THE PRINCIPLES OF MEASUREMENT AND OF CALCULATION IN THEIR APPLICATION TO THE DETERMINATION OF DIOPHANTINE QUANTITIES

By Edward W. Washburn

ABSTRACT

A Diophantine quantity is defined as a quantity which is numerically conditioned in such a way that it is a member of a set of known quantities. The measurement of such a quantity in the laboratory is therefore a problem of identification. The principles of measurement and of calculation and the precision aspects of measurements involving one or more Diophantine quantities present features quite different from those associated with ordinary physical measurements. In particular, the individual determinations should not be averaged and the most favorable experimental values are not necessarily those which are closest to the true value. The principles of "precision of measurement" as set forth in the numerous treatises on that subject are not applicable to the measurement of Diophantine quantities and may lead to erroneous conclusions.

In the present paper an experimental procedure for measuring such quantities is described, and appropriate methods for treating the experimental data are developed. The treatment of the subject is based upon the "principle of maximum error" and the methods of Diophantine analysis instead of the theory of probability and the calculus. Examples of Diophantine problems in chemistry and physics are given.

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I. INTRODUCTION

1. THE PROBLEM

In the case of most physical measurements, the value finally adopted is a mean obtained by averaging in some manner the results of a number of determinations. To this final "best" value a precision measure is attached as an index or reliability. The methods of combining experimental data so as to obtain the most reliable value

and the computation of an appropriate index of reliability are discussed in an extensive literature dealing with the theory of probability and precision of measurements.

All discussions of this character with which the author is familiar seem, however, to be confined to two types of cases which can be described somewhat as follows:

Type 1.—The "correct" value of the quantity sought is and must necessarily always remain unknown. It is, however, assumed to be a member of a continuum of "possible" values lying between the limits determined by the reliability index. Most of the so-called physical constants belong to this type. The assumption is that if the chemical composition and physical condition of the system are completely defined, then there exists a single and definite value for each of its physical constants or properties. The problem is to find a given number of significant figures of this value and to fix the limits of uncertainty.

Type 2.—In the second type of cases the individuals are either inaccessible to direct measurement or, if they are subjected to measurement, the result for a given individual is of no particular interest; that is, the actual individuals, as individuals, are inaccessible or unimportant. They are, however, members of a definable group, and the behavior of the group as found by studying a sufficient number of representative samples is capable of numerical representation by the application of statistical methods. This behavior is sometimes embodied in that fictitious entity known as the "average individual," who not infrequently has no counterpart among the actual individuals.

In addition to these two types, there exists also a third type which apparently has not been discussed by writers on this subject. This type may be described as follows:

Type 3.—The quantity sought is perfectly definite and belongs to an actual individual. The possible values of the quantity lying between the extremes determined by the accuracy of the method of measurement do not, however, form a continuum, but instead, all of these possible values, including the true or correct value, are members of a finite set or series of known values, and the problem which presents itself is the definite and certain identification of the quantity sought with one of these known quantities.

A quantity of this character may be called a "numerically conditioned" or Diophantine quantity. Following are some examples of such quantities:

The quantity is—

1. A positive integer.
2. A negative prime.
3. The logarithm of an integral multiple of 3.
4. A member of the series $2^{\frac{1}{n}}$, in which n is a positive integer.
5. A member of the set: (-37.3 ± 0.8), (-20 ± 2), (-15.3 ± 0.5), (10 ± 1), (30.1 ± 0.2), (34.8 ± 0.7).

In examples 1 and 2 each member of the set is exactly known; in examples 3 and 4 each member of the set is known (or knowable) to any desired degree of accuracy; in example 5 each member of the set is known with a stated degree of accuracy.

The principles of measurement and of calculation and the precision aspects of problems involving Diophantine quantities present features quite different from those associated with the first two types described above. It is the purpose of this paper to discuss these features for cases of this type.

Before proceeding to the detailed consideration of this type, however, it will be necessary to prepare the ground by an examination of certain features of the general problem of errors of measurement and in particular to formulate what we shall call the "principle of maximum error," a principle which in theory is applicable to any type of physical measurement, but which in practice appears to be almost valueless except in connection with the determination of Diophantine quantities or of functionally conditioned quantities.[1]

2. THE PRINCIPLE OF MAXIMUM ERROR

In the measurement of any physical quantity characteristic of a given constant (or reproducible) physical system, the accuracy of the result obtained is determined by the following elements: (1) The observer, (2) the technic, and (3) the equipment. The combination of these three elements [2] we shall designate as the "method."

The errors present in the immediate observational data will be of two classes which will be designated as vectorial and nonvectorial, respectively. The class of vectorial errors has a unidirectional (plus or minus) component. Such errors are usually called "systematic errors." They must be eliminated by proper calibration and standardization of the "method." The remaining errors are of the nonvectorial type and it is this class only which will be considered in what follows. They will be designated simply as "errors."

If, now, repeated observations of the same quantity are made, these observations will differ from one another, from the mean and from the true value of the quantity by varying amounts. Experience shows, however, that, barring mistakes, all of the observations will lie within certain finite limits above and below the true value. The deviations of the individual observations from the true value we shall call the "errors of the observations." The largest error which could conceivably be made with the "method" employed will be called the "maximum error of the method." This maximum error is a characteristic of the "method" and is an important element in connection with the measurement of numerically conditioned quantities, since it can be made the basis of a general method of treating the experimental data. In principle it could likewise be similarly applied to experimental data on nonconditioned quantities, but in practice it is of little value with such quantities because the maximum error can not ordinarily be determined with the required precision. Unless the estimated maximum error is substantially less than twice the actual maximum error, it is practically valueless in connection with nonconditioned quantities except possibly as a criterion for the exclusion of an observation from the mean.

[1] Application of the principle of maximum error to the determination of the parameters of a function connecting two or more directly measured quantities is discussed by Campbell (Norman Campbell, Measurement and Calculation, Longmans, Green & Co. (London), 1928, p. 169, et seq.). The procedures recommended by Campbell have the merit of simplicity which certainly can not be said of most of those given in the standard treatises in this field. See, however, the simple procedure suggested by Edgeworth (Hermathena, 6, No. 13; 1887).

[2] The first of these elements—the observer—is in part suppressed when recording instruments are used.

3. DETERMINATION OF THE MAXIMUM ERROR

Unfortunately no exact directions can be given for determining the "maximum error" of a "method." The best that can be done is to indicate two possible types of procedure. The final details must, however, be determined by the good judgment of the investigator.

(a) BY ANALYSIS

Resolve the "method" into its operation elements.[3] Determine for, or assign to, each operation element an appropriate "maximum error." With the aid of the functional relation connecting the operation elements with the final result, select a reasonable value for the corresponding maximum error in the result.

(b) BY TRIAL

Apply the experimental technic repeatedly to the same (preferably known) magnitude until a sufficient number of observations have been obtained to permit the construction of a satisfactory error-frequency curve. After eliminating any observations which are obviously mistakes, take as the "maximum error" the maximum observed deviation, increased by some reasonable factor of safety.

It will be noticed that in the above descriptions we have employed such expressions as "obviously," "reasonable," and "appropriate" which are not defined and which are essentially incapable of exact definition. In fact, except perhaps in purely statistical problems, no mathematical method of treating the observational data can entirely replace the scientific good judgment of the investigator. At most it can only suggest an orderly procedure and point out the logical inferences involved.

