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Validity of the Cosine-Fourth-Power Law of Illumination

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The cosine-fourth-power law states that the irradiance [1] at any point in the image formed by a photographic lens, in the absence of vignetting, is equal to $E_0 \cos^4 \beta$, where E_0 is the irradiance at the center of the field, and β is the angle between the axis of the lens and the conjugate chief ray in the object space. Although not usually so stated, this law involves the additional assumption that the lens is free from distortion. With this assumption, the law applies rigorously when the diaphragm is between the lens and object and the object is at an infinite distance. If the diaphragm is between the lens and the image plane, there may be cases in which the irradiance falls off less rapidly from the center of the field outward than is predicted by the cosine-fourth-power law. A type of negative distortion is defined for which the irradiance is uniform over the entire image. When the diaphragm is within the lens system (the more common condition for photographic lenses), one must know the distortion of the portion of the lens following the diaphragm before a definite statement regarding the irradiance of the image can be made. The departures from exactness of the cosine-fourth-power law arise partly because the effective area of the entrance pupil is a function of the obliquity of the incident chief ray. A method is given for measuring this variation in effective area.

I. Introduction

The cosine-fourth-power law may be stated as follows: In the absence of vignetting, the irradiance (or the effective exposure) for different parts of the image formed by a photographic objective varies as the fourth power of the cosine of the angle between the axis and the chief ray proceeding from the conjugate object area. The desirability of and the economic advantage resulting from the use of photographic objectives of extremely wide angle has made it important to reinvestigate the validity of this law because, unless the consequences of the cosine-fourth-power law can be evaded, negatives made with extremely wide angle photographic objectives will vary to such an extent in density from center to edge that their usefulness is greatly restricted. Although the cosine-fourth-power law is commonly set forth as a precise statement of fact, for most photographic lenses it is, at best, an approximate relation. Slussareff [2] has published a paper of fundamental importance in connection with this subject, in which he demonstrates the importance of the aberrations of the pupils, a characteristic that has been neglected in previous treatments of the cosine-fourth-power law. He concludes that the irradiance of the marginal parts of the field can be increased significantly beyond that predicted by the cosine-fourth-power law by one of the following three optical devices: appropriate aberrations of the entrance pupil; the introduction of negative distortion; and the use of a strongly curved concave image field with a concave photographic plate. Reiss [3] has taken exception to some of the methods of computation employed by Slussareff but accepts the limited validity of the cosine-fourth-power law and the importance of the aberrations of the pupils. In the present discussion the derivations of Slussareff and Reiss will be briefly repeated, and the treatment extended to cases not previously considered.

[1] Throughout the text of this paper the terms *irradiance* and *radiance* as defined by the Optical Society of America (J. Opt. Soc. Am. **34**, 184, 1944) will be used instead of the older terms *illumination* and *brightness*. Illumination and brightness usually relate to radiant energy as evaluated in terms of the luminosity function, a consideration which does not apply when dealing with an image to be recorded by photography. All the relations involved in this paper are geometrical, and hence the equations apply with equal rigor, whether applied to radiant energy, radiation evaluated in terms of theluminosity function, or radiation evaluated in terms of photographic sensitivity.

[2] J. Phys. USSR **4**, 537 (1941).

[3] J. Opt. Soc. Am. **35**, 283 (1945).

II. Magnification Ratios and Distortion of Pupils

If dF is the radiant flux proceeding from one elementary area, dA, to a second, dA' (see fig. 1), its magnitude is given by the equations

$$\left. \begin{aligned} dF &= B \cos\alpha \cos\alpha' \frac{dAdA'}{r^2} \\ &= B \cos \alpha d\Omega' dA \\ &= B \cos \alpha' d\Omega dA' \end{aligned} \right\}, \tag{1}$$

where B is the radiance of the radiating surface, r is the distance between the two elementary areas, α is the angle between the radius vector and N, the normal to dA, and α' is the similar angle between the radius vector and N', the normal to dA'. In the second and third equations, $d\Omega'$ is the solid angle subtended by the area dA' at dA and $d\Omega$ is the solid angle subtended by dA at dA'. The quantities dA and dA' enter symmetrically in these equations, and dF may be assumed to represent the flux proceeding in either direction between the two elementary areas, provided that the appropriate value of B is used. If the normals to the two elementary areas are parallel, $\alpha=\alpha'$ and eq 1 becomes

$$dF = \frac{B}{e^2} \cos^4 \alpha dAdA', \tag{2}$$

where e is the distance between the parallel planes containing dA and dA'.

