REDUCTION OF DATA ON MIXTURE OF COLOR STIMULI

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ABSTRACT

Proof is given that any set of distribution curves may be assigned luminosity coefficients giving the same luminosity sum as any other set of distribution curves embodying the same mixture data. Two methods of computing such luminosity coefficients are described and these methods are applied to four specific transformations of the O. S. A. "excitation" curves. By actual computation (for four filters) of dominant wave length, colorimetric purity, and transmission for sunlight it is demonstrated in a practical way that each of these transformations embodies the same mixture data and luminosity data as the O. S. A. "excitation" curves. One of these transformations is proposed for routine computation because it has properties which permit the adoption of simpler methods of computing than the curves now used permit.

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I. INTRODUCTION

Probably the most fundamental set of data relating to color vision is that embodied in the three "excitation"[1] curves or "visual response functions" by means of which the laws of mixture of color stimuli are applied. This set of data is obtained experimentally by introducing into one half of a photometric field light of known spectral composition, and into the other half of the field light of a different known spectral composition, so chosen that the two fields appear matched to the observer. By making a large number of such matches, information is amassed by which it is possible to tell whether any two fields of different spectral composition will appear matched to that observer or not.

It was indicated by Newton [2] and subsequently verified by others [3] that this information, or mixture data, could be embodied in a finite

[1] We say, in this instance, "excitation," because that term is a usual one. Hereafter it will be frequently convenient to substitute the term "distribution" on account of the generality of the mathematical treatment. (See footnote 36, p. 524.)

[2] Isaac Newton, Opticks, London, Innys, pp. 134–137; 1730. We find here a statement of "Newton's law of color mixture." Although this law makes no explicit mention of additive distribution curves, and does not directly state that contributions from different sources to a given primary color process are to be combined by addition, still these concepts are contained implicitly in the law. Our indebtedness to Newton in this respect was acknowledged by J. C. Maxwell (Sci. Papers, Cambridge, p. 149; 1890), who was among the first to state explicitly and use these concepts. Probably Grassman (Pogg. Ann., 89, pp. 69–84; 1853) first stated them explicitly.

[3] See, for example: A. König, Ueber Newton's Gesetz der Farbenmischung und darauf bezügliche Versuche des Hrn. Eugen Brodhun, Sitz. Akad. Wiss., Berlin, 31, pp. 311–317; 1887. Newton's law for normal observers has been amply verified by experiment for field brightnesses which are too high to involve the Purkinje effect. We restrict attention, as is usual, to the medium field brightnesses dealt with by the technique of colorimetry for which Newton's law is valid.

set of additive distribution curves.[4] From these curves directly can be obtained the prediction whether two given stimuli of different spectral composition will appear matched or not.[5] If ρ_0, γ_0, and β_0 give the distribution of three color processes [6] (say a red, a green, and a blue process) for an equal energy spectrum, and E the spectral energy distribution of the light source actually used,[7] the condition for a complete color match (hue, saturation, and brilliance) between two stimuli is:

$$\left.\begin{aligned}\int_0^\infty \rho_0\, E\,(T_1-T_2)\, d\lambda=0\\ \int_0^\infty \gamma_0\, E\,(T_1-T_2)\, d\lambda=0\\ \int_0^\infty \beta_0\, E\,(T_1-T_2)\, d\lambda=0\end{aligned}\right\} \qquad (1)$$

where T_1 and T_2 are functions of the wave length, λ; T_1 gives the ratio of the spectral energy at a given wave length of the first stimulus to the spectral energy of the source, E, at that same wave length; T_2 gives the same ratio for the second stimulus.[8] If condition (1) be not satisfied, then the stimuli specified by ET_1 and ET_2 are not a match.

It has long been recognized that if ρ_1, γ_1, and β_1 give the distribution of three primary color processes in the spectrum of energy distribution, E, we can define an infinite number of sets of distribution curves (say, for example, ρ_2, γ_2, and β_2) representing the distribu-

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[4] Newton used seven primary colors, which necessitates seven distribution curves. We now know (and Newton seems to have suspected, though he was not sure) that three independent distribution curves giving the distribution of three primary color processes throughout the spectrum are sufficient and necessary to embody the mixture data of a normal observer. More than three distribution curves may be used if desired, but they will not all be independent. The fact that three distribution curves are sufficient was probably first given its proper emphasis by Thomas Young (Lectures on Natural Philosophy, London, Savage, **I**, p. 440; 1807), who made it the basis of his color theory now widely known because of the adoption and elaboration of it by Helmholtz. Experimental evaluations of these distribution curves have been made by König and Dieterici (Sitz. Akad. Wiss., Berlin, **29**, pp. 805–829; 1886), Maxwell (Sci. Papers, Cambridge, pp. 426–444; 1890), Abney (Phil. Trans. Roy. Soc.; 1899), and others.

[5] See Nikolaus Nyberg, Zum Aufbau des Farbenkörpers im Raume aller Lichtempfindungen, Zs. f. Phys., **52**, p. 407; 1928. He gives our relation (1) as (5).

[6] "Color process" is here taken as a name for the activity, retinal and postretinal, which is in one-to-one correspondence with the incidence of radiant energy of a certain range of frequency upon the retina. If the radiant energy is specified by its distribution (E) with respect to frequency (hereafter the wave length in vacuo will be used to specify frequency), then, with the visual mechanism in the "neutral state" (see footnote 61, p. 542), there corresponds to this radiant energy a certain real color process. If E for some wave lengths be negative (which is physically impossible) still it is possible to compute a specification for the "color process" with which this imaginary stimulus is in one-to-one correspondence. Since it is convenient and useful to do this, the term "color process" is taken to refer to such activity, whether that activity actually be possible within the normal visual mechanism (as it very well may be, see footnote 61, p. 542) or not. When any three-color processes, existent or nonexistent, corresponding to any three physical stimuli, real or imaginary, are dignified by adoption, temporarily or permanently, as objects of a set of distribution curves by which the responses of an observer may be predicted, we shall denote their dignity by referring to them as "primary color processes." It is evident that this state of being "primary" either may or may not coincide with the state of being psychologically primary (unitary) or physiologically primary (invariable—independent of intensity, duration, or retinal region stimulated).

[7] ρ_0E, γ_0E, and β_0E are the distribution curves of the three primary color processes in the spectrum of the source whose spectral energy distribution is E; they are hereafter designated ρ_1, γ_1, and β_1, respectively. No restrictions as to relative size are placed on the units of ρ_0, γ_0, and β_0; but, for convenience only, ρ_1, γ_1, and β_1, are customarily expressed in comparable units. (See footnote 29, p. 522.)

[8] Note that if the two light stimuli are produced by filters (1 and 2) illuminated by the source whose spectral energy is E, then T_1 and T_2 are merely the spectral transmissions of the filters. The significance of relation (1) may be restated in experimental terms as follows:

Consider a symmetrically divided photometric field, both halves of which are illuminated by a source for which energy as a function of the wave length, λ, is given by E and let the two halves be of equal brightness. Let a filter whose transmission as a function of λ is T_1 be inserted in the path of light which illuminates one half of the field; and let a filter whose transmission as a function of λ is T_2 be inserted in the path of light which illuminates the other half of the field. If the conditions expressed by relation (1) be satisfied, the two halves of the field will then be perfectly matched in quality and brightness (hue, saturation, and brilliance); that is, they will be quite indistinguishable one from the other for the observer to whom ρ_0, γ_0, and β_0 refer.

The reason for introducing the ratios T_1 and T_2 applying to the same source of radiant energy, E, and resulting in the energy distributions ET_1 and ET_2 is that the routine use of the "excitation" curves in computation usually involves such ratios (either spectral transmission or spectral reflection). The products ET_1 and ET_2 would otherwise have more naturally been designated simply as E_1 and E_2 without indication as to the manner of their physical realization.

tions of different sets of primary color processes [9] which would equally well represent the same body of mixture data. We need only to choose at random nine finite constants (K_1 to K_9 subject to the condition:

$$\left.\begin{array}{l} \begin{vmatrix} K_1 & K_2 & K_3 \\ K_4 & K_5 & K_6 \\ K_7 & K_8 & K_9 \end{vmatrix} \neq 0^{10} \\ \\ \text{then the expressions:} \\ \\ \begin{array}{l} \rho_2 \equiv K_1\rho_1 + K_2\gamma_1 + K_3\beta_1 \\ \gamma_2 \equiv K_4\rho_1 + K_5\gamma_1 + K_6\beta_1 \\ \beta_2 \equiv K_7\rho_1 + K_8\gamma_1 + K_9\beta_1 \end{array} \end{array}\right\} \quad (2)$$

define the new set of distribution curves for each wave length.

We say that ρ_2, γ_2, and β_2 represent the same body of mixture data as do ρ_1, γ_1, and β_1 because, whenever T_1 and T_2 are so chosen that relation (1) holds, it is also true that that relation holds when ρ_2, γ_2, and β_2 are substituted for ρ_1, γ_1, and β_1, respectively.[11]

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[9] The new set of primary color processes might be any three different color processes providing the stimulus (real or imaginary) which evokes one of them can not be matched by a mixture of the stimuli which evoke the other two. The new processes might be, for example, any violet, any yellow, and any crimson process. This freedom of choice of primary color process may be stated in another way. If all color processes are represented on a mixture diagram (such as figs. 2 to 5), each process by a point, then any two points may be chosen at random for the first two primaries. The only restriction on the choice of the third primary is that its point must not lie on the straight line connecting the points for the other primaries which were chosen at random.

[10] That this determinant be different from zero is the condition which ensures that ρ_2, γ_2, and β_2 will be three independent functions provided ρ_1, γ_1, and β_1 are. In other words, it ensures that the points representing on the mixture diagram the three new primary color processes do not lie on the same straight line if the points representing the old primaries do not. In actual cases this condition is seldom tested because the collapse of the tristimulus system into a monostimulus or distimulus system is usually prevented by the nature of the problem.

[11] Although the accuracy of this statement has been regarded as so evident that writers following Newton (see footnote 12) who have made use of relation (2) have apparently never bothered to publish a proof of it, still it is, perhaps, well to indicate the proof here for the sake of completeness. We wish to show that $\int_0^\infty \rho_2\,(T_1-T_2)d\lambda=0$. From (2) we may write:

$$\int_0^\infty \rho_2\,(T_1-T_2)d\lambda = \int_0^\infty (K_1\rho_1+K_2\gamma_1+K_3\beta_1)\,(T_1-T_2)d\lambda =$$
$$K_1\int_0^\infty \rho_1\,(T_1-T_2)\,d\lambda + K_2\int_0^\infty \gamma_1\,(T_1-T_2)d\lambda + K_3\int_0^\infty \beta_1\,(T_1-T_2)d\lambda.$$

But from (1) and footnote 7, page 516, each of the three integrals is zero; hence their sum $\int_0^\infty \rho_2\,(T_1-T_2)d\lambda$ equals zero, which is what we wanted to prove. By analogous argument, $\int_0^\infty \gamma_2(T_1-T_2)d\lambda=0$, and $\int_0^\infty \beta_2\,(T_1-T_2)d\lambda=0$; hence ρ_2, γ_2, and β_2 satisfy (1) whenever ρ_1, γ_1, and β_1 do.

This form of proof can also be extended to show that whenever relation (1) is not satisfied by ρ_1, γ_1, and β_1 then neither do ρ_2, γ_2, and β_2 satisfy it. The argument is by reductio ad absurdum: Assume that ρ_2, γ_2, and β_2 do satisfy (1) when ρ_1, γ_1, and β_1 do not. Now it is possible to find constants (k_1 to k_9) such that:

$$\begin{array}{l} \rho_1 = k_1\rho_2 + k_2\gamma_2 + k_3\beta_2 \\ \gamma_1 = k_4\rho_2 + k_5\gamma_2 + k_6\beta_2 \\ \beta_1 = k_7\rho_2 + k_8\gamma_2 + k_9\beta_2 \end{array}$$

$$\begin{vmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \\ k_7 & k_8 & k_9 \end{vmatrix} \neq 0$$

(This is relation (5) of Sec. II; the constants (k_1 to k_9) of this relation may be computed from the constants (K_1 to K_9) of relation (2); see relation (7) in Sec. III.) Hence, as just shown, ρ_1, γ_1, and β_1 satisfy (1) which is contrary to hypothesis.

Hence, if ρ_2, γ_2, and β_2 are used to predict whether the stimulus specified by ET_1 matches that specified by ET_2, the same prediction is obtained as by the use of ρ_1, γ_1, and β_1.[12]

The choice of a particular set of distribution curves out of the triple infinity of sets which can all equally well embody a given set of mixture data is dictated purely by convenience. If, as in this article, the aim is prediction of whether the observer will respond "match" or "not a match" when confronted by two color stimuli of different spectral composition, a set of curves should be chosen that lends itself readily to the computation incident to this prediction.[13] If, on the other hand, the aim is to simplify the basis for speculation of a psychophysiological nature, many other quite different sets of curves would be eligible, the choice depending on the nature of the speculation.[14]

When the technique of heterochromatic photometry was worked out [15] it became convenient to use a method, long known, of specifying a color stimulus. This method is to evaluate for that stimulus three variables—dominant wave length, colorimetric purity, and luminosity.[16] In order to convert specifications of color stimuli from this system into those suggested by relation (1), namely, the

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[12] This has been recognized by nearly all writers following Newton, though they do not always give relation (2) explicitly. It is convenient at this point to introduce a partial list of the publications which have dealt, directly or indirectly, with transformations of this sort:

A. König and C. Dieterici, Die Grundempfindungen und ihre Intensitätsvertheilung im Spectrum, Sitz. Akad. Wiss., Berlin, **29**, pp. 805–829; 1886. (Or see A. König, Gesammelte Abhandlungen zur Physiologischen Optik, Leipzig Barth, pp. 60–87; 1903.) Also Die Grundempfindungen in normalen und anomalen Farbensystemen und ihre Intensitätsvertheilung im Spectrum, Zs. f. Psych. u. Physiol. der Sinnesorgane, **4**, pp. 241–347; 1892. (Or see A. König, Ges. Abh., Leipzig, Barth, pp. 214–321; 1903.)

H. v. Helmholtz. Zs. f. Psych. u. Physiol. der Sinnesorgane, **2**, pp. 1–30; **3**, pp. 1–20. (Or see H. v. Helmholtz, Handbuch der Physiologischen Optik, 2d ed., Leipzig, Voss, pp. 449–458; 1896.)

H. E. Ives, The Transformation of Color-Mixture Equations from One System to Another, J. Frank. Inst., **180**, pp. 673–701; 1915. Also, The Transformation of Color-Mixture Equations from One System to Another. II. Graphical Aids, J. Frank. Inst., **195**, pp. 23–44; 1923.

J. Guild, The Transformation of Trichromatic Mixture Data: Algebraic Method, Trans. Opt. Soc., **26**, pp. 95–108; 1924–25. Also, The Geometrical Solution of Colour Mixture Problems, Trans. Opt. Soc., **26**, pp. 139–174; 1924–25.

W. Dziobek, Ueber die Transformation von einem trichromatischen System auf ein anderes, Zs. für Instrumentenkunde, **46**, pp. 81–84; 1926.

E. Schrödinger, Grundlinien einer Theorie der Farbenmetrik im Tagessehen, Ann. d. Physik (4), **63**, pp. 397–456; 1920.

E. Schrödinger, Ueber das Verhältnis der Vierfarben- zur Dreifarbentheorie, Sitz. Akad. Wiss., Wien, Abt. IIa, **134**, pp. 471–490; 1925.