As an example of a measurement which is itself a simple operational element, let us consider the measurement of the distance between two parallel fine lines on a plane surface. We will suppose that one of the lines is brought into coincidence with a graduation mark of a standard meter divided into millimeters and that the position of the second line is estimated, to the nearest tenth millimeter or better with the aid of a cross hair, no vernier being used. It will be generally agreed that a trained observer using the above apparatus and technic could not make an error of as much as 0.5 mm. This agreement would still prevail for 0.3 mm and probably for 0.2 mm. In other words, the value 0.2 mm would be generally approved as a reasonable and conservative choice for the "maximum error" of the method. In fact, a smaller value, say 0.15 mm, would doubtless meet the approval of many investigators accustomed to making observations of this character, but this smaller value is obviously approaching the danger limit, and an investigator who proposed using it would probably be expected to justify his choice.

4. MISTAKES

Observations which differ from the true or "best" value by more than the "maximum error" may arise from any one or more of the following causes: (*a*) Mistakes by the observer, (*b*) mistakes by the apparatus, and (*c*) occasional vagaries on the part of the system.

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[3] The subject of "instrumental variance" (which is one of the elements entering into the establishment of the maximum error of the method) has been discussed by Schlink, Bull. B. S., 14, p. 741; 1919.

(a) MISTAKES BY THE OBSERVER

These include such blunders as the transposition of digits in setting down a numerical result; incorrect addition of weights during a weighing operation; omission or neglect of some item in a complicated technic; errors in computation, etc. A probable explanation can sometimes be found for such a mistake, and in that case the observer naturally has little hesitancy in excluding the observation, particularly when the discrepancy is large.

(b) MISTAKES BY THE APPARATUS

These arise from a temporary appearance of some variable which normally is absent or under control. For example, in a viscosity measurement a small dust particle might lodge temporarily in the capillary and give rise to one flow time much longer than any of the others in the series. A poor electrical contact might cause one electrical measurement to deviate widely from its fellows. A temporary stoppage of the ventilating fans of a laboratory might affect one member of a set of measurements appreciably influenced by barometric pressure.

If such sources of error affect several of the last measurements of a series, the observer naturally concludes that something has gone wrong and proceeds to look for and correct the trouble. When the difficulty occurs only once, however, its cause is not always easy to discover.

(c) VAGARIES ON THE PART OF THE SYSTEM

In certain types of systems an occasional value deviating from the mean by more than the maximum error may be obtained when no mistake has been made by either the observer or the apparatus. In other words this abnormal value actually characterized the system at the moment when it was obtained. Such a rare occurrence might possess a special interest in itself. Nevertheless the unusual value should be excluded from the mean, not because it represents a mistake but because in such a situation the mean is wanted presumably because it characterizes the normal behavior of the system, while the mean obtained by including the unusual value would not represent a quantity having any particular interest or importance. The "abnormal" value should, of course, not be rejected but should be recorded and discussed for any interest which it might possess.

Having selected an appropriate value for the maximum error of a method there are certain logical deductions therefrom which constitute a set of rules for the treatment of the observational data on Diophantine quantities. These rules will be discussed below and illustrated by numerical examples.

5. DIVISION OF THE PROBLEM

The general problems associated with measurement and calculation in dealing with Diophantine quantities may for convenience of treatment be divided into two classes as follows:

Class I.—The quantity sought is either capable of direct measurement or a value for it may be computed by ordinary mathematical methods from one or more quantities which are capable of direct measurement. Such a quantity will be called an experimentally determinable quantity. A general discussion is possible for this class of quantities.

CLASS II.—A great variety of more complicated cases can be formulated for which no general discussion is possible. Each type of case must be separately considered. Following are some examples of such cases:

Case 1.—Given

$$x = f_1(m, n, p)$$

The form of the function and its numerical parameters are known. Only the quantity x is capable of experimental determination and it is required to find the exact values of m, n, and p, each of which is a numerically conditioned quantity. For example, they might be conditioned as follows:

(*a*) m is a positive integer.
(*b*) n is a negative even integer.
(*c*) p is an integral power of 2.

Case 2.—The same as case 1, but with the following added relation.

$$y = f_2(n, p)$$

in which y is also capable of experimental determination.

Obviously a great variety of cases is possible. In all of these one or more of the quantities sought may be a Diophantine quantity. Any one or more of them may or may not belong also to Class I. The number of functional relations given may be equal to, less than, or more than the number of quantities sought. Many of the mathematical problems encountered in these more complicated cases are, in part at least, problems in Diophantine analysis, a branch of mathematics which so far as the author has been able to discover has not hitherto found much application in the physical sciences.

In the following discussion we shall first consider the simplest type as represented by Class I. This will be followed by a brief outline of the method of treating a more complicated type.

II. PROBLEMS OF CLASS I

The quantity sought is capable of experimental determination, and it is conditioned in such a way that its true value must be a member of the ascending series

$$M_0, M_1, M_2, M_3 \text{----------} M_n$$

all members of which are known.

1. NOMENCLATURE

M, The true value of the quantity sought.
M_a, The actual value found as the result of any single experimental measurement.
δM, The actual error

$$\delta M = \pm (M_a - M)$$

$(\delta M)_{\text{max.}}$, The maximum absolute error which can be made with the experimental technic employed.[4]

$p=\frac{\delta M}{M}$, The actual fractional error.

$p_{\text{max.}}=\frac{(\delta M)_{\text{max.}}}{M}$, The maximum fractional error.[4]

$\left.\begin{array}{l} M_1, \not< M-(\delta M)_{\text{max.}} \\ \text{or} \\ M_1, \not< M(1-p_{\text{max.}}) \end{array}\right\}$ The lowest possible value for M_a consistent with the accuracy of the method.

$\left.\begin{array}{l} M_2, \not> M+(\delta M)_{\text{max.}} \\ \text{or} \\ M_2, \not> M(1+p_{\text{max.}}) \end{array}\right\}$ The highest possible value for M_a consistent with the accuracy of the method.

ΔM, The difference, M_n-M_{n-1}, between two specified members of the series.

$\left.\begin{array}{l} M_{n+1} \\ M_{n-1} \end{array}\right\}$ The two members of the series which inclose the value M.

2. GENERAL CONDITIONS

1. The various possible values of M_a will ordinarily form a continuum between the limits M_1 and M_2; that is, within these limits any value is possible for M_a. Practically, however, only a finite number of values need be considered. This number will be determined by the number of significant figures[5] justified in M_a.

2. As with many cases encountered in precision-of-measurement discussions, it is necessary to know at least the order of magnitude of M in order to draw definite numerical conclusions concerning the required precision. The first step in the experimental technic is therefore the determination of an approximate value for M by some rapid method. In the present instance this determination serves to locate the region of the series in which M must lie and the remainder of the series on both sides of this region can be excluded from consideration.

3. In the following discussion the "accuracy of the experimental technic" will be understood to be measured by the magnitude of $(\delta M)_{\text{max.}}$ or $p_{\text{max.}}$ and to vary inversely with these quantities. It will also be assumed that each of these quantities is equally likely to be positive or negative.

3. APPLICATIONS OF THE PRINCIPLE OF MAXIMUM ERROR

A number of interesting problems present themselves. In discussing these problems we shall illustrate our conclusions by applying them to a specific case. For this case we shall take usually the series represented by all of the even positive integers. In other words, we shall assume that the true value of M is an even positive whole number. Evidently for such a series $\Delta M=2$ and is a constant for the series.

In general, the form in which the conclusions are expressed will depend upon whether the errors in the measurements are given in absolute terms (δM) or in relative terms (p).

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[4] It is assumed that vectorial errors have been eliminated.

[5] Thus if $M_1=240$ and $M_2=244$ and the values of M_a are read or computed only to the first decimal place, there are only 41 possible values for M_a.

