r_2} - \frac{L}{r_1} = L \left(\frac{1}{r_2} - \frac{1}{r_1} \right) = L \Delta \left(\frac{1}{r} \right) \quad (53)$$

$$\Delta \left(\frac{1}{r} \right) = \frac{D}{L} = \frac{\pi D'}{180L}$$

The expression D' refers to the angular deflection measured in degrees instead of in radians.

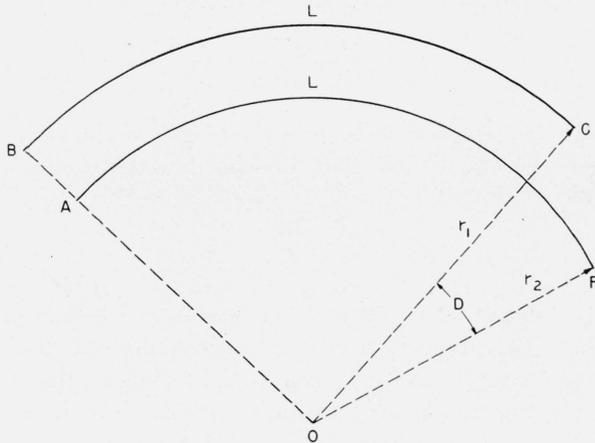


FIGURE 6. Diagram showing the relation between the radius of curvature and the angle subtended by an arc of constant length.

The two arcs in the figure are of equal length.

In this discussion the arc has been considered to lie in one plane. However, the same relation holds for a helix, if the curvature of the arc is considered with respect to the axis of the helix. This is equivalent to considering the length of the projection of the arc upon a plane normal to the axis of the helix. The projected length of the helix is approximately constant and is equal to πC times the number of turns, where C is the diameter of the helix. The number of turns of the helix is found most readily by pushing the coils together until the helix forms a cylinder and then dividing the height of the cylinder h , by the pitch, p , of the helix.

$$L = (\pi C) \left(\frac{h}{p} \right) = \frac{\pi Ch}{p} \quad (54)$$

The pitch is the width of the strip, from which the helix is made, measured in a direction parallel to the axis of the helix. It is related to the actual width b , of the strip by $b = p \cos a$, where a is the

helix angle. The projected length of the strip is related to the true length of the strip L' , by $L = L' \cos a$.

From eq 53 and 54, the expression for the change in curvature of the helix takes the form

$$\Delta \left(\frac{1}{r} \right) = D' \left(\frac{p}{180Ch} \right) \quad (55)$$

The expression in parentheses is a constant for a given helix, and needs to be determined only once.

No analysis was made of the precision of the relation (eq 55) between the change in curvature and the deflection, as was done in the discussion of the formulas applied to the flat strips. Slight errors may result from end effects, since the diameter of the helix will change slightly as its curvature changes, but the diameter of the ends is maintained constant by the fittings. Another possible source of error is the change in the helix angle as the helix coils or uncoils with the plating operation. A direct observation of the angle showed that the change in the angle in an experiment was not readily observable. It is believed that the angular deflection of the pointer is a more direct measure of the change in curvature than is the linear deflection measured on a flat strip. Therefore, the errors involved in computing the change in curvature are probably smaller than those involved in eq 50a and 52, or well under 1 percent for measurements involving less than one complete turn of the pointer (which is equivalent to a 36° turn of the helix).

One advantage of the spiral contractometer is that the unplated helix can be calibrated with known weights and thus make it unnecessary to know the modulus of the basis metal. An expression will now be derived relating the calibration constant K' , of the helix to its Young's modulus E , in order that K' can be substituted for E in the formulas that have already been developed. The deflection of the torque rod of the contractometer is proportional to the torque M , which is applied by a weight attached with a thread to the lever arm.

$$M = K'D' \quad (56)$$

The bending moment, M , transferred to the helix is given by an equation similar to eq 8

$$M = EI \Delta \left(\frac{1}{r} \right) = \frac{Et^3 p}{12} \Delta \left(\frac{1}{r} \right) \quad (57)$$

It will be noted that the moment of inertia I , of the strip composing the helix, is taken for a cross section that is parallel to the axis of the helix, because the helix curves and uncurves with respect to the axis as a center. On substituting in eq 57 for $\Delta(1/r)$, and for M , using eq 55 and 56, the expression for E is obtained

$$E = \frac{2160K'Ch}{t^3p^2}. \quad (58)$$

The Youngs modulus of the metal composing the helix has been calculated with the aid of this formula, and the agreement with the accepted value of the modulus was within 3 percent.

In calculating the stress in a deposit, Stoney's equation and eq 27 are the ones most frequently used in conjunction with the spiral contractometer. On substituting for E from eq 58 and for $\Delta(1/r)$ from eq 55, Stoney's equation takes the form

$$S_s = \frac{2K'D'}{tpd}. \quad (59)$$

Equation 27 takes the form

$$S = \frac{2K'D'(t+d)}{t^2pd} = S_s \left(1 + \frac{d}{t}\right). \quad (60)$$

In using the spiral contractometer, the angular deflection of the torque rod is not read directly, but the deflection in degrees is read from a pointer that is geared to the torque rod. The pointer makes G revolutions for one revolution of the torque rod. Therefore, in using eq 55, in which the deflection D' relates to the torque rod, the observed reading \bar{D} , of the pointer must be divided by G or

$$\Delta \frac{(1)}{(r)} = \frac{\bar{D}}{G} \frac{(p)}{(180Ch)}. \quad (61)$$

Also, the calibration constant of the helix is usually determined with direct reference to the deflection of the pointer in degrees rather than of the torque rod. The relation between the constant K , of the readings referred to the pointer, and the constant K' of those referred to the torque rod is:

$$K' = KG. \quad (62)$$

This substitution must be made in calculating E from eq 58. Since

$$K'D' = K\bar{D}, \quad (63)$$

the forms of eq 59 and 60 are unchanged if the constant and the deflection of the pointer are substituted for those of the torque rod.

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VIII. References

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