R. Luther, Aus dem Gebiet der Farbreizmetrik, Zs. f. Techn. Physik, **8**, pp. 540–558; 1927.

I. Runge, Ueber die Ermittlung der Farbkoordinaten aus den Messungen am trichromatischen Kolorimeter, Zs. f. Instrumentenkunde, **48**, pp. 387–396; 1928.

Relation (2) of this paper is given as relation (9) by Helmholtz (Handbuch der Physiologischen Optik, 2d ed., Leipzig, Voss, p. 453; 1896) and as relation (1) by Ives (J. Franklin Inst., **180**, p. 678; 1915). A restricted form of the relation is given by König and Dieterici (König, Ges. Abh., pp. 82, 305), and also by Guild as relation (4) (Trans. Opt. Soc., **26**, p. 97; 1924–25), and again as relation (10) (ibid., p. 153).

Compare also relations (12) and (13) given by Schrödinger (Ann. d. Physik, (4), **63**, p. 439; 1920).

[13] The "Elementarempfindungskurven" of König and Dieterici (A. König, Ges. Abh., Leipzig, Barth, pp. 286–288; 1903), the elementary "sensation" curves of Abney (W. de W. Abney, Researches in Colour Vision, London, Longmans-Green, pp. 229–247; 1913), and the elementary "excitations" computed by Weaver using data by König, Dieterici, and Abney (L. T. Troland, Report of the Committee on Colorimetry for 1920–21, J. Opt. Soc. Am. and Rev. Sci. Inst., **6**, p. 549; 1922) are examples of distribution curves which are convenient for computation chiefly by reason of the fact that there are no negative ordinates. The sort of computation for which these curves are used is treated by Troland (J. Opt. Soc. Am. and Rev. Sci. Inst., **6**, pp. 573–592; 1922) and by Priest (J. Opt. Soc. Am. and Rev. Sci. Inst., **9**, pp. 503–520; 1924).

[14] Thus, we might try to choose distribution curves which suggest a simple theoretical account of the facts of dichromasy just as Maxwell (On the Theory of Compound Colours, and the Relations of the Colours of the Spectrum, Sci. Papers, Cambridge, pp. 441–444; 1890), and König and Dieterici (A. König, Ges. Abh., Leipzig-Barth, pp. 303–321; 1903) did. Or we might test the applicability of an extension of Fechner's law for the prediction of sensibility to color differences as Helmholtz (Handb., d. Physiol. Optik, 2d ed., Leipzig-Voss, pp. 449–458; 1896) did. Or we might wish to show that the mixture data could suggest a four-color theory of vision as well as a three-color theory as Schrödinger (Sitz. Akad. Wiss., Wien, Abt. IIa, **134**, pp. 471–490; 1925) did.

[15] Due in large part to the research activity of Herbert E. Ives (Studies in the Photometry of Light of Different Colors, Phil. Mag., **24**, pp. 150–188, 352–370, 744–751, 845–863; 1912).

[16] I. G. Priest, Apparatus for the Determination of Color in Terms of Dominant Wave-Length, Purity and Brightness, J. Opt. Soc. Am. and Rev. Sci. Inst., **8**, pp. 173–200; 1924. (This paper gives references to previous publications on the same subject.) W. v. Bezold, Ueber das Gestez der Farbenmischung und die physiologischen Grundfarben, Pogg. Ann., **150**, pp. 71–78; 1873. We find here (p. 78) probably the first explicit definition of the concept now termed "colorimetric purity." Bezold named it "objective Reinheit."

evaluation for that stimulus of $\int_0^\infty \rho_0 ETd\lambda$, $\int_0^\infty \gamma_0 ETd\lambda$, and $\int_0^\infty \beta_0 ETd\lambda$ (and vice versa), it became necessary to evaluate a set of "luminosity coefficients" L_{r_1}, L_{g_1}, L_{b_1}, such that for each wave length:

$$\rho_1 L_{r_1} + \gamma_1 L_{g_1} + \beta_1 L_{b_1} = L_1 \tag{3}$$

where L_1 gives the distribution of luminosity of the source for which ρ_1, γ_1, and β_1 give the distribution of the primary color processes.[17] If we are to compute (for example) the colorimetric purity of a stimulus from the tristimulus specification, we must be able to compute from that specification L_1 for the wave length equal to the dominant wave length of the stimulus. The first set of distribution curves to which luminosity coefficients were attached was compiled from König's data by Dr. H. E. Ives. Ives noted that the Grundempfindungskurven of König failed to sum even approximately to the average luminosity function when weighted by any set of positive luminosity coefficients whatever.[18] Consequently, he chose a new set of primary color processes yielding a new set of distribution curves to represent the mixture data of König's observers. He found that this new set of distribution curves summed within experimental error to the average luminosity when weighted by a set of positive luminosity coefficients.

In 1920 Exner [19] independently noted the difficulties which led Ives to make the changes just described. Exner, however, proposed to obtain by purely arbitrary changes a substantial agreement of the mixture data with the luminosity data through the use of positive coefficients. These arbitrary changes (not being a transformation of coordinates) fail to satisfy (2). The resulting set of distribution curves, therefore, does not embody the same set of mixture data as the average König curves, and if used for computation does not give the same numerical results that the König curves give. These curves have attracted the well-deserved criticism of Schrödinger.[20]

It has been shown by the author [21] that one set of distribution curves (the O. S. A. "excitation" curves)[22] will sum, when properly weighted, to a satisfactory approximation of the accepted luminosity function of the source. Throughout the present paper these distribution curves will be used as a basis for computation because they represent in the author's judgment the best reduction of mixture data available. They are somewhat more congruent with the luminosity function than the König-Ives curves, due partly to the

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[17] $\rho_1 \equiv \rho_0 E$, $\gamma_1 \equiv \gamma_0 E$, and $\beta_1 \equiv \beta_0 E$. See footnote 7, p. 516.

[18] J. Frank. Inst., **180**, pp. 693–694; 1915. Ives noted that a negative luminosity coefficient for the "blue" curve would be required; but apparently he regarded a negative coefficient unacceptable because he remarks in italics that "* * * no positive luminosity values for the blue sensation will satisfy the condition" and later (p. 694) implies that the König curves fail to yield consistent luminosity values on this account. Of course, to attach a negative coefficient to the blue curve involves a difficulty on the basis of psychophysiological theory, namely, that the mechanism which is responsible for "blue" must also partially inhibit the "brightness" response; this consideration was probably the reason for Ives's rejection of negative coefficients.

[19] F. Exner, Zur Kenntnis der Grundempfindungen im Helmholtz'schen Farbensystem, Sitz. Akad. Wiss., Wien, Abt. IIa, **129**, pp. 27–46; 1920.

[20] E. Schrödinger, Ueber das Verhältnis der Vierfarben- zur Dreifarbentheorie, Sitz. Akad. Wiss., Wien, Abt. IIa, **134**, p. 479; 1925. He remarks in a footnote, "Von der Korrektur der König'schen Grundempfindungskurven, die Exner weiterhin vornimmt, konnte ich aus dem Grunde keinen Gebrauch machen, weil sie Schönheitsfehler (Konkavitäten nach aussen) der Spectralkurve in der Farbentafel nach sich zieht."

[21] D. B. Judd, Chromatic Visibility Coefficients by the Method of Least Squares, J. Opt. Soc. Am. and Rev. Sci. Inst., **10**, pp. 635–651; 1925.

[22] See footnote 35, p. 524.

more refined method of computing luminosity coefficients but more particularly to the different choice of mixture data they embody and to the fact that more extensive luminsoity data have since become available.

The purposes of the present paper are:

To show that any one of the infinite number of sets of distribution curves representing a given body of mixture data will sum equally well to the luminosity function. (Sec. II.)

To outline two short methods for computing the luminosity coefficients of any of these sets of distribution curves. (Sec. III.)

To present four examples of this problem solved by the methods outlined in Section III. (Sec. IV.)

To present a demonstration that each of the four sets of distribution curves used embody the same mixture data (that of the O. S. A. "excitation" curves) by showing that each of the four sets gives the same results when used for the computation of the dominant wave length, colorimetric purity, and luminosity of color stimuli. (Sec. V.)

To show that one of these sets of distribution curves embodies greater computational convenience than any set heretofore proposed. (Sec. VI.)

II. CHOICES OF DISTRIBUTION CURVES AND THE INDEPENDENCE OF THE LUMINOSITY FUNCTION

Theorem.—Any set of distribution curves may be assigned luminosity coefficients giving the same luminosity sum as any other set of distribution curves embodying the same mixture data.

The theorem restated is: If ρ_1, γ_1, β_1 and ρ_2, γ_2, β_2 are sets of distribution curves which satisfy relation (2) for each wave length, and L_{r1}, L_{g1}, and L_{b1} are three numbers satisfying relation (3) for each wave length, then three numbers, L_{r2}, L_{g2}, and L_{b2}, exist such that, for each wave length:

$$\rho_2 L_{r2}+\gamma_2 L_{g2}+\beta_2 L_{b2}=\rho_1 L_{r1}+\gamma_1 L_{g1}+\beta_1 L_{b1}=L_1 \tag{4}$$

Proof.—Since these two sets are connected by relation (2) it is possible to find constants (k_1 to k_9) such that:

$$\left.\begin{array}{c} \rho_1=k_1\rho_2+k_2\gamma_2+k_3\beta_2 \\ \gamma_1=k_4\rho_2+k_5\gamma_2+k_6\beta_2 \\ \beta_1=k_7\rho_2+k_8\gamma_2+k_9\beta_2 \\ \begin{vmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \\ k_7 & k_8 & k_9 \end{vmatrix} \neq 0 \end{array}\right\} \tag{5}$$

by solving for ρ_1, γ_1, and β_1 from (2).[23] From (3) and (5) we may write:

$$L_{r1}\,(k_1\rho_2+k_2\gamma_2+k_3\beta_2)+L_{g1}(k_4\rho_2+k_5\gamma_2+k_6\beta_2)+L_{b1}(k_7\rho_2+k_8\gamma_2+k_9\beta_2)=L_1$$

[23] The result of this solution is given as relation (7) in Sec. III. Compare relation (15) given by Schrödinger (Ann. d. Physik, (4), 63, p. 440; 1920). This solution is also given by Guild (Trans. Opt. Soc., 26, p. 100; 1924–25) as relation (10), though he does not evaluate the determinant we abbreviate as Δ. (See relation (7).)

or, rearranging terms:

$$\rho_2(L_{r_1}k_1+L_{g_1}k_4+L_{b_1}k_7)+\gamma_2(L_{r_1}k_2+L_{g_1}k_5+L_{b_1}k_8)+\beta_2(L_{r_1}k_3+L_{g_1}k_6+L_{b_1}k_9)=L_1$$

whence it is evident that numbers, L_{r_2}, L_{g_2}, and L_{b_2}, which satisfy (4) do exist because they are:

$$\left.\begin{aligned} L_{r_2}&=L_{r_1}k_1+L_{g_1}k_4+L_{b_1}k_7\\ L_{g_2}&=L_{r_1}k_2+L_{g_1}k_5+L_{b_1}k_8\\ L_{b_2}&=L_{r_1}k_3+L_{g_1}k_6+L_{b_1}k_9\end{aligned}\right\} \qquad (6)^{24}$$

It may easily be shown that the sum of the luminosity coefficients applying to the second set of distribution curves is equal to the sum of the luminosity coefficients applying to the first set of distribution curves provided all distribution curves of both sets have equal areas. It may be seen from relation (6) that if we wish to choose any set of luminosity coefficients whatever which satisfies the restriction just stated, it will be possible to derive a set of distribution curves to which that set of luminosity coefficients apply.[25] It is further evident from relation (6), since we have nine constants out of which to build but three luminosity coefficients whose nature imposes but three added restrictions,[26] that not only will it be possible to find a set of distribution curves embodying the mixture data of the first set of distribution curves to which the arbitrarily chosen set of luminosity coefficients apply, but that it will be possible to find many such sets. We shall have occasion to make use of this possibility in Sections V and VI. (See footnote 52, p. 533, and footnote 74, p. 544.)

It is of interest to indicate the steps in the actual discovery of the truth of our theorem because the discovery involves a special case of relation (2) which is in itself worthy of attention. Consider a set of distribution curves ρ_3, γ_3, and β_3, computed from ρ_1, γ_1, and β_1, by the relations:

$$\left.\begin{aligned} \rho_3&=\rho_1\\ \gamma_3&=\gamma_1\\ \beta_3&=\frac{D\rho_1}{3-D}+\frac{D\gamma_1}{3-D}+\frac{(3-3D)\beta_1}{3-D}\end{aligned}\right\} \qquad (2a)$$

where D is any finite constant except $+1$ and $+3$ [27] independent of wave length; D may be thought of as fractional deficiency of primary color process;[28] here D refers to fractional deficiency of "blue" process since it is attached to β_1, but it may, of course, be attached equally well to ρ_1 or γ_1. It may be noted that relation (2a) is a special case of relation (2); hence ρ_3, γ_3, and β_3 represent the same body of mixture data as ρ_1, γ_1, and β_1.

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[24] I am indebted to Dr. L. B. Tuckerman for this elegantly simple proof.

[25] We have merely to choose the constants (k_1 to k_9) which, according to (6) give the desired values of L_{r_2}, L_{g_2}, and L_{b_2}. These constants (k_1 to k_9) lead, according to (5), from the new distribution curves to the old; hence, they may helpfully be called the "reverse" coefficients. But we can find the direct coefficients (K_1 to K_9) of relation (2) from the reverse coefficients in the same way that the reverse coefficients are found from the direct coefficients. (See relation (7).)

[26] The case of interest is that in which the luminosity coefficients refer to distribution curves of equal area. This condition requires that: $k_1+k_2+k_3=k_4+k_5+k_6=k_7+k_8+k_9=1$.

[27] $D=+3$ makes β_3 infinite. It is ruled out because relation (2) applies to a choice of finite constants only. $D=+1$ is ruled out because the determinant of relation (2) then becomes zero.

[28] Thus, for $D=+1$, we find no trace of the β_1 curve in the β_3 curve because β_3 is simply the sum of half of each of the other two. This corresponds, of course, to the collapse of the tristimulus into a distimulus system in which the orginal "blue" process is missing; hence, we may say that for $D=+1$, the system is wholly "blue" deficient.

It was known that the theorem held for this special case of relation (2) because, by trial and error, the luminosity coefficients, L_{r3}, L_{g3}, and L_{b3}, had been found to be:

$$\left.\begin{aligned} L_{r3} &= L_{r1} - DL_{b1}/3(1-D), \\ L_{g3} &= L_{g1} - DL_{b1}/3(1-D), \\ L_{b3} &= L_{b1}/(1-D) - DL_{b1}/3(1-D) \end{aligned}\right\} \tag{6a}$$

Examination of (2a) and (6a) shows that ρ_3, γ_3, and β_3 have the following properties: (1) They differ from ρ_1, γ_1, and β_1 in respect to one variable only (here β_1), (2) they yield curves of equal area provided ρ_1, γ_1, and β_1 do, and (3) they have luminosity coefficients which sum to unity provided ρ_1, γ_1, and β_1 have such luminosity coefficients.