In practice, one form of expression can be frequently translated into the other, but in some cases this is not possible, and, to make the treatment perfectly general, it should cover both forms of expression. In discussing the various problems we shall, therefore, include both forms as subcases (1) and (2), respectively, under each problem. After discussing each problem in analytical terms we shall then present a discussion of subcase (1) in graphical form.

Owing to the simplicity of the graphical presentation, it is suggested that the reader turn to II, 4, on p. 233, and read the description of the graphical presentation before undertaking a study of the more exact and complete analytical presentation which we shall now proceed to discuss.

(a) M_a GIVEN. TO DETERMINE THE POSSIBLE VALUES OF M.

(1) MAXIMUM ERROR GIVEN AS $(\delta M)_{\text{max.}}$—Obviously the only possible values for M are those members of the series which meet at least one of the following conditions:

$$M+(\delta M)_{\text{max.}} \nless M_a \nless M-(\delta M)_{\text{max.}} \tag{1}$$

(2) MAXIMUM ERROR GIVEN AS $p_{\text{max.}}$—The corresponding condition is

$$M\,(1+p_{\text{max.}}) \nless M_a \nless M\,(1-p_{\text{max.}}) \tag{2}$$

Example.—Given $M_a=241.4$ and $(\delta M)_{\text{max.}}=4.2$ or $p_{\text{max.}}=2$ per cent. Applying the above relations gives us the following table:

TABLE 1

M	236	238	240	242	244	246	248
$M+(\delta M)_{\text{max}}$	240.2	242.2	244.2	246.2	248.2	250.2	252.2
M_a	241.4	241.4	241.4	241.4	241.4	241.4	241.4
$M-(\delta M)_{\text{max}}$	231.8	233.8	235.8	237.8	239.8	241.8	243.8
$M(1+p_{\text{max.}})$	240.7	242.8	244.8	246.8	248.9	250.9	253.0
M_a	241.4	241.4	241.4	241.4	241.4	241.4	241.4
$M\,(1-p_{\text{max.}})$	231.3	233.2	235.2	237.1	239.1	241.1	243.0

Consequently, if $(\delta M)_{\text{max.}}=4.2$, M must be 238, 240, 242, or 244; or, if $p_{\text{max.}}=2$ per cent, M must be 238, 240, 242, 244, or 246.

Relations (1) and (2) may also be written in the forms

$$M_{\text{min.}} \nless M_a-(\delta M)_{\text{max.}} \tag{1a}$$

$$M_{\text{max.}} \ngtr M_a+(\delta M)_{\text{max.}} \tag{1b}$$

$$M_{\text{min.}} \nless M_a/(1+p_{\text{max.}}) \tag{2a}$$

$$M_{\text{max.}} \ngtr M_a/(1-p_{\text{max.}}) \tag{2b}$$

which are convenient for calculating the extreme values for M.

Example.—If we apply these relations to the above example we have

(*a*)

$$M_{\text{min.}} \nless 241.4-4.2 \nless 237.2 \nless 238$$

$$M_{\text{max.}} \ngtr 241.4+4.2 \ngtr 245.6 \ngtr 244$$

or (*b*)

$$M_{\text{min.}} \nless 241.4/(1.02) \nless 236.7 \nless 238$$

$$M_{\text{max.}} \ngtr 241.4/(0.98) \ngtr 246.3 \ngtr 246$$

(b) $(\delta M)_{\text{max.}}$ OR $p_{\text{max.}}$ GIVEN. WHAT SINGLE VALUES OF M_a WILL LEAD TO THE DEFINITE EVALUATION OF M?

(1) $(\delta M)_{\text{max.}}$ GIVEN.—The condition is obviously that M_a shall be included within $M \pm (\delta M)_{\text{max.}}$ for a single member of the series or, what amounts to the same thing, that $M_a \pm (\delta M)_{\text{max.}}$ shall include a single member of the series. In other words, in order to definitely evaluate M, M_a must lie between the limits

$$M_{n+1} - (\delta M)_{\text{max.}} \text{ and } M_{n-1} + (\delta M)_{\text{max.}}$$

which may be abbreviated thus

$$M_{n\pm 1} \mp (\delta M)_{\text{max.}} \tag{3}$$

For a series in which ΔM is a constant, this is equivalent to

$$(\delta M)_{\text{max.}} < \Delta M - \delta M \tag{4}$$

in which δM is the actual error taken with a positive sign.

(2) $p_{\text{max.}}$ GIVEN.—The condition is that M_a shall be included between the limits

$$M_{n\pm 1}(1 \mp p_{\text{max.}}) \tag{5}$$

for a single member of the series. For a series in which ΔM is a constant, this reduces to

$$(M \pm \Delta M)\ (1 \mp p_{\text{max.}}) \tag{6}$$

Example.—Given $(\delta M)_{\text{max.}} = 1.3$ or $p_{\text{max.}} = 0.5$ per cent. If $M = 242$ and $\Delta M = 2$, then M will be definitely evaluated if M_a has any value not less than

$$M - (\Delta M - (\delta M)_{\text{max.}}) = 242 - 2 + 1.3 = 241.3$$

or not greater than

$$242 + 2 - 1.3 = 242.7$$

Or using $p_{\text{max.}}$, M will be definitely evaluated if M_a has any value between

$$(242 - 2)\ (1 + 0.005) = 241.20$$

and

$$(242 + 2)\ (1 - 0.005) = 242.78$$

The above limits are noninclusive in both cases.

(c) WITH A GIVEN VALUE FOR M_a WHAT ARE THE MAXIMUM ALLOWABLE VALUES FOR $(\delta M)_{\text{max.}}$ AND $p_{\text{max.}}$, IF M IS TO BE DEFINITELY EVALUATED?

(1) FOR $(\delta M)_{\text{max.}}$—The condition is that

$$(\delta M)_{\text{max.}} \text{ shall be} < \pm M_{n\pm 1} \mp M_a \tag{7}$$

For a series in which ΔM is constant, this reduces to

$$(\delta M)_{\max.} \text{ shall be} < \pm M + \Delta M \mp M_a \tag{8}$$

which is equivalent to relation (4).

(2) For $p_{\max.}$—The condition is

$$p_{\max.} \text{ shall be} < \frac{\mp M_a \pm M_{n\pm1}}{M_{n\pm1}} \tag{9}$$

For a series in which ΔM is constant this reduces to

$$p_{\max.} \text{ shall be} < \frac{\mp M_a \pm M + \Delta M}{M + \Delta M} \tag{10}$$

which is equivalent to

$$p_{\max.} \text{ shall be} < \frac{-\frac{\delta M}{M} + \frac{\Delta M}{M}}{1 + \frac{\Delta M}{M}} \tag{11}$$

Examples.—

(1) For $(\delta M)_{\max.}$

(*a*) Suppose $M=242$ and given $M_a=242.0$. From relation (8) we have

$$(\delta M)_{\max.} \text{ shall be} < \pm 242 + 2 \mp 242 < 2$$

(*b*) If $M_a=241$, we have

$$(\delta M)_{\max.} \text{ shall be} < \pm 242 + 2 \mp 241 < 1$$

This involves a contradiction since if the actual error (that is, 242−241) is 1, the maximum error can not be less than 1. In other words, if M_a differs from M by as much as one-half ΔM, definite evaluation of M will not be possible from a single value of M_a.

(2) For $p_{\max.}$

(*a*) Suppose $M=242$ and given $M_a=242.0$. From relation (11) we have

$$p_{\max.} \text{ shall be} < \frac{-\frac{0}{242} + \frac{2}{242}}{1 + \frac{2}{242}} < 0.82 \text{ per cent.}$$

(*b*) If $M_a=241$, we have

$$p_{\max.} \text{ shall be} < \frac{\frac{-1+2}{242}}{1 + \frac{2}{242}} < 0.41 \text{ per cent.}$$

The actual error is $\frac{1}{242}=0.413$ per cent. Consequently $p_{\max.}$ can not

be less than 0.41 per cent; that is, evaluation of M will fail, if $M_a = 241$.

(d) WHAT IS THE MOST FAVORABLE VALUE FOR M?

(1) Error Given as $(\delta M)_{\text{max.}}$—The most favorable value is given by

$$2\ M_a = M_{n+1} + M_{n-1} \tag{12}$$

since then and only then will the allowable $(\delta M)_{\text{max.}}$ be a maximum. From this relation it is obvious that the most favorable value is not in general the true value, M. The most favorable value for M_a is the true value M only when

$$M_{n+1} + M_{n-1} = 2M \tag{13}$$

which is equivalent to, $\Delta M =$ a constant. For example, given the series 2, 4, 8, 16, 32, 64, etc. Suppose the true value is 32. The most favorable value for M_a is then $\frac{64+16}{2} = 40$, since this value will permit $(\delta M)_{\text{max.}}$ to have any value less than $32 - 8 = 24$. In other words, if the true value of M is 32 and the accuracy of the experimental technic is $\pm$ say 20 units, then M will be definitely evaluated, if the value obtained in the measurement is in error by $+18$ units, but the evaluation will fail if the investigator is so unfortunate as to obtain the correct value in the determination. A similar situation would exist if the correct value of M were 15, 20, 25, 30, 35, or 40, etc., and the series to which M belonged were the following: 10, 11, 12, 13, 14, 15, 20, 21, 22, 23, 24, 25, 30, 31, 32, 33, 34, 35, 40, 41, etc.

(2) Error Given as $p_{\text{max.}}$—The most favorable value for M_a is given by

$$M_a = M_{n+1}(1 - p_{\text{max.}}) = M_{n-1}(1 + p_{\text{max.}}) \tag{14}$$

since under these conditions the allowable $p_{\text{max.}}$ has its maximum value which is

$$p_{\text{max.}} < \frac{M_{n+1} - M_{n-1}}{M_{n+1} + M_{n-1}} \tag{15}$$

Combining this with (14) gives the most favorable value of M_a

$$M_a = \frac{2M_{n+1}M_{n-1}}{M_{n+1} + M_{n-1}} \tag{16}$$

For a series in which ΔM is constant relation (15) becomes

$$p_{\text{max.}} < \frac{\Delta M}{M} \tag{17}$$

and equation (16) becomes

$$M_a = \frac{M^2 - (\Delta M)^2}{M}$$

The most favorable value is the true value only when [6]

$$M=\frac{2M_{n+1}M_{n-1}}{M_{n+1}+M_{n-1}} \tag{18}$$

or for a series in which ΔM is constant, only when

$$M=M-\frac{(\Delta M)^2}{M} \tag{19}$$

which is equivalent to the condition

$$\frac{(\Delta M)^2}{M}=0 \tag{20}$$

In words, the smaller $\frac{(\Delta M)^2}{M}$, the more nearly will the most favorable value approach the true value.

Examples.—If $M=4$ and $\Delta M=2$, then the most favorable value for M_a is

$$M_a=4-\frac{4}{4}=3$$

If $M=32$ and the series is 2, 4, 8, 16, 32, 64, 128 . . . . , the most favorable value for M_a is

$$M_a=\frac{2\times 64\times 16}{64+16}=25.6$$ [7]

If $M=100$ and the series is 95, 96, 97, 98, 99, 100, 110, 111, 112, 113, 114, 115, 120, 121, ... , the most favorable value for M_a is

$$M_a=\frac{2\times 110\times 99}{99+100}=104$$

(e) LEAST FAVORABLE VALUE OF M_a

(1) Error Given as $(\delta M)_{\max.}$—There are two values of M_a which are in general equally unfavorable and which represent the limits at which evaluation of M from a single value of M_a just fails of accomplishment. They are respectively given by

$$2M_a=M+M_{n-1} \tag{21}$$

and

$$2M_a=M_{n+1}+M \tag{22}$$

While, in general, both values are equally unfavorable from the standpoint of the definite evaluation of M from a single value of M_a, there is a special case in which one of them disappears. For example, if the series starts with M, the value $M_a=\frac{1}{2}(M_{n+1}+M)$, is the most unfavorable value. Similarly if the series ends with M, the value, $M_a=\frac{1}{2}(M_{n-1}+M)$, is the most unfavorable value.

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[6] This is ordinarily an impossible condition, since it imposes an arbitrary mathematical restriction on the series to which M belongs.

[7] Note that here the most favorable value is less than the true value while in (1) above the most favorable value was greater than the true value.

(2) Error Given as $p_{\max.}$—The two least favorable values are:

$$M_a = \frac{2M \times M_{n-1}}{M + M_{n-1}} \tag{23}$$

and

$$M_a = \frac{2M \times M_{n+1}}{M + M_{n+1}} \tag{24}$$

As in the above case, one of these values disappears when M is the end member of a series.

(f) WHAT ARE THE CONDITIONS WITH RESPECT TO M_a AND $(\delta M)_{\max.}$ OR $p_{\max.}$ WHICH WILL LEAD TO DEFINITE EVALUATION OF M FROM A PAIR OF VALUES OF M_a?

(1) For $(\delta M)_{\max.}$—Definite evaluation of M from a pair of values of M_a is frequently possible when neither value alone will lead to evaluation. If M_a and M'_a are the two values, then definite evaluation will result, if

$$M_a - (\delta M)_{\max.} > M_{n-1} \tag{25}$$

$$M'_a + (\delta M)_{\max.} < M_{n+1} \tag{26}$$

and

$$M_a > M > M'_a \tag{27}$$

(2) For $p_{\max.}$—The corresponding conditions are

$$M_a - Mp_{\max.} > M_{n-1} \tag{28}$$

$$M'_a + Mp_{\max.} < M_{n+1} \tag{29}$$

and

$$M_a > M > M'_a \tag{30}$$

Example.—Using our type series 238, 240, 242, 244, etc., suppose $M=242$ and $(\delta M)_{\max.}=3$. It is obvious that no single value of M_a can lead to definite evaluation of M. From relation (25) we have

$$M_a > 240+3 > 243$$

$$M'_a < 244-3 < 241$$

That is, if two values are obtained for M_a, one of which is greater than 243 and the other less than 241, M will be definitely evaluated. Evaluation by this method is mathematically possible no matter how great $(\delta M)_{\max.}$ or $p_{\max.}$ may be. It is of no practical importance, however, when the steps in the series to which M belongs are small in comparison with $(\delta M)_{\max.}$ (See further below, pp. 237 to 241.)

4. GRAPHICAL REPRESENTATION

The relations deduced in the foregoing pages may be illustrated graphically. We shall discuss this method only for $(\delta M)_{\max.}$ Its extension to cover the case in which the error is given as $p_{\max.}$ will be obvious.

Imagine the members of the group in the neighborhood of M_a to be located on a uniform scale. We may now proceed as follows:

1. From the locus of a given value of M_a as a center draw a circle with the radius $(\delta M)_{\max.}$ If the circle contains only one of the M

points, M is definitely identified. (The word "contains" will be here understood to include values lying upon the circumference as well as those lying within the circumference.)