After we have made one such transformation with a given value (positive or negative) of D applied to a particular distribution curve (say β_1), we may follow that transformation with another which involves a different value of D applied to the ρ_1 or the γ_1 curve; and so on. But since for each of these transformations it is possible to find luminosity coefficients which satisfy (3) merely by computation according to (6a), and since apparently any set of distribution curves embodying the same mixture data as ρ_1, γ_1, and β_1 (and having equal areas)[29] may be obtained by successive transformations of this kind, it was supposed that the theorem held for all cases.[30]

III. TWO METHODS OF CALCULATING THE LUMINOSITY COEFFICIENTS OF A SET OF DISTRIBUTION CURVES FROM THE LUMINOSITY COEFFICIENTS OF ANOTHER SET OF DISTRIBUTION CURVES EMBODYING THE SAME MIXTURE DATA

If a set of distribution curves is given and the luminosity curve of the source is at hand which the sum of the distribution curves, properly weighted by luminosity coefficients, ought to approximate, the luminosity coefficients which give the best approximation may be computed by the method of least squares.[31] It was apparently assumed by Ives [32] that a new choice of primary color processes giving a new set of distribution curves embodying the same mixture data might improve the goodness of fit of those data to the luminosity data, and such an implication was accepted by the author until the present work showed it to be untenable. The incorrectness of this view can now be conveniently demonstrated by an important corollary of the theorem proven in Section II, namely, that a new choice of distribution curves results in exactly the same agreement with the luminosity data, no better and no worse.[33] For all sets of stimulus primaries,

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[29] It is customary to adjust the ordinates of the three distribution curves so that the areas under the curves are equal. Such an adjustment is convenient because it causes the point representing the color of the light stimulus from the source (whose energy distribution is E) to fall at $r=1/3$, $g=1/3$, $b=1/3$. (See figs. 2 to 5, also relation (9) et seq.)

[30] Tnis argument, as given, does not amount to a proof, as does the one previously given, because it does not show that every set of distribution curves satisfying (2) can be approximated by the successive application of (6a). It has merely shown that many such sets can be obtained.

[31] See footnote 21, p. 519.

[32] See footnote 18, p. 519.

[33] The argument is by reductio ad absurdum: The theorem states that there exist luminosity coefficients which give a new distribution curve sum identical with the old (relation 4). Thus, if a new choice of distribution curves (obtained by a new choice of stimulus primaries) gives a better fit than the best obtainable by the old choice, we can transfer back to the original primaries and find coefficients which give the same sum, a sum which must agree better than the best obtainable, which is impossible.

then, the least square solution results in exactly the same degree of agreement; this method of solving for luminosity coefficients is, of course, always applicable. There are shorter ways, however, of obtaining the best luminosity coefficients (L_{r2}, L_{g2}, L_{b2}) for a set of distribution curves (ρ_2, γ_2, β_2) provided a least square solution for L_{r1}, L_{g1}, and L_{b1} has already been carried out on another set (ρ_1, γ_1, β_1) representing the same mixture data. Two methods will be given; the first method, which requires only that the theorem of Section II be true, involves the following steps:

1. Compute $\rho_1 L_{r1}+\gamma_1 L_{g1}+\beta_1 L_{b1}$ for any three convenient values of the wave length (such as 450, 550 and 650 mμ).[34]

2. Set these numbers equal to $\rho_2 L_{r2}+\gamma_2 L_{g2}+\beta_2 L_{b2}$ for the corresponding three wave lengths and solve for L_{r2}, L_{g2}, and L_{b2}.

By the theorem of Section II, the values thus found will be the luminosity coefficients which give the same luminosity sum ($\rho_2 L_{r2}+\gamma_2 L_{g2}+\beta_2 L_{b2}$) as the original distribution curves gave ($\rho_1 L_{r1}+\gamma_1 L_{g1}+\beta_1 L_{b1}$), not only for the three wave lengths chosen for computation, but for all wave lengths.

The second method is based on relation (6). If the constants k_1 to k_9 be expressed in terms of the constants K_1 to K_9 by the method indicated in Section II (see relation (5) et seq.), there results:

$$\left.\begin{array}{lll} k_1=\frac{1}{\Delta}\begin{vmatrix} K_5 & K_8 \\ K_6 & K_9 \end{vmatrix} & k_4=\frac{1}{\Delta}\begin{vmatrix} K_7 & K_4 \\ K_9 & K_6 \end{vmatrix} & k_7=\frac{1}{\Delta}\begin{vmatrix} K_4 & K_7 \\ K_5 & K_8 \end{vmatrix} \\ k_2=\frac{1}{\Delta}\begin{vmatrix} K_8 & K_2 \\ K_9 & K_3 \end{vmatrix} & k_5=\frac{1}{\Delta}\begin{vmatrix} K_1 & K_7 \\ K_3 & K_9 \end{vmatrix} & k_8=\frac{1}{\Delta}\begin{vmatrix} K_7 & K_1 \\ K_8 & K_2 \end{vmatrix} \\ k_3=\frac{1}{\Delta}\begin{vmatrix} K_2 & K_5 \\ K_3 & K_6 \end{vmatrix} & k_6=\frac{1}{\Delta}\begin{vmatrix} K_4 & K_1 \\ K_6 & K_3 \end{vmatrix} & k_9=\frac{1}{\Delta}\begin{vmatrix} K_1 & K_4 \\ K_2 & K_5 \end{vmatrix} \end{array}\right\} \tag{7}$$

where Δ is an abbreviation for: $\begin{vmatrix} K_1 & K_2 & K_3 \\ K_4 & K_5 & K_6 \\ K_7 & K_8 & K_9 \end{vmatrix}$

From (6) we may write:

$$\left.\begin{array}{l} L_{r2}=\frac{1}{\Delta}\left[L_{r1}\begin{vmatrix} K_5 & K_8 \\ K_6 & K_9 \end{vmatrix}+L_{g1}\begin{vmatrix} K_7 & K_4 \\ K_9 & K_6 \end{vmatrix}+L_{b1}\begin{vmatrix} K_4 & K_7 \\ K_5 & K_8 \end{vmatrix}\right] \\ L_{g2}=\frac{1}{\Delta}\left[L_{r1}\begin{vmatrix} K_8 & K_2 \\ K_9 & K_3 \end{vmatrix}+L_{g1}\begin{vmatrix} K_1 & K_7 \\ K_3 & K_9 \end{vmatrix}+L_{b1}\begin{vmatrix} K_7 & K_1 \\ K_8 & K_2 \end{vmatrix}\right] \\ L_{b2}=\frac{1}{\Delta}\left[L_{r1}\begin{vmatrix} K_2 & K_5 \\ K_3 & K_6 \end{vmatrix}+L_{g1}\begin{vmatrix} K_4 & K_1 \\ K_6 & K_3 \end{vmatrix}+L_{b1}\begin{vmatrix} K_1 & K_4 \\ K_2 & K_5 \end{vmatrix}\right] \end{array}\right\} \tag{8}$$

and compute L_{r2}, L_{g2}, and L_{b2} directly.

[34] In actual cases, the errors of rejection become troublesome if the values of wave length taken differ only slightly.

IV. CALCULATION OF LUMINOSITY COEFFICIENTS FOR FOUR SETS OF DISTRIBUTION CURVES EMBODYING THE SAME MIXTURE DATA AS THE O. S. A. "EXCITATION" CURVES

The mixture data which are embodied in each of the four sets of distribution curves for which luminosity coefficients are to be calculated are those of the O. S. A. "excitation" curves in the form (ρ_0, γ_0, β_0) extrapolated by Priest and Gibson.[35] Table 1 gives (columns 2, 3, and 4) the distribution curves [36] ρ_1, γ_1, and β_1 referring to Abbot-Priest sunlight according to the mixture data embodied in the O. S. A. "excitation" curves. The spectral energy distribution (E) of Abbot-Priest sunlight [37] is given in column 1. The distribution curves of columns 2, 3, and 4 are computed from the O. S. A. "excitation" curves (ρ_0, γ_0, β_0) as follows:

$$\left.\begin{aligned}\rho_1 &= 0.17547\ \rho_0 E\\ \gamma_1 &= 0.17802\ \gamma_0 E\\ \beta_1 &= 0.17482\ \beta_0 E\end{aligned}\right\} \tag{9}$$

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[35] J. Opt. Soc. Am. and Rev. Sci. Inst., **10**, p. 230; 1925.

[36] The function of wave length, ρ_1, gives the distribution of a certain sort of "redness" or "red process" throughout the spectrum of Abbot-Priest sunlight; γ_1 gives the distribution of a certain "green process," and so on. There is no satisfactory proof that these processes have a separate physical existence; hence, the distribution curves given probably do not even approximately characterize any retinal activity (photochemical activity or decomposition of sensitive substance) nor any neural activity (either in the optic nerve, the occipital lobe, or elsewhere); the processes are called primaries by virtue of their position in a mathematical structure (see footnote 6, p. 516) whose present chief justification is that it makes colorimetric computation possible. Although ρ_1, γ_1, and β_1 represent the distribution of certain processes throughout the spectrum of Abbot-Priest sunlight whose claim to the term "primary" is purely mathematical, still it is probable that the distribution of some aspect of the color processes which are really "primary" or "fundamental" may be calculated from ρ_1, γ_1, and β_1 by relation (2). Our present ignorance concerning color vision is such that we know neither the constants of relation (2) nor the aspect of the color processes really "primary" to which these unknown constants refer.

The practice in this paper of referring to ρ, γ, and β briefly as "distribution curves" rather than the more widely used "excitation curves" is due primarily to the fact that the mathematical treatment is general and need not be restricted to the distribution curves of real or imaginary color processes. It is further true that the term "excitation curves" used thus briefly is objectionable to some who wish to reserve that term to apply to processes which actually exist in a physically separated state. Ladd-Franklin has already used "distribution curves" as a substitute term. (See, for example, Appendix to English Edition of the Helmholtz Physiological Optics, **2**, p. 461; 1924.)

[37] I. G. Priest, A Precision Method for Producing Artificial Daylight, Phys. Rev. (2), **11**, p. 502; 1918. J. Opt. Soc. Am. and Rev. Sci. Inst., **12**, p. 479; 1926. Standards Yearbook 1927, p. 204 and Figure 4.

NOTE.—The values of E introduced by Priest into the work of the colorimetry section, Bureau of Standards, some years ago, now in current use, and reproduced in Table 1 of the present paper are not exactly the same as proposed in his earliest publication cited above. Except for change of scale (100 at wave length 590 mμ instead of 100 at wave length 560 mμ) the values given here are as shown in Figure 4 of the Standards Yearbook for 1927. They are given by:

$$E \propto E_{2848} \sin^2\alpha/2$$

where E_{2848} is energy per unit wave length as a function of wave length for the Planckian radiator at 2,848° K. for c_2 at 14,350 micron degrees, and α is the rotation of the plane of polarization of light, as a function of wave length, per millimeter in crystalline quartz.

TABLE 1.—*The distributions of the O. S. A. "Excitations" throughout the spectrum of Abbot-Priest sunlight*

λ in mμ	E (Abbot-Priest sunlight)	ρ_1	γ_1	β_1	$\rho_1 L_{r1}$ ($L_{r1}=0.45$)	$\gamma_1 L_{g1}$ ($L_{g1}=0.54$)	$\beta_1 L_{b1}$ ($L_{b1}=0.01$)	$\rho_1 L_{r1}+\gamma_1 L_{g1}+\beta_1 L_{b1}$
	1	2	3	4	5	6	7	8
380	50.60	--------	--------	442	--------	--------	4.4	4.4
90	56.05	--------	--------	1,274	--------	--------	12.7	12.7
400	61.42	--------	--------	2,717	--------	--------	27.2	27.2
10	66.65	--------	--------	5,045	--------	--------	50.4	50.4
20	71.54	--------	13	7,679	--------	7.0	76.8	83.8
30	76.18	--------	41	12,186	--------	22.1	121.9	144.0
40	80.56	--------	100	14,351	--------	54.0	143.5	197.5
450	84.46	--------	240	14,027	--------	129.6	140.3	269.9
60	88.03	15	595	12,958	6.8	321.3	129.6	457.7
70	91.08	64	1,313	11,098	28.8	709.0	111.0	848.8
80	93.93	231	2,040	7,767	104.0	1,101.6	77.7	1,283.3
90	96.29	693	2,897	3,703	311.8	1,564.4	37.0	1,913.2
500	98.22	1,430	4,546	2,112	643.5	2,454.8	21.1	3,119.4
10	99.77	2,644	6,944	1,517	1,189.8	3,749.8	15.2	4,954.8
20	100.90	4,125	9,160	1,076	1,856.2	4,946.4	10.8	6,813.4
30	101.71	5,479	10,357	765	2,465.6	5,592.8	7.7	8,066.1
40	102.08	6,681	10,958	517	3,006.4	5,917.3	5.2	8,928.9
550	102.20	7,604	11,134	322	3,421.8	6,012.4	3.2	9,437.4
60	101.81	8,325	10,475	196	3,746.2	5,656.5	2.0	9,404.7
70	101.44	8,989	9,336	124	4,045.0	5,041.4	1.2	9,087.6
80	100.93	9,210	7,456	71	4,144.5	4,026.2	.7	8,171.4
90	100.00	9,388	5,269	35	4,224.6	2,845.3	.4	7,070.3
600	98.79	8,841	3,447	17	3,978.4	1,861.4	.2	5,840.0
10	97.45	7,900	1,960	--------	3,555.0	1,058.4	--------	4,613.4
20	95.61	6,291	1,004	--------	2,831.0	542.2	--------	3,373.2
30	93.78	4,690	484	--------	2,110.5	261.4	--------	2,371.9
40	92.38	3,161	164	--------	1,422.4	88.6	--------	1,511.0
650	90.35	1,871	48	--------	842.0	25.9	--------	867.9
60	88.62	1,057	16	--------	475.6	8.6	--------	484.2
70	86.29	606	--------	--------	272.7	--------	--------	272.7
80	84.33	326	--------	--------	146.7	--------	--------	146.7
90	81.99	187	--------	--------	84.2	--------	--------	84.2
700	79.57	105	--------	--------	47.2	--------	--------	47.2
10	77.31	54	--------	--------	24.3	--------	--------	24.3
20	75.21	33	--------	--------	14.8	--------	--------	14.8

The constants in relation (9) are chosen so that (except for errors of rejection of about 5 parts in 100,000 which can not conveniently be avoided): $\Sigma\rho_1\Delta\lambda=\Sigma\gamma_1\Delta\lambda=\Sigma\beta_1\Delta\lambda=1{,}000{,}000$, for $\Delta\lambda=10$ mμ.[38] Although ρ_1, γ_1, and β_1 are specified only at intervals of 10 mμ,[39] other values required may be found by interpolation and extrapolation.[40]

L_{r1}, L_{g1}, and L_{b1} are taken as 0.45, 0.54, and 0.01, respectively;[41] the products $\rho_1 L_{r1}$, $\gamma_1 L_{g1}$, and $\beta_1 L_{b1}$, and the sum of the products appear in columns 5, 6, 7, and 8. In order to show that the differences between this weighted distribution curve sum and standard luminosity data are negligible, these sums[42] divided by the spectral energy, E, of Abbot-Priest sunlight (column 1) are compared in Figure 1 to visibility data taken from the work of Gibson and Tyndall.[43] Visibility via the weighted distribution curve sums is shown by circles with tags; the experimental average visibility reported by Gibson and Tyndall from their data referring to 52 observers is

[38] See footnote 29, p. 522.

[39] This method of specification has thus far been found sufficient; perhaps intervals of 5 mμ (or less) will be justified by subsequent advances in colorimetric technique.

[40] Note that ρ_1, γ_1, and β_1 are to be taken as zero for the 10 mμ intervals for which no value is given in Table 1; a similar remark applies to Tables 2 to 5.

[41] Least square computation shows these to be a reasonable choice for representing luminosity data. (See footnote 21, p. 519.)

[42] Multiplied by the constant, 0.010764, which makes the maximum ordinate approximately unity and, on this account, all the ordinates comparable to the standard visibility data. (See footnote 43.)