2. If the above procedure is followed for two or more values of M_a, then M will be definitely identified, if any two of the circles contain only one M point in common, even though each circle contains more than one M point.

3. Identification will follow from *any* value of M_a, if no center can be found for a circle which will inclose more than one M point.

4. Identification by 1 above *may* follow, if any center can be found for the circle which will inclose only one M point.

5. The locus of the most favorable M_a value is a point from which as a center the largest possible circle containing only one M point can be drawn. The radius of the smallest of these "most favorable circles" is the value below which $(\delta M)_{\text{max.}}$ must lie, if identification is to be possible from a single value of M_a for the most unfavorable case in the group of possible M values.

6. A least favorable M_a value is similarly a point which is the center of the circle (containing only one M value) which has the smallest radius.

5. A SPECIAL CASE—THE KNOWN MEMBERS OF THE SERIES ARE NOT EXACTLY KNOWN

In all of the preceding discussion we have assumed that the members of the set of known quantities were exactly known (or knowable). If this is not the case, we have the situation illustrated by example 5, p. 222, and the foregoing discussion must be modified. The necessary modifications can best be indicated by using the graphical method. In this method it is necessary only to replace the M points by circular areas, the radius in each case being taken equal to the maximum uncertainty in the value of the corresponding M, the center being the "best value" of the M. Instead of M points we now have M areas and the preceding discussion is applicable, if for "M point" we substitute "any part of an M area" in all cases in which the "containing" of an M point is involved.

A numerical example will illustrate this case. The unknown quantity is known to be one of the members of the following set:

$$(200 \pm 5), \qquad (210 \pm 1), \qquad (212.0 \pm 0.2)$$

1. Identification will be certain from any value of M_a, if $(\delta M)_{\text{max.}}$ is less than the smaller of the two quantities

$$\frac{(210-1)-(200+5)}{2}=2$$

and

$$\frac{(212.0-0.2)-(210+1)}{2}=0.4$$

that is, less than 0.4.

2. If $(\delta M)_{\text{max.}}$ is less than 2, definite identification will result or the value 200 will be eliminated.

3. If the actual value is 210, definite identification will result, if $(\delta M)_{\text{max.}}<0.8$, provided the value obtained for M_a is less than $211.8-(\delta M)_{\text{max.}}$

4. If 210 is the actual value, the most favorable value for M_a is 210.4, since with this value for M_a definite identification will result, if $(\delta M)_{\text{max.}} < 1.4$.

It is obvious that if any two of the M areas have one or more points in common, identification with one of such a pair of M areas can never be certain.

6. EXPERIMENTAL PROCEDURE

An appropriate experimental procedure to be followed in cases belonging to the subject of this paper will now be described. Each step of the procedure will be given, together with an illustrative example.

(a) PRELIMINARY DETERMINATION OF M

Determine M approximately by any convenient rapid method and write down the corresponding members of the series.

Example.—Suppose we find $M = 242 \pm 5$ per cent and that the members of the series in this neighborhood are the following: . . . . 216, 220, 224, 228, 232, 236, 240, 242, 244, 246, 248, 250, 254, 258, 262, 266, 270 . . . .

(b) COMPUTATION OF VALUES OF $(\delta M)_{\text{max.}}$ AND OF $p_{\text{max.}}$ WHICH WILL MAKE CERTAIN THE EVALUATION OF M FROM A SINGLE VALUE OF M_a

1. Value of $(\delta M)_{\text{max.}}$—For the above series the smallest value of ΔM is evidently 2. Consequently we have (see p. 230, supra)

$$(\delta M)_{\text{max.}} \text{ must be} < 1 \tag{31}$$

2. Value of $p_{\text{max.}}$—Since M is unknown it is necessary to provide for the least favorable case. This is the case which will give the smallest value of p in the following expression:

$$p = \frac{(M - M_{n\pm1})}{(M + M_{n\pm1})} \tag{32}$$

in which the $\pm$ sign is to be taken in the sense which gives the smaller value for p.

Example.—For the above series the values of p so calculated are shown in Table 2 in the column headed p_1. For the least favorable case it is obvious that

$$p_{\text{max.}} \text{ must be} < 0.4 \text{ per cent}$$

(c) COMPUTATION OF VALUES OF $(\delta M)_{\text{max.}}$ FOR WHICH EVALUATION OF M IS POSSIBLE FROM A SINGLE VALUE OF M_a

(1) For $(\delta M)_{\text{max.}}$—From the general relation

$$(\delta M)_{\text{max.}} < \Delta M - \delta M \tag{4}$$

it is obvious that evaluation of M is possible whenever

$$(\delta M)_{\text{max.}} < \Delta M \tag{33}$$

In the above series the smallest value for ΔM is obviously 2.

(2) For $p_{\text{max.}}$—The condition is expressed by relation (15) or (17).

Example.—For the series under consideration these values for $p_{max.}$ are shown in Table 2 in the column headed p_2. For the least favorable case it is obvious that

$$p_{max.} \text{ must be } < 0.8 \text{ per cent}$$

TABLE 2

For M=	p_1	p_2	For M=	p_1	p_2
	Per cent	*Per cent*		*Per cent*	*Per cent*
220	0.901	1.818	244	0.408	0.820
224	.875	1.785	246	.406	.813
228	.870	1.754	248	.402	.806
232	.854	1.724	250	.402	1.195
236	.840	1.695	254	.782	1.575
240	.415	1.255	258	.769	1.550
242	.411	.826	262	.758	1.526

(d) SELECTION OF $(\delta M)_{max.}$ OR OF $p_{max.}$

If the conditions set down in 2 above can be readily met, this should of course be done and M can be definitely evaluated from a single determination. Even in this case, however, check determinations would ordinarily be made to eliminate the possibility of "mistakes."

If this condition can not be readily met, then the smallest practicable value of $(\delta M)_{max.}$ or of $p_{max.}$ should be selected and repeated observations made.

7. TREATMENT OF THE OBSERVED VALUES

(a) GENERAL METHOD

From the first value obtained for M_a deduce (as explained above, p. 228) the possible values of M. Do the same for the second value of M_a. Strike out of the two sets of possible M's all values not common to both sets. Proceed similarly with each M_a as it is obtained until only one value of M remains as a possibility. Obviously the true value of M must be contained in all of the sets of possible M values. As soon as M has been identified in this way the result may be checked by additional determinations of M_a treated in the same way. A check is obtained as soon as the same M emerges a second time as the only value common to all the sets.

If a number of determinations (say seven or more) have been made without obtaining a definite identification of M by the above procedure, or if the chance of obtaining identification by this procedure is small, the investigator may prefer to treat his measured values by the target procedure which is described below. This procedure may also be used in addition to that described above, if desired.

Example.—Given $p_{max.}=0.7$ per cent. Suppose our first experiment gives $M_a=240.4$. We now prepare the following table:

TABLE 3

M	240	242	244
$M\ (1+p_{max.})$	241.68	243.70	245.71
$M\ (1-p_{max.})$	238.32	240.30	242.29

Evidently the only possible values for M are 240 and 242.

Second experiment.—$M_a = 240.5$. The only possible values for M are 240 and 242.

Third experiment.—$M_a = 241.9$. The only possible value for M is 242. Hence, M must be 242.

This is a complete identification. If treated in the usual way these measurements would give as an average 240.9 with an average deviation of ± 0.9 and the identification would be in doubt.[8]

In this example the value 241.9 is alone sufficient for identification. If instead of 241.9 we had obtained say 243.0 in our third experiment, the corresponding possible values of M would be 242 and 244. This result combined with either of the preceding ones would also give 242 as the only possible value.

The above procedure is in sharp contrast with that ordinarily followed in physical measurements, in that (1) the individual measurements are not averaged, and (2) the most favorable values are not necessarily those which are closest to the true value.

Mistakes.—In the above procedure nothing has been said concerning the detection and elimination of values of M_a which are affected by mistakes, nor is it necessary to include any special machinery for this purpose in the case of large mistakes. The procedure automatically eliminates such values. For example, suppose the first value of M_a yielded 230, 232, and 234 as possible values for M and the second determination gave 240, 242, and 244. Since the true value of M must belong to every set of possible values, one of the above values of M_a must contain a mistake. It is not necessary, however, to make special provision for detecting and rejecting large mistakes of this character, since the procedure automatically eliminates them.

The situation which arises in the case of small mistakes, however, requires further examination. A small mistake may yield an M_a value differing from the true M by an amount only a little greater than $(\delta M)_{\text{max.}}$. This result is equivalent to, and in practice can not be distinguished from, the selection of a value of $(\delta M)_{\text{max.}}$ slightly smaller than the true value. The two effects may, therefore, be discussed together.

Such a situation, if it occurs early in a set of M_a determinations, can under some circumstances lead to an erroneous conclusion. For example, using our type series, suppose that the correct M is 242 and that $(\delta M)_{\text{max.}}$ is estimated to be 1.5. Given

1. $M_a = 240.4$, hence $M = 240$.

This value is incorrect. The only recourse is to make a sufficient number of check determinations. For example:

2. $M_a = 241.1$, hence $M = 240$, or 242.

3. $M_a = 242.3$, hence, $M = 242$.

4. $M_a = 243.2$, hence $M = 242$, or 244.

[8] This is admittedly an improbable case; that is, in an actual set of measurements it is unlikely (but not impossible) that the first two results obtained will be low by amounts approaching the maximum error of the method.

It is now clear that the first observation contained a mistake, or that the value selected for $(\delta M)_{\max.}$ was not chosen with sufficient conservatism.

If we assume that the first value of M_a was affected by a mistake, and we must make this assumption if we retain our confidence in the value selected for $(\delta M)_{\max.}$, then the other three values check one another in yielding 242 as the value for M. If, however, we reexamine our grounds for selecting $(\delta M)_{\max.}$ and decide, let us say, to raise our estimated value to $(\delta M)_{\max.}=2.0$, then the above observations would give

1. $M=240$ or 242
2. $M=240$ or 242
3. $M=242$ or 244
4. $M=242$ or 244

and M is evidently 242.

In other words, if $(\delta M)_{\max.} \not> 2.0$ the above set of observations gives two independent evaluations of M, since 1 and 4 together and 2 and 3 together both yield $M=242$. The same is, of course, true of 1 with 3 and 2 with 4. If, however, $(\delta M)_{\max.}=3.1$, then the above set of observations would still yield $M=242$, but the evaluation would now come only from the two extreme values, 1 and 4, and if either of these values contained a mistake the conclusion might be in error. In other words, an evaluation which depends solely on one M_a value or solely on the two extreme members of the set should, in general, be checked by additional determinations or confirmation should be sought from the target method.

(b) THE TARGET METHOD

We have now to consider the treatment of a considerable number of observations which, by the methods outlined above, have failed to yield a definite evaluation of M.

The problem may be illustrated by the following situation. Assume a marksman whose shots have the property of randomness and whose shooting has an accuracy which can be represented by a circle of radius R inclosing the target; that is, no shots will fall outside of this circle. Place before this marksman a set of targets only one of which is exposed. The marksman now makes a series of shots at the exposed target. The problem is to determine, from the distribution of the hits, the target which was exposed during the firing.

Bring the center of the circle of radius R into coincidence with each target in turn. For the actual target fired at, this circle must inclose all of the hits. This test will eliminate all but a small number of targets and the problem is to select the actual target from this small number of possibilities.

In the case of physical measurements of numerically conditioned quantities the possible values of M remaining after treatment of the observations by the methods of the preceding pages correspond to this small number of possible targets. For selecting the correct value from among these possible values the following methods suggest themselves.

Exclude any values of M_a whose corresponding set of possible M values does not include any members of the final set of possible M values. Treat the remaining M_a values by one or more of the following procedures.

1. Each value of M_a considered by itself yields a set of possible values for M. A given M will occur n times in the group of these sets. Using values of M as abscissæ and corresponding values of n as ordinates, construct the frequency-of-occurrence curve. This curve will have a flat maximum on which will be found the only values of M consistent with the whole set of observations. Locate the ordinate which bisects the area under the curve. The most probable value for M is the value closest to this ordinate. This

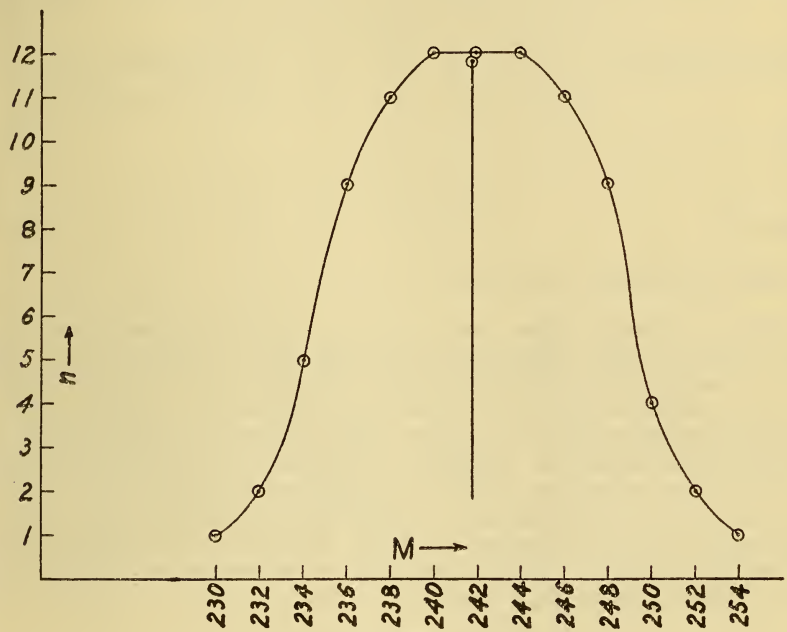

FIGURE 1

procedure should be applied only when the set of M values is characterized by a constant value of ΔM.

If a sufficient number of observations have been made, the flat maximum should always contain an odd number of points, the midpoint being the true value of M. This value should, in general, be also the median of the whole set.

2. (*a*) Average the M_a values. (*b*) Take as the most probable value of M that member of the M series which is closest to the average M_a.[9]

[9] If the values of M_a are directly measured or are computed from measurements of a single variable, the average value is usually taken as the arithmetic mean. If, however, M_a is a function of two or more measured quantities from which it is computed, then a method of averaging appropriate to the situation should be employed. Such methods are described in treatises on precision of measurement. That described by Campbell (*see* p. 162 of reference in footnote 1, p. 223), for example, employs the principle of maximum error in obtaining the average and its precision measure.

3. Take as the most probable value of M that member of the M series which is closest to the median of the set of M_a values.

Example.—Given $(\delta M)_{\text{max.}}=7$, $\Delta M=2$, and the values of M_a shown in Table 4. This table illustrates the result of applying procedures 2 and 3. Procedure 1 gives the curve shown in Figure 1.