[43] K. S. Gibson and E. P. T. Tyndall, The Visibility of Radiant Energy, B. S. Sci. Paper No. 475, pp. 156–159, 174; 1923.

shown by small circles; their adopted or recommended visibility[44] is shown by large circles. It is to be noted that at about half the wave lengths at which this comparison is made, the visibility via the distribution curve sum (small circles with tags) either falls between the Gibson and Tyndall experimental mean (small circles) and the

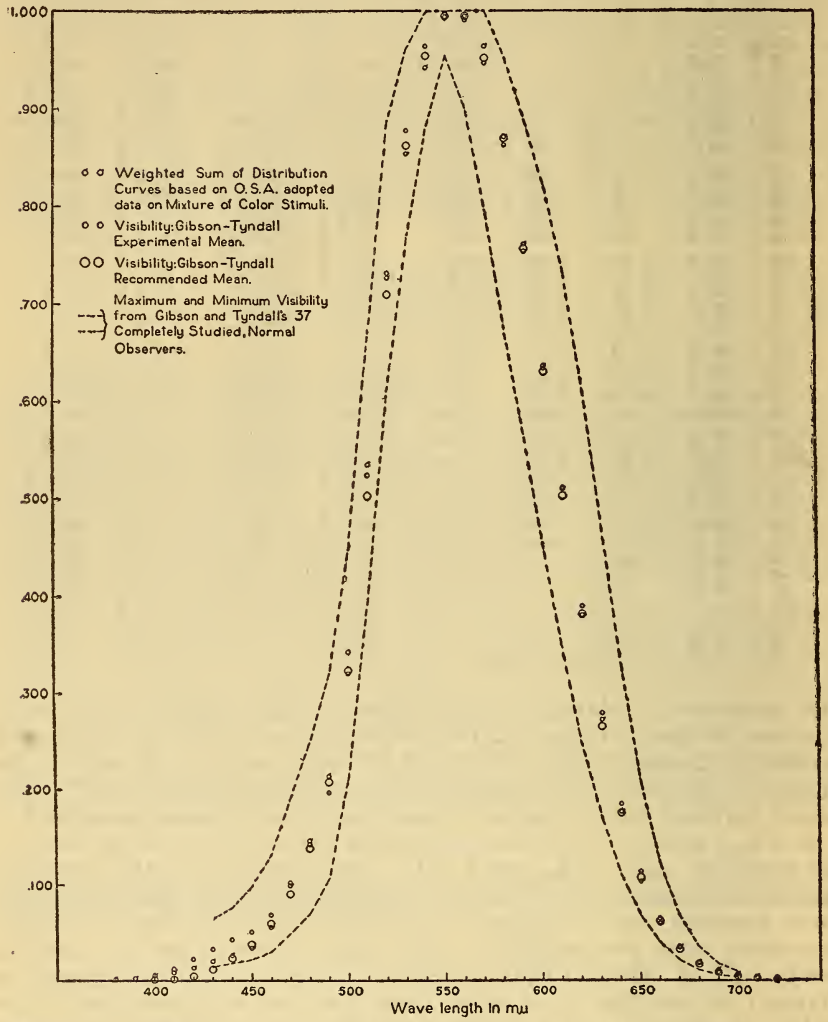

FIGURE 1.—*Demonstration that the distribution curves expressing mixture data are not inconsistent with standard visibility data*

The weighted distribution curve sum is compared (1) to the Gibson-Tyndall experimental mean visibility (B. S. Sci. Paper No. 475, Table 3, second column), (2) to the recommended mean (ibid, Table 3, last column), and (3) to the highest and lowest visibility found experimentally among the 37 completely studied, normal observers (ibid, Table 2). The weighted distribution curve sum is seen to agree closely with visibility directly determined.

values recommended by them (large circles) or coincides with one of them. To make further evident that the discrepancies which do occur are negligible in comparison to individual differences, the

[44] Subsequently adopted as standard by the International Commission on Illumination, 6th meeting, Geneva, July, 1924; Proceedings, pp. 67 and 232.

maximum and minimum visibilities found by Gibson and Tyndall among their 37 completely studied, normal observers[45] are indicated also (points connected by straight, dotted lines). It may be seen from Figure 1 that the differences which do exist between the values of visibility and the sums of the weighted distribution curve ordinates are negligible in comparison to differences between individual visibilities.[46] On this account the failure of the distribution curves to sum exactly to the standard visibility function need cause no further concern. It is true, of course, that the standard visibility differs significantly for some purposes from the distribution curve sums; but the difference is significant because standard values have been formally adopted, not because sufficient experimental data exists to prove that the distribution curve sums fail to represent normal visibility.[47] For this reason, the luminosity of Abbot-Priest sunlight given by the weighted distribution curve sums (column 8 of Table 1) is taken as "standard" in the present paper for the mixture data of the O. S. A. "excitation" curves; and there need be here no more discussion concerning the "degree of approximation" yielded by a given set of luminosity coefficients. If the values for the luminosity coefficients yield the luminosity given in column 8 of Table 1, they are the correct values for that set of distribution curves and constitute the luminosity coefficients; and if the values do not, they are wrong, and do not constitute the luminosity coefficients.

We now propose to take four different sets of distribution curves representing the distribution of four new sets of "red," "green," and "blue" primary color processes throughout the spectrum of Abbot-Priest sunlight according to the mixture data embodied in the O. S. A. "excitation" curves. These four new sets of distribution curves ($\rho_2, \gamma_2, \beta_2$; $\rho_3, \gamma_3, \beta_3$; $\rho_4, \gamma_4, \beta_4$; and $\rho_5, \gamma_5, \beta_5$) are given by columns 2, 3, and 4 of Tables 2 to 5, and are plotted near the bottom of Figures 2 to 5. The distribution curves have been computed from ρ_1, γ_1, and β_1 (see Table 1) in the following way:

$$\left.\begin{aligned}\rho_2 &\equiv 1.0499\rho_1 - 0.1552\gamma_1 + 0.1054\beta_1\\ \gamma_2 &\equiv 0.2023\rho_1 + 0.7977\gamma_1 + 0.0000\beta_1\\ \beta_2 &\equiv 0.0000\rho_1 + 0.0000\gamma_1 + 1.0000\beta_1\end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned}\rho_3 &\equiv 1.0000\rho_1 + 0.0000\gamma_1 + 0.0000\beta_1\\ \gamma_3 &\equiv 0.0000\rho_1 + 1.0000\gamma_1 + 0.0000\beta_1\\ \beta_3 &\equiv 0.3636\rho_1 + 0.3636\gamma_1 + 0.2727\beta_1\end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned}\rho_4 &\equiv 0.9783\rho_1 + 0.0000\gamma_1 + 0.0217\beta_1\\ \gamma_4 &\equiv 0.0000\rho_1 + 1.0000\gamma_1 + 0.0000\beta_1\\ \beta_4 &\equiv 0.0000\rho_1 + 0.0000\gamma_1 + 1.0000\beta_1\end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}\rho_5 &\equiv 1.0000\rho_1 + 0.0000\gamma_1 + 0.0000\beta_1\\ \gamma_5 &\equiv 0.4500\rho_1 + 0.5400\gamma_1 + 0.0100\beta_1\\ \beta_5 &\equiv 0.0000\rho_1 + 0.0000\gamma_1 + 1.0000\beta_1\end{aligned}\right\} \tag{13}$$

[45] One observer was rejected because he was known to be color blind; 14 observers of the 52 were also rejected because they were not studied for the entire visible spectrum, but only for its more luminous portion (490 to 680 mμ). Considering the data referring to the remaining 37 observers for one value of the wave length at a time (at intervals of 10 mμ) the highest value and the lowest value (as recorded in Table 2 of Gibson and Tyndall's paper) for that wave length were plotted.

[46] Comparison has here been only with values of visibility found or adopted by Gibson and Tyndall because that is the most recent work on visibility that has received formal recognition. The same sort of spread among individuals has, however, been found by previous observers; for example, by Coblentz and Emerson (B. S. Sci. Paper No. 303; 1917; see fig. 13).

[47] Note that, since the O. S. A. "excitation" curves are based on so few observers (only three—Koenig, Dieterici, and Abney), they are open to attack as being not truly representative of normal mixture data. It is probable that at some future time the O. S. A. curves will be supplanted by curves more truly representative of normal mixture data. They are not open to attack, however, on the ground of being inconsistent with normal visibility because the differences of the weighted sums of the O. S. A. curves from the standard visibility function (perhaps by good luck) are insignificant relative to individual differences.

TABLE 2.—*The distributions of the König "Grundempfindungen" throughout the spectrum of Abbot-Priest sunlight according to the mixture data embodied in the O. S. A. "excitation" curves*

λ in mμ	ρ_2	γ_2	β_2	$\sigma_2=\rho_2+\gamma_2+\beta_2$	$R_2=\rho_2/\sigma_2$	$G_2=\gamma_2/\sigma_2$	$B_2=\beta_2/\sigma_2$
1	2	3	4	5	6	7	8
380	47	----------	442	489	----------	----------	----------
90	134	----------	1,274	1,408	----------	----------	----------
400	286	----------	2,717	3,003	----------	----------	----------
10	532	----------	5,045	5,577	0.0954	0.0000	0.9046
20	807	10	7,679	8,496	.0950	.0012	.9038
30	1,278	32	12,186	13,496	.0946	.0024	.9029
40	1,497	80	14,351	15,928	.0940	.0050	.9010
450	1,441	192	14,027	15,660	.0920	.0123	.8957
60	1,289	478	12,958	14,725	.0875	.0325	.8800
70	1,033	1,061	11,098	13,192	.0783	.0804	.8413
80	744	1,674	7,767	10,185	.0730	.1644	.7626
90	668	2,451	3,703	6,822	.0979	.3593	.5428
500	1,019	3,916	2,112	7,047	.1446	.5557	.2997
10	1,857	6,074	1,517	9,448	.1965	.6429	.1606
20	3,022	8,142	1,076	12,240	.2469	.6652	.0879
30	4,225	9,370	765	14,360	.2942	.6525	.0533
40	5,369	10,092	517	15,978	.3360	.6316	.0324
550	6,288	10,420	322	17,030	.3692	.6119	.0189
60	7,135	10,040	196	17,371	.4107	.5780	.0113
70	8,001	9,266	124	17,391	.4601	.5328	.0071
80	8,518	7,811	71	16,400	.5194	.4763	.0043
90	9,042	6,102	35	15,179	.5957	.4020	.0023
600	8,748	4,538	17	13,303	.6576	.3411	.0013
10	7,990	3,162	----------	11,152	.7165	.2835	.0000
20	6,449	2,074	----------	8,523	.7567	.2433	.0000
30	4,848	1,335	----------	6,183	.7841	.2159	.0000
40	3,293	771	----------	4,064	.8103	.1897	.0000
650	1,957	417	----------	2,374	.8243	.1757	.0000
60	1,108	226	----------	1,334	.8306	.1694	.0000
70	636	122	----------	758	.8391	.1609	.0000
80	342	66	----------	408	----------	----------	----------
90	196	38	----------	234	----------	----------	----------
700	110	21	----------	131	----------	----------	----------
10	57	11	----------	68	----------	----------	----------
20	35	7	----------	42	----------	----------	----------

TABLE 3.—*The distributions throughout the spectrum of Abbot-Priest sunlight according to the mixture data embodied in the O. S. A. "excitation" curves of primary color processes which suggest a degree of blue-yellow deficiency* [1]

λ in mμ	ρ_3	γ_3	β_3	$\sigma_3=\rho_3+\gamma_3+\beta_3$	$R_3=\rho_3/\sigma_3$	$G_3=\gamma_3/\sigma_3$	$B_3=\beta_3/\sigma_3$
1	2	3	4	5	6	7	8
380	----------	----------	121	121	----------	----------	----------
90	----------	----------	347	347	----------	----------	----------
400	----------	----------	741	741	----------	----------	----------
10	----------	----------	1,376	1,376	0.0000	0.0000	1.0000
20	----------	13	2,099	2,112	.0000	.0062	.9938
30	----------	41	3,338	3,379	.0000	.0121	.9879
40	----------	100	3,951	4,051	.0000	.0247	.9753
450	----------	240	3,913	4,153	.0000	.0578	.9422
60	15	595	3,756	4,366	.0034	.1363	.8603
70	64	1,313	3,528	4,905	.0130	.2677	.7193
80	231	2,040	2,944	5,215	.0443	.3912	.5645
90	693	2,897	2,316	5,906	.1173	.4905	.3921
500	1,430	4,546	2,749	8,725	.1639	.5210	.3151
10	2,644	6,944	3,901	13,489	.1960	.5148	.2892
20	4,125	9,160	5,124	18,409	.2241	.4976	.2783
30	5,479	10,357	5,967	21,803	.2513	.4750	.2737
40	6,681	10,958	6,555	24,194	.2761	.4529	.2709
550	7,604	11,134	6,902	25,640	.2966	.4343	.2692
60	8,325	10,475	6,890	25,690	.3241	.4077	.2682
70	8,989	9,336	6,697	25,022	.3592	.3731	.2676
80	9,210	7,456	6,080	22,746	.4049	.3278	.2673
90	9,388	5,269	5,340	19,997	.4695	.2635	.2670
600	8,841	3,447	4,473	16,761	.5275	.2057	.2669
10	7,900	1,960	3,585	13,445	.5876	.1458	.2666
20	6,291	1,004	2,653	9,948	.6324	.1009	.2667
30	4,690	484	1,881	7,055	.6648	.0686	.2666
40	3,161	164	1,209	4,534	.6972	.0362	.2667
650	1,871	48	698	2,617	.7149	.0183	.2667
60	1,057	16	390	1,463	.7225	.0109	.2666
70	606	----------	220	826	.7337	.0000	.2663
80	326	----------	119	445	----------	----------	----------
90	187	----------	68	255	----------	----------	----------
700	105	----------	38	143	----------	----------	----------
10	54	----------	20	74	----------	----------	----------
20	33	----------	12	45	----------	----------	----------

[1] The distribution curves were calculated from the O. S. A. "Excitation" curves according to relation (6a) by setting $D=4/5$.