TABLE 4

M_a	M_a—mean	Mean—M_a	M_a—240	240—M_a	M_a—242	242—M_a	M_a—244	244—M_a
241		0.8	1			1		3
242	0.2		2		0			2
247	5.2		7		5		3	
244	2.2		4		2		0	0
240		1.8	0	0		2		4
241		.8	1			1		3
239		2.8		1		3		5
243	1.2		3		1			1
242	.2		2		0	0		2
237		4.8		3		5		7
245	3.2		5		3		1	
242	.2		2		0	0		2
Mean=241.8.	1.8	2.2	2.7	1.3	1.6	1.7	1.3	2.9
Median=242.	a. d.=1.9$_5$		=2.6		=1.9$_2$		=2.7$_5$	

Possible values by methods of preceding pages, 240, 242, 244.

If the procedures check one another in clearly designating one and the same M value, there is considerable probability that it is the correct value. If the procedures do not check one another, or if, for example, the mean M_a is about halfway between two members of the M set, identification is in doubt and a more accurate method of measurement should be employed. Obviously the results obtained by the target procedure will be materially strengthened, if it is first applied to the data obtained by applying the experimental "method" to one or more known members of the M series in the neighborhood of the value sought. This amounts to a calibration or standardization of the experimental "method."

8. RÉSUMÉ FOR CLASS I

The procedure for the identification of a numerically conditioned quantity open to direct experimental determination may be summed up as follows:

1. Approximate determination of the quantity and consequent limitation to a small number of possibilities.

2. If possible and convenient, select $(\delta M)_{\text{max.}}$ or $p_{\text{max.}}$ so that a single determination will be certain to yield positive identification.

3. If this is impossible or inconvenient, then if feasible, select $(\delta M)_{\text{max.}}$ or $p_{\text{max.}}$ so that a single determination may lead to positive identification. In any case proceed with the determinations until the combined conclusions obtained by treating each determination separately lead to positive identification or until the investigator prefers to resort to the target method.

When positive identification has not been attained at the end of a number of determinations (say 7 or more), the investigator may prefer to treat his combined results by the target method instead of con-

tinuing his determinations. This situation is likely to arise (*a*) when the investigator has been too conservative in estimating the value of $(\delta M)_{\max.}$ or $p_{\max.}$ for the technic and apparatus employed; or (*b*) when the requisite values for M_a lie in the neighborhood of $M_a \pm (\delta M)_{\max.}$ Under these circumstances the probability of obtaining sufficiently favorable values of M_a may be small.

In general, the target method will have to be resorted to whenever $(\delta M)_{\max.}$ is large in comparison with ΔM, since with increase in the ratio, $(\delta M)_{\max.}/\Delta M$, the situation approaches that which prevails in physical measurements of nonconditioned quantities. The target method will also be frequently employed for purposes of confirmation

III. PROBLEMS OF CLASS II

1. A PROBLEM INVOLVING FUNCTIONS OF TWO VARIABLES

We shall now consider briefly the following case:
Given

$$M = 14n + x \tag{34}$$

$$r = \frac{n}{x} \tag{35}$$

and the following conditions:
(*a*) n is a positive integer,
(*b*) x is a member of the closed series, $2, 0, -2, -4, -6, \cdots (2-2n)$
(*c*) M and r only are capable of experimental measurement. The general problem is to determine the exact values of n and x.

It is obvious to begin with that M is an even positive integer. Also up to any given value of n there are only a finite number of values possible for r and these are all calculable. M and r are, therefore, Diophantine quantities belonging to Class I.

Belonging to each possible value of M or of r there are only certain possible values for n and x. Thus there are 42 values of M for each of which only one value of n and only one value of x is possible. If, therefore, M is definitely identified as one of these 42 values, n and x will be also determined thereby and the experimental determination of r will be unnecessary. There are also 42 values of M for each of which 2 values are possible for n and x, 42 for which 3 values are possible, etc. There are also 15 members of the M series of even positive integers for which no values of n and x are possible.

For each finite value of r there exists a minimum value for n and a minimum value for x, the other possible values being the integral multiples of these values. The corresponding values of M belong to the series $(14\ n_{\min.} + x_{\min.}) \times I$ in which I is a positive integer. This is a series in which ΔM is constant and equal to $(14\ n_{\min.} + x_{\min.})$.

Thus for example, if r is found to be -0.875, $n_{\min.} = 7$, $x_{\min.} = -8$. The n series is defined by $7 \times I$, the x series is in correspondence and is defined by $-8 \times I$, and the M series is defined by $90 \times I$. This last is a series in which $\Delta M = 90$. Consequently, M will be identified with certainty from any single value of M_a, if $(\delta M)_{\max.} < 45$.

In many cases, however, the exact value of r can not be determined, but only certain limiting values determined by the magnitude of $(\delta r)_{\max.}$ For example, suppose we find

$$r = -0.875 \pm 0.003$$

For values of n not exceeding 100 it can be shown that the possible values of n must belong to the series

$$(7\times I),\ (72+7\times I) \text{ or } (75+7\times I)$$

the possible values of $-x$ are the corresponding (1 to 1) members of the series

$$(8\times I),\ (82+8\times I) \text{ or } (86+8\times I)$$

and the possible values of M belong to the series

$$(90\times I),\ (926+90\times I) \text{ or } (964+90\times I)$$

It is obvious, therefore, that if M_a is determined with any given accuracy $(\delta M)_{\text{max.}}$, the possible values of M, n, and x will be limited to a small number of known quantities and one can then easily compute the accuracy necessary in r in order to select from these values the correct ones. Similarly if r is determined with a given accuracy, $(\delta r)_{\text{max.}}$, the possible values of M, n, and x will be limited to a known set from which the correct values can be selected provided M is determined with an accuracy, $(\delta M)_{\text{max.}}$, which can be readily calculated. The errors in the determinations of M_a and r_a can, therefore, be assigned to meet the convenience of the investigator; that is, within certain limits (which can be computed in any given case), $(\delta M)_{\text{max.}}$ can be made large and $(\delta r)_{\text{max.}}$ small, or vice versa, or any desired distribution can be made.

For any given accuracy in M_a, r_a being undetermined, or for any given accuracy in r_a, M_a being undetermined, or for any given values of $(\delta M)_{\text{max.}}$ and $(\delta r)_{\text{max.}}$, both M_a and r_a being determined, it is possible to compute the limits between which n and x must lie.

In other words, it is possible to compute for each value of M_a the corresponding set of possible values of M, n, and x. Similarly for each value of r_a it is possible to compute the corresponding set of possible values of M, n, and x. The true values of n and x must belong to both of the sets of possible values so computed. The computations of the sets of possible values are problems in Diophantine analysis.

This case will not be discussed in any further detail in this paper, since the detailed discussion of an artificial case would have an academic interest only. There are many actual scientific problems which involve one or more numerically conditioned quantities, and one of these problems will be fully discussed in a later paper.

IV. EXAMPLES FROM CHEMISTRY

As examples of scientific problems involving numerically conditioned quantities, the following may be cited:

Example 1.—Given, a sample of a pure chemical substance containing only carbon, hydrogen, and oxygen. Required, its empirical formula.

The formula may be written, $C_aH_bO_c$. The laws of valency and of atomic proportions impose the following conditions:

(1) a is a positive integer.
(2) b is a positive even integer.
(3) c is a positive integer.
(4) $b \ngtr 2a+3$.

We have also

$$(5)\quad M=12a+b+16c+(0.0077b)$$

where M is the molecular weight. M is experimentally determinable and is itself a numerically conditioned quantity belonging to Class I. It differs from an even whole number only by the small quantity $0.0077b$.