TABLE 4.—*The distributions of primary color processes such that the "blue" process contributes nothing to the luminosity of the spectrum*

[Mixture data as of O. S. A. "excitation" curves; spectrum of Abbot-Priest sunlight]

λ in mμ	ρ_4	γ_4	β_4	$\sigma_4=\rho_4+\gamma_4+\beta_4$	$R_4=\rho_4/\sigma_4$	$G_4=\gamma_4/\sigma_4$	$B_4=\beta_4/\sigma_4$
1	2	3	4	5	6	7	8
380	10	----------	442	452	----------	----------	----------
90	28	----------	1,274	1,302	----------	----------	----------
400	59	----------	2,717	2,776	----------	----------	----------
10	110	----------	5,045	5,155	0.0213	0.0000	0.9787
20	167	13	7,679	7,859	.0212	.0017	.9771
30	265	41	12,186	12,492	.0212	.0033	.9755
40	312	100	14,351	14,763	.0211	.0068	.9721
450	305	240	14,027	14,572	.0209	.0165	.9626
60	296	595	12,958	13,849	.0214	.0430	.9357
70	304	1,313	11,098	12,715	.0239	.1033	.8728
80	395	2,040	7,767	10,202	.0387	.2000	.7613
90	758	2,897	3,703	7,358	.1030	.3937	.5033
500	1,445	4,546	2,112	8,103	.1783	.5610	.2606
10	2,620	6,944	1,517	11,081	.2364	.6267	.1369
20	4,059	9,160	1,076	14,295	.2839	.6407	.0753
30	5,377	10,357	765	16,499	.3259	.6277	.0464
40	6,547	10,958	517	18,022	.3633	.6080	.0287
550	7,446	11,134	322	18,902	.3939	.5890	.0170
60	8,148	10,475	196	18,819	.4330	.5566	.0104
70	8,796	9,336	124	18,256	.4818	.5114	.0068
80	9,011	7,456	71	16,538	.5449	.4508	.0043
90	9,185	5,269	35	14,489	.6339	.3637	.0024
600	8,649	3,447	17	12,113	.7140	.2846	.0014
10	7,728	1,960	----------	9,688	.7977	.2023	.0000
20	6,154	1,004	----------	7,158	.8597	.1403	.0000
30	4,588	484	----------	5,072	.9046	.0954	.0000
40	3,092	164	----------	3,256	.9496	.0504	.0000
650	1,830	48	----------	1,878	.9744	.0256	.0000
60	1,034	16	----------	1,050	.9848	.0152	.0000
70	593	----------	----------	593	1.0000	.0000	.0000
80	319	----------	----------	319	----------	----------	----------
90	183	----------	----------	183	----------	----------	----------
700	103	----------	----------	103	----------	----------	----------
10	53	----------	----------	53	----------	----------	----------
20	32	----------	----------	32	----------	----------	----------

TABLE 5.—*The distribution throughout the spectrum of Abbot-Priest sunlight of one of the sets of primary color processes for which neither the "red" nor the "blue" primary color process contributes to the luminosity of the spectrum. The distribution of the "green" primary color process is the same as the distribution of luminosity.*

λ in mμ	ρ_5	γ_5	β_5	$\sigma_5=\rho_5+\gamma_5+\beta_5$	$R_5=\rho_5/\sigma_5$	$G_5=\gamma_5/\sigma_5$	$B_5=\beta_5/\sigma_5$
1	2	3	4	5	6	7	8
380	----------	4	442	446	----------	----------	----------
90	----------	13	1,274	1,287	----------	----------	----------
400	----------	27	2,717	2,744	----------	----------	----------
10	----------	50	5,045	5,095	0.0000	0.0098	0.9902
20	----------	84	7,679	7,763	.0000	.0108	.9892
30	----------	144	12,186	12,330	.0000	.0117	.9883
40	----------	198	14,351	14,549	.0000	.0136	.9864
450	----------	270	14,027	14,297	.0000	.0189	.9811
60	15	458	12,958	13,431	.0011	.0341	.9648
70	64	849	11,098	12,011	.0053	.0707	.9240
80	231	1,283	7,767	9,281	.0249	.1382	.8369
90	693	1,913	3,703	6,309	.1098	.3032	.5869
500	1,430	3,119	2,112	6,661	.2147	.4682	.3171
10	2,644	4,955	1,517	9,116	.2900	.5435	.1664
20	4,125	6,813	1,076	12,014	.3433	.5671	.0896
30	5,479	8,066	765	14,310	.3829	.5637	.0535
40	6,681	8,929	517	16,127	.4143	.5537	.0321
550	7,604	9,437	322	17,363	.4379	.5435	.0185
60	8,325	9,405	196	17,926	.4644	.5247	.0109
70	8,989	9,088	124	18,201	.4939	.4993	.0068
80	9,210	8,171	71	17,452	.5277	.4682	.0041
90	9,388	7,070	35	16,493	.5692	.4287	.0021
600	8,841	5,840	17	14,698	.6015	.3973	.0012
10	7,900	4,613	----------	12,513	.6313	.3687	.0000
20	6,291	3,373	----------	9,664	.6510	.3490	.0000
30	4,690	2,372	----------	7,062	.6641	.3359	.0000
40	3,161	1,511	----------	4,672	.6766	.3234	.0000
650	1,871	868	----------	2,739	.6831	.3169	.0000
60	1,057	484	----------	1,541	.6859	.3141	.0000
70	606	273	----------	879	.6894	.3106	.0000
80	326	147	----------	473	----------	----------	----------
90	187	84	----------	271	----------	----------	----------
700	105	47	----------	152	----------	----------	----------
10	54	24	----------	78	----------	----------	----------
20	33	15	----------	48	----------	----------	----------

It is to be noted that each of these four new sets of distribution curves represents the same body of mixture data as ρ_1, γ_1, and β_1, because each set satisfies (2).[48] The theorem of Section II must, therefore, hold for these four sets of curves; hence, the methods of computing the luminosity coefficients (Sec. III) which depend on that theorem can be used. We proceed to apply those methods.

The first step in the first method is to compute $\rho_1 L_{r1}+\gamma_1 L_{g1}+\beta_1 L_{b1}$ for three convenient wave lengths. This has already been done for all wave lengths (column 8 of Table 1); we choose 450, 550, and 650 mμ, which give 269.9, 9,437.4, and 867.9, respectively. Then we set up the following four triplets of simultaneous equations:

$$\left.\begin{array}{rrrl} 1{,}441\ L_{r2}+ & 192\ L_{g2}+ & 14{,}027\ L_{b2}= & 269.9 \\ 6{,}288\ L_{r2}+ & 10{,}420\ L_{g2}+ & 322\ L_{b2}= & 9{,}437.4 \\ 1{,}957\ L_{r2}+ & 417\ L_{g2} & = & 867.9 \end{array}\right\}$$

$$\left.\begin{array}{rrrl} & 240\ L_{g3}+ & 3{,}913\ L_{b3}= & 269.9 \\ 7{,}604\ L_{r3}+ & 11{,}134\ L_{g3}+ & 6{,}902\ L_{b3}= & 9{,}437.4 \\ 1{,}871\ L_{r3}+ & 48\ L_{g3}+ & 698\ L_{b3}= & 867.9 \end{array}\right\}$$

$$\left.\begin{array}{rrrl} 305\ L_{r4}+ & 240\ L_{g4}+ & 14{,}027\ L_{b4}= & 269.9 \\ 7{,}446\ L_{r4}+ & 11{,}134\ L_{g4}+ & 322\ L_{b4}= & 9{,}437.4 \\ 1{,}830\ L_{r4}+ & 48\ L_{g4} & = & 867.9 \end{array}\right\}$$

$$\left.\begin{array}{rrrl} & 270\ L_{g5}+ & 14{,}027\ L_{b5}= & 269.9 \\ 7{,}604\ L_{r5}+ & 9{,}437\ L_{g5}+ & 322\ L_{b5}= & 9{,}437.4 \\ 1{,}871\ L_{r5}+ & 868\ L_{g5} & = & 867.9 \end{array}\right\}$$

and solve for the luminosity coefficients.[49]

The second method consists merely of the substitution of the values of L_{r1}, L_{g1}, L_{b1} (which are 0.4500, 0.5400, and 0.0100, respectively, in each case) and of the values of the constants, K_1 to K_9 (which are given in relations (10) to (13)) in relation (8).

The results of computation by these two methods appear in Table 6. It may be seen that the first method (which is, perhaps, less provocative of arithmetical blunders, though it takes about ten times as long) yields computational accuracy up to two or three in the fourth decimal place. Rejection error in the second method may readily be made negligible.

TABLE 6.—*Results of solutions for the luminosity coefficients*

Distribution curves	L_r		L_g		L_b	
	Method 1	Method 2	Method 1	Method 2	Method 1	Method 2
$\rho_2, \gamma_2, \beta_2$	0.2872	0.2875	0.7330	0.7327	−0.0203	−0.0203
$\rho_3, \gamma_3, \beta_3$	.4367	.4367	.5268	.5267	.0366	.0367
$\rho_4, \gamma_4, \beta_4$	.4601	.4600	.5401	.5400	.0000	.0000
$\rho_5, \gamma_5, \beta_5$	.0000	.0000	1.0000	1.0000	.0000	.0000

[48] We know, without troubling to integrate with respect to wave length, that each curve has the same area as ρ_1, γ_1, and β_1 because the sum of the coefficients of ρ_1, γ_1, and β_1 in each of the 12 equations given in relations (10), (11), (12), and (13) is unity (except for discrepancies of one in the last place due to errors of rejection in the computation).

[49] The numbers in the left members of the four triplets of equations may be obtained, of course, from columns 2, 3, and 4 of Tables 2, 3, 4, and 5 for wave lengths 450, 550, and 650 mμ.

L_{r2}, L_{g2}, and L_{b2} were further checked by computation of $\rho_2 L_{r2} + \gamma_2 L_{g2} + \beta_2 L_{b2}$ which was found to agree for all wave lengths with $\rho_1 L_{r1} + \gamma_1 L_{g1} + \beta_1 L_{b1}$ in accordance with relation (4); hence, the values found, $L_{r2} = 0.2875$, $L_{g2} = 0.7327$, and $L_{b2} = 0.0203$ are the luminosity coefficients for the distribution curves of the König "Grundempfindungen" according to the mixture data embodied in the O. S. A. "excitation" curves.[50] L_{r3}, L_{g3}, and L_{b3} were checked by computation from relation (6a)[51] and found to be correct. There is no need of checking L_{r4}, L_{g4}, and L_{b4} or L_{r5}, L_{g5}, and L_{b5} because the constants, K_1 to K_9, by which ρ_4, γ_4, and β_4 (relation (12)) and ρ_5, γ_5, and β_5 (relation (13)) were computed, were chosen so as to obtain the luminosity coefficients found. The correctness of this choice can be seen by inspection of relations (12)[52] and (13).

V. DEMONSTRATION THAT THE NEW SETS OF DISTRIBUTION CURVES EMBODY THE MIXTURE DATA OF THE O. S. A. "EXCITATION" CURVES

Since from their derivation (see relations (10), (11), (12), and (13)) the new sets of distribution curves all satisfy relation (2), we know that they must embody the same mixture data as the O. S. A. "excitation" curves. A demonstration more convincing than this formal proof, perhaps, is to compute by each set of curves the dominant wave length, purity, and luminosity of the color stimuli obtained by filters illuminated by Abbot-Priest sunlight, and compare the values obtained. If all five sets of distribution curves yield the same values of dominant wave length, we could say that the identity of their mixture data had been checked. Since luminosity coefficients have been found for each set of distribution curves, we can also compute the colorimetric purity and transmission of the filters (which gives a measure of the luminosity of the color stimulus) for Abbot-Priest sunlight. If all five sets of distribution curves yield the same values of colorimetric purity and transmission for Abbot-Priest sun-

[50] It is convenient to point out here that the computation of the luminosity coefficients for this approximation to the Grundempfindungskurven demonstrates that the failure reported by Ives (see footnote 18, p. 519) of the König curves to yield consistent luminosity values was due (as he implies but does not state explicitly) to his rejection of a negative value for the "blue" coefficient. With a negative value, Ives could have obtained as close agreement with König's original Grundempfindungskurven as he did with his transformed curves and positive coefficients. It should be noted that the convexity between 400 and 450 mμ, which the spectrum locus according to the König-Ives curves shows, is not due to the transformation of the coordinates, nor to the avoidance of a negative luminosity coefficient, nor to luminosity considerations at all; it is due to the fact that the original mixture data on which the König-Ives curves are based is different from that on which the O. S. A. curves are based. The essential difference lies in the inclusion in the König-Ives curves of data by partially color-blind observers. There is fairly good ground to believe that such inclusion is justifiable when the König Grundempfindungskurven are being considered because it was shown by König that the color responses of his protanopic and deuteranopic observers could be predicted closely from the Grundempfindungskurven of the normal observers merely by the omission of either the red or the green curve. Although it is the author's opinion that such inclusion is still questionable, nevertheless there is some reason to suspect that an improvement was wrought by the introduction of data by the partially color-blind observers because Guild (A Critical Survey of Modern Developments in the Theory and Technique of Colorimetry and Allied Sciences, Proc. Optical Convention, London, Pt. I, p. 57; 1926) who uses the König-Ives curves for routine computation criticizes the O. S. A. "excitation" curves in these terms, "My own view is that these are in some respects less satisfactory than the König-Ives curves. One serious defect is the absence of any reappearance of red in the violet region, which on Weaver's basis would be all of one color from 440 mμ onwards." This seems to imply that Guild has evidence that the König-Ives curves, which give a marked convexity in the spectrum locus on the mixture diagram from 400 to 450 mμ, agree with experimental results by his observers better than the O. S. A. "excitation" curves which do not have that convexity (compare Guild's fig. 12 with figs. 2 to 5 of the present paper). It is also true that Houstoun (Phil. Mag. (6), **45**, p. 176; 1923) with rather crude apparatus found this convexity, but it is probable that stray light was present in sufficient quantity in the photometric field of Houstoun's apparatus to give his result. (For the effect of stray light under these conditions, see Abney, on the Change of Hue of Spectrum Colors by Dilution with White Light, Proc. Roy. Soc., A., **83**, 1909.)

[51] Relation (6a) proves, of course, to be a special case of relation (6).

[52] Note that 45/46 is approximately 0.9783, and that 1/46 is approximately 0.0217. See relation (6) et seq.

light, we could say that not only had the identity of their mixture data been checked a second time, but also had we checked (for a third time) the accuracy of the luminosity coefficients, since those coefficients enter into the computation. We proceed to make these computations, and for the purpose of checking we choose four (hypothetical, nonexistent) filters defined by their spectral transmissions which are zero for all wave lengths except as follows:

Filter No. (m)	Spectral transmission (T_λ) [1]	
1	$T_{490}=1.00000$	$T_{610}=0.37680$
2	$T_{640}=1.00000$	$T_{500}=.99291$
3	$T_{590}=1.00000$	$T_{440}=.72571$
4	$T_{460}=1.00000$	$T_{550}=.47877$

[1] Of course, these values of spectral transmission were not chosen at random. The large wave-length regions of zero transmission (all but two intervals of 10 mμ for each hypothetical filter) result in the avoidance of much computation and, at the same time, provide a more rigorous test of the colorimetric identity of the new sets of distribution curves than that afforded by actual filters. The transmissions which are not zero were so chosen that the dominant wave length computed in the usual, routine way from the O. S. A. "excitation" curves for each filter ($m=1$, $m=2$, $m=3$, $m=4$) is as nearly as convenient an exact multiple of 10 (530, 570, 510, and 480, respectively) when expressed in millimicrons. Such dominant wave lengths may be computed with more certainty (say to 0.01 instead of to 0.1 mμ) than values of dominant wave length for which the ordinates of the distribution curves are not specified explicitly but only indirectly by interpolation.

TABLE 7.—*Computation leading up to the evaluation of dominant wave length, colorimetric purity, and transmission of filters ($m=1$ to 4) illuminated by Abbot-Priest sunlight*

Distribution curves	Filter 1 $m=1$			Filter 2 $m=2$			Filter 3 $m=3$			Filter 4 $m=4$		
$\rho_1, \gamma_1, \beta_1$ $n=1$	[1] 693	2,897	3,703	1,420	4,514	2,097	0	73	10,415	15	595	12,958
	[1] 2,977	739	0	3,161	164	0	9,388	5,269	35	3,641	5,331	154
	[2] 3,670	3,636	3,703	4,581	4,678	2,097	9,388	5,342	10,450	3,656	5,926	13,112
	[3] 0.3334	0.3303	----------	0.4034	0.4119	----------	0.3728	0.2122	----------	0.1611	0.2611	----------
$\rho_2, \gamma_2, \beta_2$ $n=2$	668	2,451	3,703	1,012	3,888	2,097	1,086	58	10,415	1,289	478	12,958
	3,011	1,191	0	3,293	771	0	9,042	6,102	35	3,011	4,989	154
	3,679	3,642	3,703	4,305	4,659	2,097	10,128	6,160	10,450	4,300	5,467	13,112
	0.3337	0.3304	----------	0.3892	0.4212	----------	0.3788	0.2304	----------	0.1879	0.2390	----------
$\rho_3, \gamma_3, \beta_3$ $n=3$	693	2,897	2,316	1,420	4,514	2,730	0	73	2,867	15	595	3,756
	2,977	739	1,351	3,161	164	1,209	9,388	5,269	5,340	3,641	5,331	3,304
	3,670	3,636	3,667	4,581	4,678	3,939	9,388	5,342	8,207	3,656	5,926	7,060
	0.3345	0.3314	----------	0.3471	0.3544	----------	0.4093	0.2329	----------	0.2197	0.3561	----------
$\rho_4, \gamma_4, \beta_4$ $n=4$	758	2,897	3,703	1,435	4,514	2,097	226	73	10,415	296	595	12,958
	2,912	739	0	3,092	164	0	9,185	5,269	35	3,565	5,331	154
	3,670	3,636	3,703	4,527	4,678	2,097	9,411	5,342	10,450	3,861	5,926	13,112
	0.3334	0.3303	----------	0.4005	0.4139	----------	0.3734	0.2120	----------	0.1686	0.2588	----------
$\rho_5, \gamma_5, \beta_5$ $n=5$	693	1,913	3,703	1,420	3,097	2,097	0	144	10,415	15	458	12,958
	2,977	1,738	0	3,161	1,511	0	9,388	7,070	35	3,641	4,518	154
	3,670	3,651	3,703	4,581	4,608	2,097	9,388	7,214	10,450	3,656	4,976	13,112
	0.3329	0.3312	----------	0.4059	0.4083	----------	0.3470	0.2667	----------	0.1681	0.2288	----------

[1] Products. [2] Sums. [3] r, g.

The method of computing dominant wave length, colorimetric purity and transmission follows closely that recommended by the O. S. A. Committee on Colorimetry for 1920–21 [53] except for the computation of purity for which papers by Priest [54] and Judd [55] should be consulted. Table 7 shows (1) the products of the distribution curve ordinates (ρ_n, γ_n, β_n) by transmissions (T_m) [56] of filter, m; (2) the sums of these products which when multiplied by 10 constitute the evaluation of $\int_0^\infty \rho_n T_m d\lambda$, $\int_0^\infty \gamma_n T_m d\lambda$, and $\int_0^\infty \beta_n T_m d\lambda$; and (3) two of the three trilinear coordinates (r, g, b) [57] of the colors whose dominant wave length and colorimetric purity we are computing. The trilinear coordinates of filter, m, via distribution curves ρ_n, γ_n, and β_n are defined as:

$$\left.\begin{aligned} r &= \frac{\int_0^\infty \rho_n T_m d\lambda}{\int_0^\infty \rho_n T_m d\lambda + \int_0^\infty \gamma_n T_m d\lambda + \int_0^\infty \beta_n T_m d\lambda} \\ g &= \frac{\int_0^\infty \gamma_n T_m d\lambda}{\int_0^\infty \rho_n T_m d\lambda + \int_0^\infty \gamma_n T_m d\lambda + \int_0^\infty \beta_n T_m d\lambda} \\ b &= \frac{\int_0^\infty \beta_n T_m d\lambda}{\int_0^\infty \rho_n T_m d\lambda + \int_0^\infty \gamma_n T_m d\lambda + \int_0^\infty \beta_n T_m d\lambda} \end{aligned}\right\} \tag{14}$$

Figures 2 to 5 show mixture diagrams on which the spectrum locus has been plotted, the r-coordinate of the spectrum, R_n, being plotted against the g-coordinate, G_n. (See columns 6 and 7 of Tables 2 to 5.)[58] On these diagrams, every color is represented by a point; these diagrams possess the property [59] that the points representing the colors which result from the additive mixture of two light stimuli producing any two colors fall upon the straight line connecting the points representing those colors. The "neutral" ($r=1/3$, $g=1/3$) point is connected by straight lines to the points representing the colors of the extremes of the spectrum; these lines serve to divide the diagram into the spectral colors and the nonspectral, or purple, colors. The points representing the colors of filters Nos. 2, 3, and 4 ($m=2$, 3, and 4) are also shown; the color of filter No. 1 ($m=1$) is so close to the "neutral" color that its point has been omitted for the sake of

[53] L. T. Troland, J. Opt. Soc. Am. and Rev. Sci. Inst., **6**, pp. 527–596; 1922.

[54] I. G. Priest, The Computation of Colorimetric Purity, J. Opt. Soc. Am. and Rev. Sci. Inst., **9**, pp. 503–520; 1924. Also The Computation of Colorimetric Purity II, J. Opt. Soc. Am. and Rev. Sci. Inst., **13**, pp. 123–132; 1926.

[55] D. B. Judd, The Computation of Colorimetric Purity, J. Opt. Soc. Am. and Rev. Sci. Inst., **13**, pp. 133–152; 1926.

[56] The subscript, m, here denotes the filter, not (as is usual) the wave length.

[57] r and g are given in Table 7, although b is not. But from relation (14) we know that $r+g+b=1$; hence we may easily find b from r and g since it is $1-r-g$.

[58] The trilinear coordinates (R_n, G_n, B_n) of the spectrum referred to Abbot-Priest sunlight as the "neutral" ($r=1/3$, $g=1/3$) stimulus have been computed only for wave lengths between 410 and 670 mμ, inclusive, because beyond these wave-length limits according to the mixture data of the O. S. A. "excitation" curves in their extrapolated form the trilinear coordinates are constant. If, for example, the computation were carried out for 380, 390, and 400 mμ, the trilinear coordinates found would differ from those of 410 mμ only by reason of computational errors of rejection.

[59] This property constitutes one expression of Newton's law of color mixture; on this account, the diagrams are sometimes called "Newton's mixture diagrams."

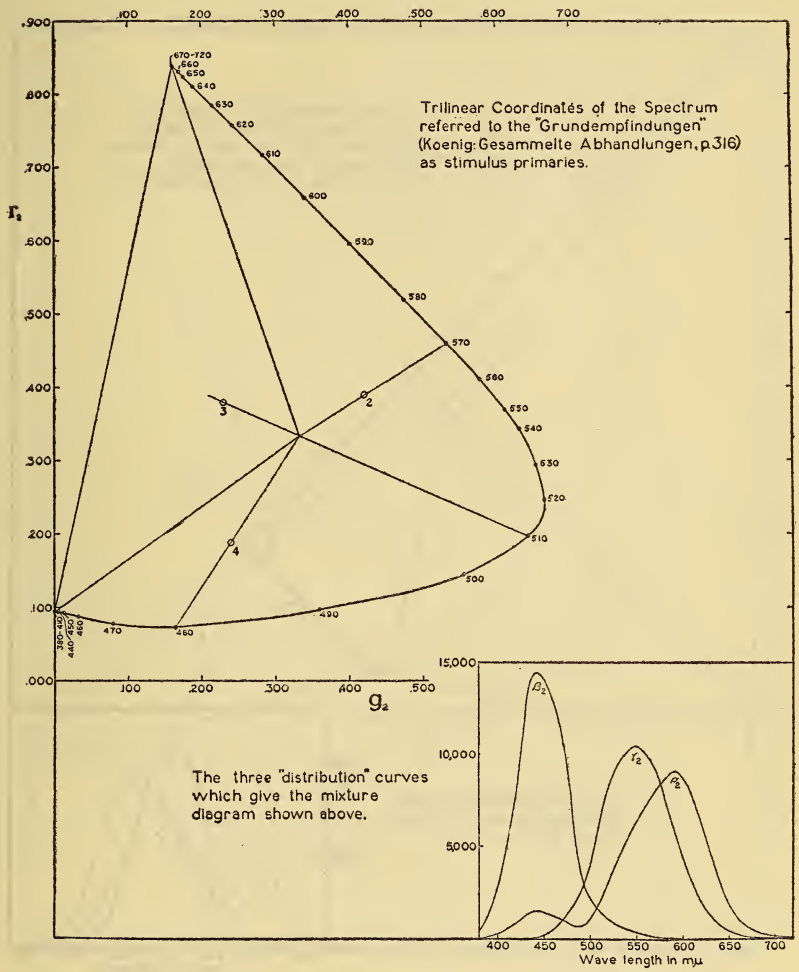

FIGURE 2.—*The distribution curves, ρ_2, γ_2, and β_2, and the mixture diagram showing the spectrum locus given by ρ_2, γ_2, and β_2*

Note that the points representing the colors of filters 2, 3, and 4 (see Table 7, *r* and *g*) fall on the straight lines connecting the "neutral" point with 570, 510, and 480 mμ, respectively.

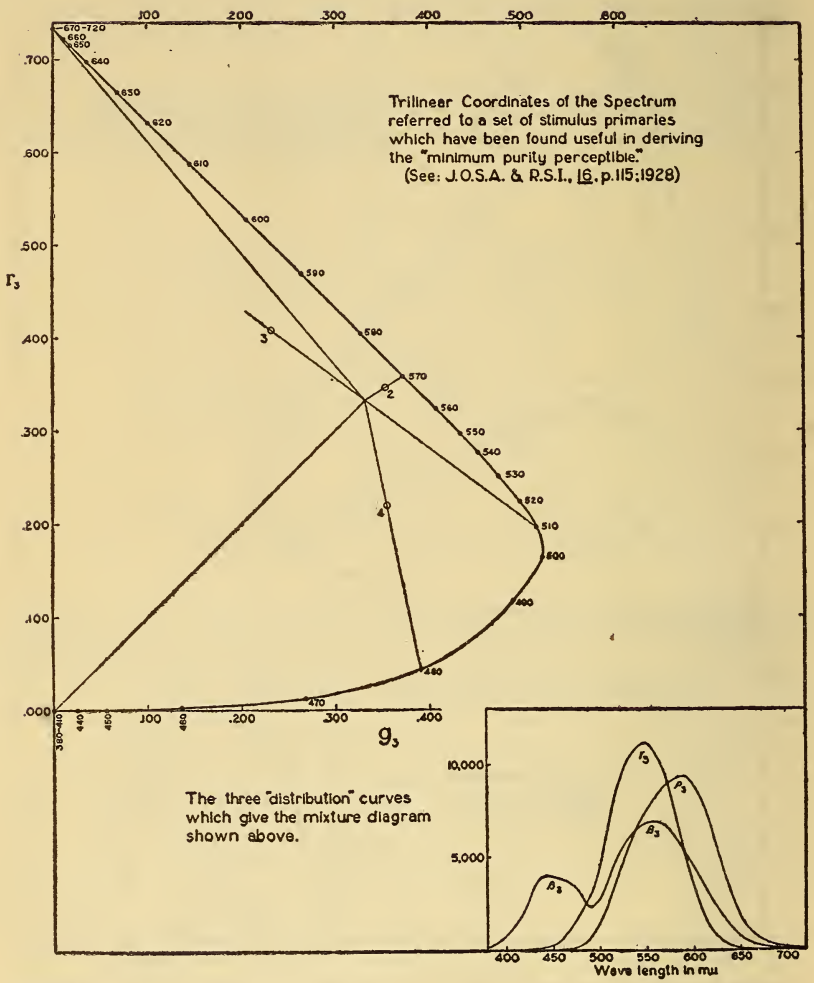

FIGURE 3.—*The distribution curves, ρ_3, γ_3, and β_3 and the mixture diagram showing the spectrum locus given by ρ_3, γ_3, and β_3*

Note that the points representing the colors of filters 2, 3, and 4 (see Table 7, *r* and *g*) fall on the straight lines connecting the "neutral" point with 570, 510, and 480 mμ, respectively.

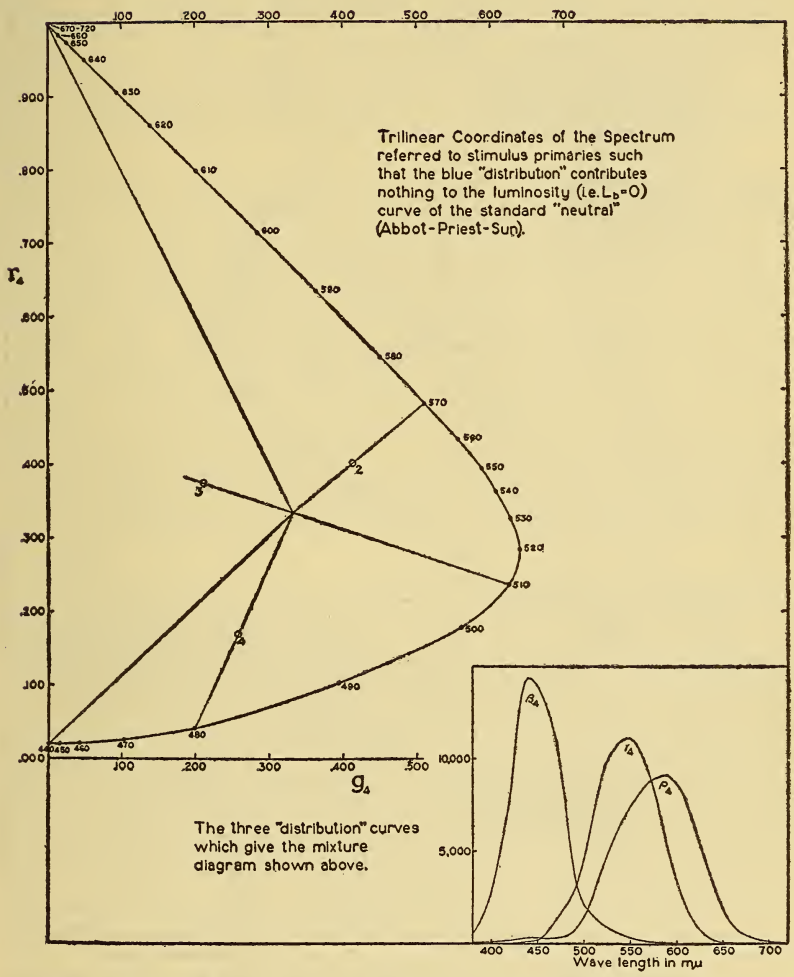

FIGURE 4.—*The distribution curves, ρ_4, γ_4, and β_4, and the mixture diagram showing the spectrum locus given by ρ_4, γ_4, and β_4*

Note that the points representing the colors of filters 2, 3, and 4 (see Table 7, *r* and *g*) fall on the straight lines connecting the "neutral" point with 570, 510, and 480 mμ, respectively.