We may, if we wish, have at our disposal also or instead, either or both of the following relations:

$$(6)\quad r_C=\frac{12a}{M}$$

$$(7)\quad r_H=\frac{1.0077b}{M}$$

in which r_C and r_H are experimentally determinable numerically conditioned quantities; that is, they belong to Class I.

Example 2.—Given a sample of a pure chemical substance known to have the empirical formula C_7H_8O. Required to determine which of the various possible isomers it is.

If some one property has been measured for each isomer and if these values are available in the literature, the identification can be made by determining this property for the substance in question. It is obvious that under these conditions we are dealing with a numerically conditioned quantity belonging to Class I and of the kind discussed on page 234, section 5.

If the corresponding necessary conditions for definite identification can not be met, then we may take also (or instead) some other property for which data are available for the different isomers. For absolute certainty of identification it is essential that every possible isomer be included at least once in the group of sets of property values taken. If several sets of property values are available, however, identification will in many cases be practically complete, if the isomer in question is included in all of the sets, even though one or a few of the isomers may not appear in every one of the sets.

The accuracy necessary in the measurement of the property will depend upon the number of properties which are measured, the accuracy with which the values in the sets are known, the number of sets of properties in which a given isomer appears, the nature of these sets, and the distribution of the known values in the sets.

Example 3.—Given a pure chemical substance. Required to identify it by means of its properties. This example is similar to the preceding one except that the possibilities, instead of being restricted to a comparatively small number, include all chemical substances known and unknown. The success of the identification is materially affected by the choice of the property or properties to be measured. The principles governing this choice will not be discussed here. We shall content ourselves with an example based upon the use of the following extensively available properties: Freezing point, normal boiling point, and density. Starting with the freezing point suppose that we find $-25\pm1°$ C.

From some compendium of freezing-point data we now select those substances which have recorded freezing points within say $-25 \pm 6°$. If we use International Critical Tables for this purpose, we obtain a list of 69 substances. We now determine the normal boiling point of our substance and find $81 \pm 1°$. We now obtain a similar list of substances boiling within say 81 ± 6. We compare the two lists and eliminate all substances not common to both lists except those appearing in one list which have no value recorded for the property corresponding to the other list.

This procedure reduces our list to 16 items. We now measure the density of our substance at 20° and find 0.690 ± 0.001. From our 16 items we now discard all having density values outside the limits of $0.690 \pm$ say 0.005. This leaves us with the following items:

TABLE 5

Substance	Mol. wt.	M. P.	B.P.	$d_4^{20°}$
(1) Cl_2O_7	183	−30	82	
(2) SCl_4	174	−30		
(3) 2, 2, 3—Trimethylbutane	100	−25.0	80.9	0.6900
(4) 2, 2, 3—Trimethyl-1-butene	98		80	
(5) $ClCH_2CNO$	91.5		81	
(6) Crotonylamine, C_4H_9N	71		81	
(7) $(CH_3)_2Te$	157.5		82	
(8) $H_2C{:}NOH$	45		84	
(9) 2, 4—Dimethyl-2-pentene	98		84	$0.699^{25/4}$
(10) $(C_2H_5)_2PH$	90		85	

We are now justified in concluding that our substance is one of the 10 substances appearing in the above table, or it is a substance which did not appear in the lists employed. Undoubtedly most of the above substances have been retained in our final list solely because only one property is recorded for them. To reduce our list still further we must resort to chemical means or to the measurement of another characteristic.[10] Fortunately there is one property which is accurately known for all of these substances, that is the molecular weight. Suppose, therefore, we make a molecular weight determination and find 99 ± 5. Our list is now reduced to items (3), (4), and (9). A combustion analysis or a bromine or iodine number determination will eliminate either (3), or both (4) and (9). If No. (3) is eliminated in this way, we may resort to chemical methods to distinguish between (4) and (9); or, if the recorded data are trustworthy, to a more accurate determination of boiling point, density, or some other physical property.

It is obvious that the atomic weights of the elements, the atomic numbers of the elements, and the proportions with which the elements combine are all examples of Diophantine quantities. Consequently, any property or numerical characteristic of a substance which is a known function of any one or more of these quantities is itself a Diophantine quantity, and the problem of determining its value by measurement in the laboratory is a problem belonging to the subject of this paper.

[10] The chemist would materially reduce this list on the basis of obvious chemical characteristics or simple chemical tests.

V. EXAMPLES FROM PHYSICS

In the measurement of any magnitude composed of quanta, the problems involved in the precision aspects of the measurements are, in the last analysis, Diophantine in nature. In principle, nearly all physical quantities seem to be acquiring a Diophantine character with the continued extension of the process of quantization in physics. In the vast majority of cases, however, the quantization is as yet too fine-grained to be distinguishable from perfect continuity in so far as it can influence problems in precision of measurement, but with the continued development of quantum theory and the ability to measure small magnitudes, more practical examples are likely to be found. The method of treatment of such problems can be generalized as follows:

Given a class of magnitudes composed of quanta of the same kind. Required to determine (*a*) the number of quanta in a given magnitude and/or (*b*) the most accurate value of the magnitude of a single quantum. We proceed as follows:

1. Determine as accurately as convenient by any available method the approximate magnitude of the quantum.
2. Measure as accurately as possible the value of a magnitude composed of approximately n quanta, n being appropriately chosen (in accordance with the result obtained in 1) so as to make possible operation 3.
3. Divide the result obtained in 2 by that obtained in 1 and take the integer nearest the quotient. This will be the exact value [11] of n.
4. Now divide the result obtained in 2 by this exact value of n. This will give a new and more exact value for the magnitude of the quantum.

Using this new value, the above series of operations can now be repeated with a larger value of n and a still more accurate value for the magnitude of the quantum obtained. Continued repetition will result in continued improvement in accuracy as long as the new magnitude composed of the n quanta can be measured with a higher percentage accuracy than can the corresponding smaller magnitude in the preceding series. The accuracy required in the various steps of the above procedure can be readily computed by the methods outlined in the preceding pages.

Having once determined as accurately as possible by the above procedure the value, q, of a single quantum, then at any future time the value of any magnitude composed of n quanta can be found to the same accuracy with which q is known merely by measuring the magnitude with an accuracy sufficient to evaluate n.

For example, suppose it is desired to obtain, with an accuracy of about 0.1 per cent, the magnitude of an electrical charge which has been measured approximately and found to be (94 ± 2 per cent) $\times 10^{-10}$ esu. Dividing by the value of $e = 4.770 \times 10^{-10}$, we obtain, $n = 19.7 \pm 0.4 = 20$ and the value of our magnitude is $20 \times 4.770 \times 10^{-10} = (95.40 \pm 0.1 \text{ per cent}) \times 10^{-10}$ esu.

The above principles are applied for the purpose of determining the period of an undamped oscillator (such as a pendulum, for

[11] Method of counting by measurement. Used, for example, by Millikan to determine the number of elementary charges of electricity on a charged oil drop (Phys. Rev., 2, p. 124; 1913). Used also in a variety of industrial machines in which, by automatic or semiautomatic devices, weighing operations are utilized for counting industrial quanta, such as pins, nuts, bolts, coins, sheets of paper, etc.

example). The period is here the quantum and the total elapsed time between any two passages in the same direction through a given point is the larger magnitude composed of the n quanta.

If the "period" of the oscillator is accurately known then it might be employed for the accurate measurement of a time period composed of a whole number of such quanta. The period, or rather the corresponding wave length, is in the case of light waves employed in measuring distances with an interferometer, the principle of the method being substantially that used in the example of the electrical charge given above.

WASHINGTON, July 18, 1929.