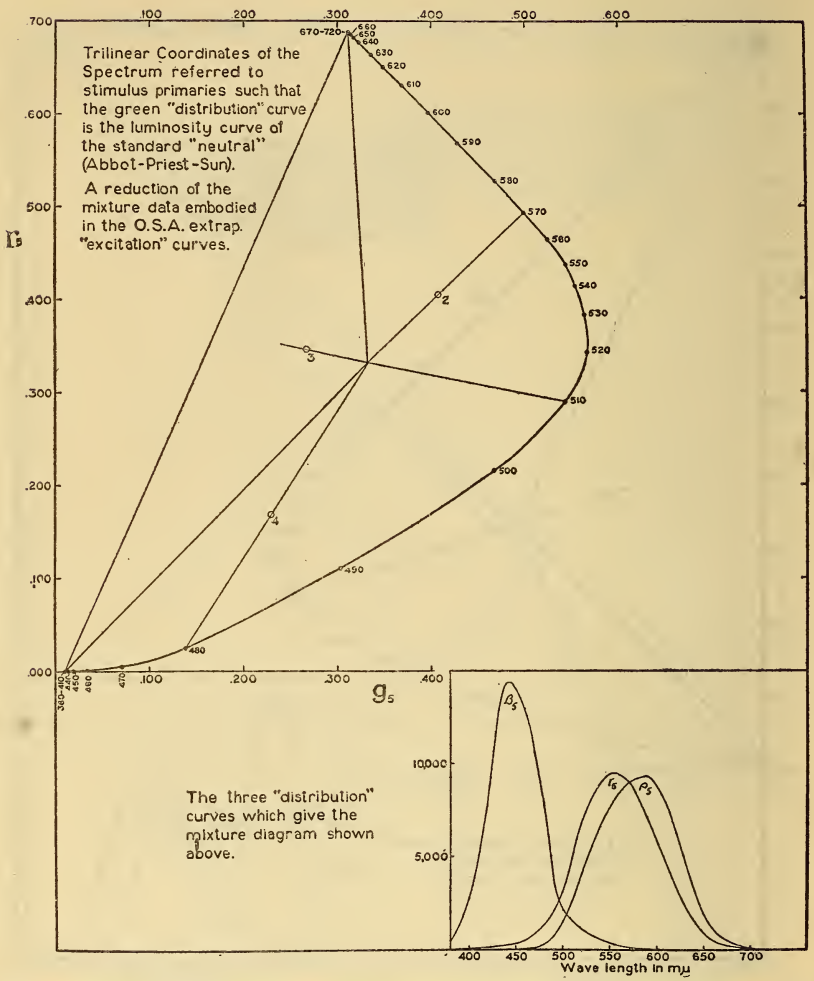

FIGURE 5.—*The distribution curves, ρ_5, γ_5, and β_5, and the mixture diagram showing the spectrum locus given by ρ_5, γ_5, and β_5*

Note that the points representing the colors of filters 2, 3, and 4 (see Table 7, *r* and *g*) fall on the straight lines connecting the "neutral" point with 570, 510, and 480 mμ, respectively.

clarity. The fact that all these apparently different mixture diagrams (figs. 2 to 5), represent the same body of mixture data may be approximately demonstrated by the fact that the points representing the colors of filters Nos. 2, 3, and 4 fall in each case on the straight line connecting the "neutral" point with the point representing the color produced by light of wave length 570, 510, and 480 mμ, respectively.[60] We may also see now what the primary color processes are whose distributions throughout the spectrum are given by ρ_n, γ_n, and β_n ($n=1$ to 5). The "red" primary process for set, n, of distribution curves is that designated on the mixture diagram by $r_n=1$, $g_n=0$; the "green" primary process is that designated by $r_n=0$, $g_n=1$; and the "blue" primary process is that designated by $r_n=g_n=0$.[61]

We proceed with the more exact method for attaining the demonstration that the several mixture diagrams embody identical mixture data (computation of dominant wave length and purity of the filter). The condition that a point (r, g) on the mixture diagram lies on the same straight line with the "neutral" point ($r=1/3$, $g=1/3$) and a point (R, G) of the spectrum locus is that: $(r-1/3)/(g-1/3)=(R-1/3)/(G-1/3)$. We proceed to compute $(r-1/3)/(g-1/3)$ for each of the four filters ($m=1$ to 4) and for each of the five sets of distribution curves ($n=1$ to 5) and to compare it with $(R-1/3)/(G-1/3)$ for the proper wave length and set of distribution curves. Table 8 shows this comparison. It is seen that discrepancies exist in every case; examination shows, however, that these discrepancies may be ascribed to rejection error, since they are smaller than the change introduced by an error of one in the fourth place of the values of r and g.

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[60] It is also true for each diagram (figs. 2 to 5) that the points representing the colors of filters Nos. 1 to 4 ($m=1$ to 4) lie upon the straight lines connecting the points representing the colors produced by light of wave lengths 490 and 610 mμ, 500 and 640 mμ, 440 and 590 mμ, and 460 and 550 mμ, respectively. These straight lines were omitted for the sake of clarity.

[61] Reference to fig. 3 shows that, for $n=3$, the "blue" primary process is that which is evoked by homogeneous radiation of wave length 380 to 410 mμ, when the visual mechanism is in such a state of adaptation that Abbot-Priest sunlight evokes the "white" or "neutral" process. Similarly fig. 4 shows that, for $n=4$, the "red" primary process is that evoked by homogeneous radiation of wave length 670 to 720 mμ. The other primary processes plainly can not be evoked by homogeneous radiation of any wave length for a state of adaptation for which Abbot-Priest sunlight yields the "white" response because the points representing them do not fall on the spectrum locus. It is equally plain that they can not be evoked by a mixture of homogeneous radiation with Abbot-Priest sunlight because they lie outside the area bounded by the spectrum locus and the straight line joining its extremes. Take, for example, the "blue" primary process for $n=4$ (fig. 4); this process could be evoked by a stimulus of dominant wave length about 452 mμ (since the point representing it lies on the same straight line as the points representing the colors evoked by Abbot-Priest sunlight and homogeneous radiation of wave length about 452 mμ) and of purity greater than unity (since the point lies farther from the "neutral" point than does that of the color evoked by homogeneous radiation of wave length 452 mμ). Such a stimulus, of course, does not exist; hence, it is proper to say that this color process which we have happened to choose as a primary process in one case ($n=4$) to place at $r_4=0$, $g_4=0$ of our mixture diagram corresponds to an imaginary stimulus if the visual mechanism of the observer be in the "neutral state"; that is, in such a state of adaptation that Abbot-Priest sunlight evokes the "white" response. It is not proper to say, however, that the color process, itself, is nonexistent, or virtual, because it is possible to evoke colors for the visual mechanism in one state of adaptation which can not be evoked if the visual mechanism is in a different state. We say, therefore, that some of the color processes which we have happened to choose for primaries correspond to imaginary stimuli for the visual mechanism in the "neutral state"; whether these processes, themselves, are real or not, is not definitely known nor is it, for our purposes, important.

TABLE 8.—*Results of computation incident to evaluation of dominant wave length and colorimetric purity of the four filters (m=1 to 4) illuminated by Abbot-Priest sunlight*

Set (n) of distribution curves	Filter 1 $m=1$			Filter 2 $m=2$			Filter 3 $m=3$			Filter 4 $m=4$		
	$\frac{r-1/3}{g-1/3}$	$\frac{R-1/3}{G-1/3}$	Difference	$\frac{r-1/3}{g-1/3}$	$\frac{R-1/3}{G-1/3}$	Difference	$\frac{r-1/3}{g-1/3}$	$\frac{R-1/3}{G-1/3}$	Difference	$\frac{r-1/3}{g-1/3}$	$\frac{R-1/3}{G-1/3}$	Difference
$n=1$	−0.0333	−0.0114	0.0219	+0.8919	+0.8911	0.0008	−0.3262	−0.3260	0.0002	+2.3850	+2.3851	0.0001
$n=2$	−.1379	−.1225	.0154	+.6359	+.6356	.0003	−.4422	−.4419	.0003	+1.5419	+1.5411	.0008
$n=3$	−.6316	−.5787	.0529	+.6540	+.6508	.0032	−.7570	−.7565	.0005	−4.9825	−4.9914	.0089
$n=4$	−.0333	−.0251	.0082	+.8337	+.8338	.0001	−.3306	−.3303	.0003	+2.2107	+2.2101	.0006
$n=5$	+.1905	+.2153	.0248	+.9680	+.9675	.0005	−.2057	−.2060	.0003	+1.5809	+1.5807	.0002

Table 9 shows the final results of computation of dominant wave length, purity, and transmission of the four filters illuminated by Abbot-Priest sunlight. The dominant wave lengths are computed from the differences recorded in Table 8, together with the known rate of variation of dominant wave length with $(r-1/3)/(g-1/3)$ for the wave lengths in question.[62] The purity, p, is computed from:

$$p=\frac{RL_r+GL_g+BL_b}{RL_r+GL_g+BL_b+(g-G)/(1-3g)} \qquad (15)$$ [63]

TABLE 9.—*Computed values of the dominant wave length, colorimetric purity, and transmission of filters (m=1 to 4) illuminated by Abbot-Priest sunlight*

Set (n) of distribution curves	Filter 1 $m=1$			Filter 2 $m=2$			Filter 3 $m=3$			Filter 4 $m=4$		
	Λ in mμ	p	T_s	Λ in mμ	p	T_s	Λ in mμ	p	T_s	Λ in mμ	p	T_s
$n=1$	528.33	−0.0153	0.03652	570.01	+0.5521	0.04608	509.99	−0.6461	0.07215	480.00	+0.3235	0.04976
$n=2$	528.91	−.0156	.03651	570.01	+.5524	.04609	509.98	−.6463	.07213	480.01	+.3235	.04976
$n=3$	525.18	−.0152	.03652	570.02	+.5511	.04609	509.96	−.6463	.07214	480.01	+.3235	.04976
$n=4$	529.39	−.0152	.03651	570.00	+.5524	.04608	509.99	−.6459	.07214	480.00	+.3236	.04976
$n=5$	528.45	−.0158	.03651	570.01	+.5518	.04608	510.01	−.6461	.07214	480.00	+.3236	.04976

The transmission (T_s) for Abbot-Priest sunlight is computed as:

$$T_s=\frac{L_{r_n}\int_0^\infty \rho_n T_m d\lambda+L_{g_n}\int_0^\infty \gamma_n T_m d\lambda+L_{b_n}\int_0^\infty \beta_n T_m d\lambda}{L_{r_n}\int_0^\infty \rho_n d\lambda+L_{g_n}\int_0^\infty \gamma_n d\lambda+L_{b_n}\int_0^\infty \beta_n d\lambda} \qquad (16)$$

where L_{r_n}, L_{g_n}, and L_{b_n} are given in Table 6, method 2; one-tenth of $\int_0^\infty \rho_n T_m d\lambda$, $\int_0^\infty \gamma_n T_m d\lambda$, and $\int_0^\infty \beta_n T_m d\lambda$ are given in Table 7; $\int_0^\infty \rho_n d\lambda$, $\int_0^\infty \gamma_n d\lambda$, and $\int_0^\infty \beta_n d\lambda$ are taken as 1,000,000,[64] and their

[62] This method is applicable because the dominant wave lengths fall so closely on the even 10 mμ where the distribution curve ordinates (and, hence $R-1/3$ and $G-1/3$) are definitely specified. The transmissions of the four filters were chosen with this end in view. See footnote 1 to tables, p. 534.

[63] D. B. Judd, The Computation of Colorimetric Purity, J. Opt. Soc. Am. and Rev. Sci. Inst., **13**, p. 135; 1926. Relation (15) is form "Judd G." In 3 cases out of the 20, the "Judd R" form (Judd, loc. cit.) was used because of its greater convenience.

[64] See footnote 38, p. 525.

weighted sum is 1,000,000, since the luminosity coefficients have been adjusted so as to sum to unity.

It is seen that (with one exception, the dominant wave length of filter No. 1, $m=1$ [65]) to a high degree of accuracy the dominant wave length, colorimetric purity, and transmission for Abbot-Priest sunlight of the four filters is independent of the choice (among the five, $n=1$ to 5) of a set of distribution curves. It has been demonstrated, therefore, that the four new sets ($n=2$ to 5) of distribution curves embody the same mixture data as the O. S. A. "excitation" curves ($n=1$).

VI. CHOICE OF A SET OF DISTRIBUTION CURVES

As has been pointed out in the introduction, convenience in discussion of psychophysiological theory may justify the temporary adoption of any one of a wide range of sets of distribution curves. A set very similar to ρ_2, γ_2, and β_2 was chosen by König and Dieterici because it suggested a simple way of explaining the facts of congenital dichromasy. ρ_2, γ_2, and β_2 have also served to suggest (to the present writer) a simple way of explaining the "hue change" which is observed when "white" light is added to certain portions of the spectrum.[66] ρ_3, γ_3, and β_3 were used by the present writer [67] to represent the experimentally known facts concerning sensibility to purity change.[68] Neither of these sets of distribution curves could be used advantageously in the routine computation of dominant wave length, colorimetric purity, and transmission because both are less convenient for computation than the O. S. A. "excitation" curves (ρ_0, γ_0, β_0) from which they were derived.[69] A set of distribution curves proposed by Luther [70] refers to such a choice of stimulus primaries that $L_b=0$. The computational advantage of this choice of primaries is quite evident. These same advantages may be obtained by the use of ρ_4, γ_4, and β_4, which also have a further advantage over those of Luther because they involve more ordinates having the value zero.

It seems, however, that there exists a choice of stimulus primaries which expresses the mixture data of the O. S. A. "excitation" curves in a way that affords still greater convenience in the routine computation of colorimetric purity, and transmission than any choice of primary color processes heretofore proposed. The set of distribution curves resulting from this choice of primary color processes is ρ_5, γ_5, and β_5 defined in relation (13). The computational superi-

[65] The explanation for the deviation of the dominant wave length of this filter from 530 mμ by more than 4 mμ lies, of course, in the close approach to zero of the purity of that filter (−0.015). It has already been shown (Table 8) that these discrepancies are ascribable to errors of rejection in computing.

[66] W. de W. Abney, On the Change of Hue of Spectrum Colours on Dilution with White Light, Proc. Roy. Soc., A., **83**, 1909. Also Abney, Researches in Colour Vision, London, Longmans-Green, pp. 255–266; 1913. D. B. Judd, The Empiric Relation between Dominant Wave Length and Purity, J. Opt. Soc. Am. and Rev. Sci. Inst., **14**, p. 475; 1927.

[67] D. B. Judd, Sensibility to Color Change Determined from the Visual Response Functions: Extension to Complete and Partial Dichromasy, J. Opt. Soc. Am. and Rev. Sci. Inst., **16**, p. 115; 1928.

[68] Reference to fig. 3 will show one of the characteristic properties of this set of distribution curves. It is seen that the spectrum locus (at about 572 mμ) approaches much closer to the "neutral" point ($r=1/3$, $g=1/3$) than it does in figs. 2, 4, or 5. This is the dominant wave length of the color stimulus which is, for normal observers, most confusible with the "neutral" stimulus. (See I. G. Priest and F. G. Brickwedde, The Minimum Perceptible Colorimetric Purity as a Function of Dominant Wave Length with Sunlight as Neutral Standard, J. Opt. Soc. Am. and Rev. Sci. Inst., **13**, p. 306; 1926.)

[69] They are less convenient chiefly because they involve fewer ordinates having the value zero.

[70] R. Luther, Aus dem Gebiet der Farbreizmetrik, Zs. f. Techn. Physik, **8**, p. 544; 1927.

ority of this set of distribution curves can best be established by enumerating the conveniences which it involves:

1. No negative ordinates.[71]
2. Twenty out of one hundred and five ordinates specified are zero.[72]
3. $L_r = L_b = 0$.[73]

Although the advantage of having $L_{r_5} = L_{b_5} = 0$ (which makes $L_{g_5} = 1$, and makes γ_5 identical with the luminosity function) [74] will scarcely be denied, still it is of interest to point out exactly how simple the formulas to be used in routine computation of transmission and colorimetric purity become:

$$T_s = \int_0^\infty \gamma_5 T_m d\lambda \Big/ \int_0^\infty \gamma_5 d\lambda \qquad (17)\ ^{75}$$

$$p = \frac{G(1-3g)}{g(1-3G)} = \frac{3-1/g}{3-1/G} \qquad (18)\ ^{76}$$

We do not propose that ρ_5, γ_5, and β_5 be adopted forthwith because the confusion of changing from the O. S. A. "excitation" curves in their extrapolated form (ρ_0, γ_0, β_0) would probably more than outweigh the subsequent gain in computational convenience. However, when it does become desirable to adopt a new set of mixture data which is more representative of average, normal vision than that embodied in the O. S. A. "excitation" curves, it is evident that the primaries embodied in ρ_5, γ_5, and β_5 could be taken with advantage for routine computation rather than those embodied in the O. S. A. "excitation" curves.

I wish again to express my indebtedness to Dr. L. B. Tuckerman for his proof of the theorem of Section II. Doctor Tuckerman's suggestions have resulted in a considerably improved presentation of the introduction.

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[71] In this respect ρ_5, γ_5, and β_5 are no better than a number of other sets of distribution curves. (See footnote 13, p. 518.)

[72] In this respect, these distribution curves are second only to the O. S. A. "excitation" curves themselves (ρ_0, γ_0, β_0), which have 30 out of 105 ordinates specified equal to zero. They are, therefore, somewhat less convenient than the O. S. A. "excitation" curves in their extrapolated form for the computation of dominant wave length.

[73] There exists an infinity of sets of distribution curves which have this property; this set is the most advantageous with respect to the two points already mentioned. Another of this infinity of sets was approximately evaluated by Schrödinger (Ueber das Verhältnis der Vierfarben- zur Dreifarbentheorie, Sitz. Akad. Wiss., Wien, Abt. IIa, **134**, pp. 471–490; 1925) in order to demonstrate that the mixture data may suggest a four-color as well as a three-color theory of vision. In fact, a four-color theory is mathematically equivalent to a three-color theory in which one of the three primary color processes is taken as "white." We say the set of distribution curves was "approximately evaluated" because the mixture data embodied in Schrödinger's curves are those of the König Grundempfindungskurven which are not quite the same as those of the O. S. A. "excitation" curves which we have used. Furthermore the luminosity curve which Schrödinger derives from the König Grundempfindungskurven is distinctly too high for short wave lengths. The discrepancies are, in fact, the same ones which led Exner (see footnote 19, p. 519) to make unwarranted arbitrary changes in the König Grundempfindungskurven. Though it was not essential to his purpose, Schrödinger could have derived an acceptable luminosity curve from the König Grundempfindungskurven (see Sec. IV especially footnote 50, p. 533) by using a negative luminosity coefficient (his term, γ, see his relation (1)) of the proper magnitude for the blue curve.

[74] It has already been pointed out that if any luminosity coefficients be chosen whose sum is unity, we may find many sets of distribution curves to which they apply (see relation (6) et seq). This set (ρ_5, γ_5, β_5) is one for which $L_r=0$, $L_g=1$ and $L_b=0$, the choice being made purely for the purpose of simplifying computation.

[75] Compare with the general formula (16).

[76] Compare with the general formula (15). Of course, this form is not always applicable to computation because when G approaches 1/3, it is difficult to avoid serious errors of rejection. In these cases the equally simple form, $p=G(r-g)/g(R-G)$, may be used.

VII. APPENDIX—A GRAPHICAL INTERPRETATION

It has been pointed out [77] that the existence of an infinite number of sets of distribution curves satisfying relation (2) corresponds to the possibility of choosing an infinite number of sets of primary color processes in terms of which to express a given body of mixture data. If all color processes are represented on a mixture diagram (such as figs. 2 to 5), each process by a point, this possibility corresponds to that of choosing any three points in the plane as reference points (with trivial exceptions already noted). It may be easily shown that if the triangle formed by connecting these three points by straight lines incloses the spectrum locus, then the distribution curves of the primary color processes represented by the three reference points will have positive ordinates throughout. Reference to Figures 2 to 5 shows that the four sets of distribution curves we have taken as examples are cases of this kind.[78]

Similarly the freedom of choice for the luminosity coefficients (see relation (6), et seq.) may be interpreted in terms of the selection of reference points on the mixture diagram which represent the primary color processes. We proceed to develop the basis for such an interpretation. The equation

$$rL_r+gL_g+bL_b=0 \tag{19}$$

defines a straight line on the mixture diagram which represents color processes of zero luminosity; hence, we shall call it after Schrödinger [79] "Alychne" (lightless). Naturally this line does not intersect the spectrum locus; the stimuli corresponding to the color processes represented by points on the Alychne are all imaginary. If one of these points be chosen to represent one of the primary color processes of a three-color system, the luminosity coefficient attached to that color process will be zero.[80] By analogy it is natural to suppose that all points which fall on the same side of the Alychne as the spectrum locus are associated with positive luminosity while those which fall on the opposite side refer to color processes of negative luminosity; hence, we might guess that if one of the first group of points were chosen to represent a primary color process the luminosity coefficient would be positive and if one of the second group of points were chosen the luminosity coefficient would be negative. We proceed to substantiate this guess.

Theorem.[81]—If a luminosity coefficient is positive (negative), the point representing the corresponding primary color process lies on the same (opposite) side of the Alychne as the spectrum locus.

Proof.—As already noted, the Alychne does not intersect the spectrum locus; hence, if we can show that any point lying within the spectrum locus and the straight line joining its extremes lies on the same (opposite) side of the Alychne as the point representing the

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[77] See footnotes 9 and 10, p. 517.

[78] The O. S. A. "excitation" curves, the Elementarempfindungskurven of König, the König-Ives curves, and the "sensation" curves of Abney also satisfy this condition. See footnotes 13 and 18, pp. 518 and 519.

[79] E. Schrödinger, Ueber das Verhältnis der Vierfarben- zur Dreifarbentheorie, Sitz. Akad. Wiss., Wien, Abt. IIa, **134**, p. 476; 1925.

[80] Although this was pointed out by Schrödinger without proof, the accuracy of his statement may readily be checked. Assume (for example) $L_r=0$. We must see whether the point corresponding to the red primary process falls on the Alychne; that is, we must see whether the coordinates ($r=1$, $g=0$, $b=0$) satisfy relation (19). Substitution shows this to be the case.

[81] For simplicity this theorem is stated in a form applicable only to trilinear diagrams in which the spectrum locus does not pass through infinity. This requires that $\rho+\gamma+\beta$ be not less than zero for any wave length.

primary color process having a positive (negative) luminosity coefficient, the theorem will be proven. We elect for the sake of convenience to show that the "white" point ($r=1/3$, $g=1/3$, $b=1/3$) is so situated.[82]

Assume (for example) L_b greater than (less than) zero; we must show that the point ($r=0$, $g=0$, $b=1$) lies on the same (opposite) side of the Alychne as the "white" point. Now the Alychne may be defined in terms of r, g, L_r, and L_g alone since:

$$r+g+b=1 \qquad \text{(See relation (14))}$$

and

$$L_r+L_g+L_b=1 \qquad \text{(See Table 6)}^{83}$$

thus

$$(2L_r+L_g-1)r+(L_r+2L_g-1)g+1-L_r-L_g=0 \qquad (20)$$

Furthermore, we know that any point (r_1, g_1) is on the same (opposite) side of the line $Ar+Bg+C=0$ as the origin provided:

$$\frac{Ar_1+Bg_1+C}{\pm\sqrt{A^2+B^2}}$$

is positive (negative), where the denominator of the fraction is given the same algebraic sign as C.[84]

Now, from (20), $C=1-L_r-L_g=L_b$, and since, by hypothesis, L_b is greater than (is less than) zero, the theorem is proved (both cases) if for $r_1=g_1=1/3$, Ar_1+Bg_1+C is greater than zero. But, from (20);

$$A=2L_r+L_g-1$$
$$B=L_r+2L_g-1$$
$$C=1-L_r-L_g$$

and substitution shows that when $r_1=g_1=1/3$, $Ar_1+Bg_1+C=1/3$ which is greater than zero.

Therefore, for sets of distribution curves like those of the present paper, it has been proven that the spectrum locus is on the same side of the Alychne as the point representing the primary color process if the corresponding luminosity coefficient is positive, and on the opposite side, if the luminosity coefficient is negative.

In Figure 6 are shown the same trilinear coordinates of the spectrum (R_2, G_2; see columns 6 and 7 of Table 2) that are shown in Figure 2; but the scale is so chosen that the Alychne and the points representing the primary color processes of some of the other sets of distribution curves dealt with may also be shown. From (20) and from the values of the luminosity coefficients found for ρ_2, γ_2, and β_2 (see Table 6), the equation of the Alychne is:

$$r_2=0.066-2.446\ g_2$$

The points representing the primary color processes whose distributions according to wave length are evaluated by ρ_2, γ_2, and β_2 are, of

[82] See footnote 29, p. 522. Since we make use of the "white" point at ($r=1/3$, $g=1/3$, $b=1/3$) this proof is good for only those sets of distribution curves (like those of the present paper) whose "white" point falls at that place on the mixture diagram. Similar proof for sets of distribution curves not satisfying this restric tion can readily be formulated.

[83] The condition that the luminosity coefficients sum to unity is imposed for the sake of convenience throughout the present paper. It makes the present proof easier to state, but similar proof can readily be given for sets of distribution curves whose luminosity coefficients sum to any positive value.

[84] Ashton, Plane and Solid Analytic Geometry, New York, Scribners, p. 54; 1902.

course ($r=1$, $g=0$, $b=0$), ($r=0$, $g=1$, $b=0$), and ($r=0$, $g=0$, $b=1$), respectively. These points have been labeled R'_2, G'_2, and B'_2, respectively. The analogous points for the O. S. A. "excitation" curves (see Table 1) have been labeled R'_1, G'_1, and B'_1. R'_1 and B'_1 are located at the extremes of the spectrum locus while G'_1 falls at the intersection of the tangents to the spectrum locus at R'_1 and B'_1. The points representing the primary color processes whose distributions throughout the spectrum are given by ρ_3, γ_3, and β_3

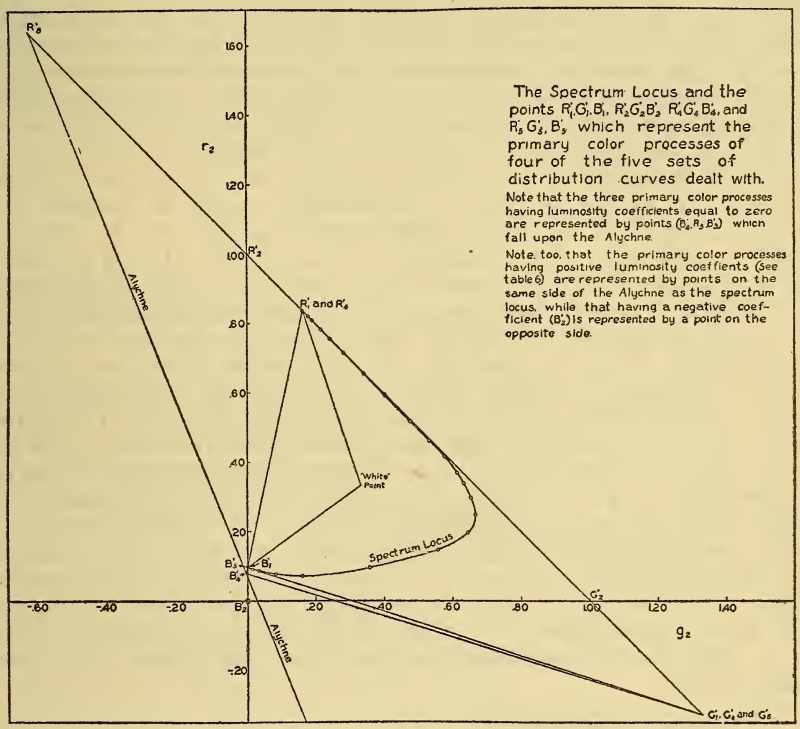

FIGURE 6.—*The mixture diagram and spectrum locus of Figure 2 to such a scale that the Alychne (lightless line) may be shown together with the points representing some of the primary color processes dealt with*

The relation between the algebraic sign of the luminosity coefficient and the position of the point representing the corresponding primary color process with reference to the spectrum locus and the Alychne is exemplified.

are not shown because two of these points (those representing the red and green primary processes) are located too far from the origin to be shown with any convenient scale for r_2 and g_2. The points representing the primary color processes of ρ_4, γ_4, and β_4 (see Table 4) are labeled R'_4, G'_4, and B'_4. R'_4 like R'_1 is located at the long-wave-length end of the spectrum locus. G'_4 does not coincide exactly with G'_1, but it is so nearly coincident that the departure can not be conveniently shown on Figure 6. B'_4 is located at the intersection of the Alychne with the straight line joining the extremes of the spectrum locus. R'_5, G'_5, and B'_5, like R'_1, G'_1, and B'_1 all fall upon the tangents to the extremes of the spectrum locus, the difference being that while R'_1 and B'_1 are the points of tangency

R'_5 and B'_5 are the intersections of the tangents with the Alychne. It is to be noted, of course, that, in accord with the theorem just proven, the points (R'_1, G'_1, B'_1; R'_2, G'_2; R'_4, G'_4; and G'_5) representing primary color processes to which positive luminosity coefficients are attached (see Table 6) fall on the same side of the Alychne as the spectrum locus; also the point (B'_2) representing the only primary color process to which a negative luminosity coefficient is attached is the only point which falls on the side of the Alychne opposite to the spectrum locus.

If, then, we would avoid negative ordinates in the distribution curves of the three primary color processes we must choose processes such that the resulting triangle completely incloses the spectrum locus (as, for example, any of the five sets of distribution curves dealt with in the present paper). If we would make as many ordinates as possible of the distribution curves zero, we should choose processes such that the sides of the resulting triangle are coincident with the spectrum locus for as great a wave-length range as possible (for example, the O. S. A. "excitation" curves, ρ_1, γ_1, and β_1, best, see Table 1; and ρ_5, γ_5, and β_5, next best, see Table 5). If we would avoid negative luminosity coefficients, we must avoid, in our selection of primary color processes, all processes represented by points which fall on the side of the Alychne opposite to the spectrum locus (as has been done, for example, for all choices of primary process shown on Figure 6 except B'_2).[85] If we would make two of the luminosity coefficients equal to zero, we must choose two of the primary color processes such that the points which represent them fall on the Alychne (for example, Schrödinger's Vierfarbentheorie curves see footnote 73, p. 544; and ρ_5, γ_5, and β_5). If two of the luminosity coefficients are zero, the distribution curve of the remaining primary color process coincides with the luminosity curve.[86] ρ_5, γ_5, and β_5 are the distribution curves of a choice of primary color processes which to a considerable degree combine all the properties which have just been enumerated. It is by virtue of these properties that ρ_5, γ_5, and β_5 are advantageous from the standpoint of routine computation.

WASHINGTON, August, 1929.

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[85] We may note that the König-Grundempfindungskurven refer to a "blue" primary process whose point is separated from the spectrum locus by the Alychne. In so far, then, as it is justifiable from the standpoint of psychophysiological interpretation to take the König primary color processes as identical with some aspect of the retinal or post-retinal activity which actually occurs, it is justifiable to conclude that the activity of the "blue" component of the visual mechanism involves an inhibition of the brightness aspect of the response. In other words, we may obtain from König's work a suggestion of the "specific darkening power" of blue light which was advocated by König's theoretical enemies (the Hering school).

Ives believed (see footnote 18, p. 519) that König's blue primary process had a sufficiently arbitrary basis to justify another choice of primary that is, a choice which does not involve a negative luminosity coefficient. The König-Ives curves embody a blue primary which falls on the same side of the Alychne as the spectrum locus.

[86] The fact that a certain green process for the sake of computational convenience has been endowed with a distribution curve throughout the spectrum which is identical with the luminosity curve of the spectrum has, of course, no importance from the standpoint of psychophysiological theory whatever. The endowment could be granted any color process represented by a point on the mixture diagram not on the Alychne. If of this infinity of color processes there is one process which seems to have from the psychophysical standpoint a real claim to a distribution curve identical with the luminosity curve, that is the "white" process; and in Schrödinger's Vierfarbentheorie curves (see footnote 73, p. 544) the "white" process is so endowed. These curves embody properties which permit them to be given important theoretical interpretations; but it is not surprising that they should be distinctly inferior to ρ_5, γ_5, and β_5 from the standpoint of convenience in routine computation.