In figure 2 a photographic objective is represented with the conjugate areas dA and dA' in the object and image spaces respectively. At S and S', are two additional planes normal to the axis of the lens, one being in the object space and the other in the image space. The shaded pencil includes all rays proceeding from dA through the lens to dA'. The cross sections of this pencil, in the planes S and S' are dS and dS', respectively. If, for the moment, absorption and reflection losses in the lens are neglected, B is the same for each of the four elementary areas under consideration and the flux through all the cross sections of the pencil is the same. Therefore, applying eq 2, one may write

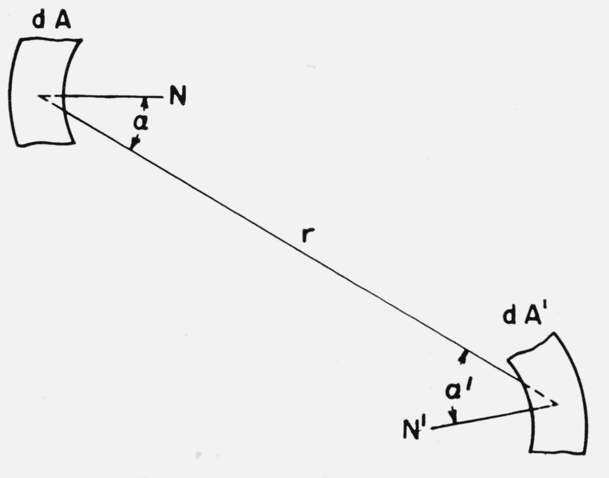

FIGURE 1. *Geometrical variables that determine the flux radiated from one elementary area to a second.*

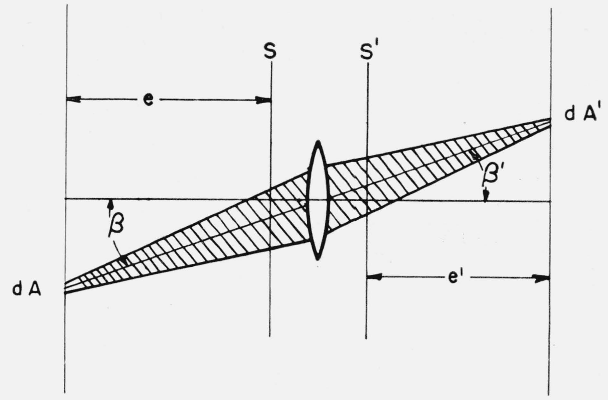

FIGURE 2. *Geometrical variables governing the transfer of radiant energy from the elementary area dA in the object space to the conjugate area dA' in the image space.*

$$\frac{B}{e^2} \cos^4 \beta dAdS = \frac{B}{e'^2} \cos^4 \beta' dA' dS', \tag{3}$$

where β is the angle between the elementary pencil in the object space and the normals to the parallel elements of surface dS and dA. In the image space, β' is the corresponding angle. From eq 3

$$\frac{dS'}{dS} = \frac{e'^2}{e^2} \frac{dA}{dA'} \frac{\cos^4 \beta}{\cos^4 \beta'}. \tag{4}$$

If it be assumed that the photographic lens is free from distortion, $dA/dA'=1/M^2$, where M is the magnification. Equation 4 can, therefore, be written

$$dS'/dS = C \cos^4 \beta / \cos^4 \beta', \tag{5}$$

where $C=e'^2/e^2M^2$ is a constant independent of β and β'. It will now be assumed that S and S' are the planes of the entrance and exit pupils, in which case dS and dS' are conjugate elementary areas in the entrance and exit pupils, respectively.

Equation 5 now has a twofold interpretation. If it be assumed that the conjugate areas dA and dA' remain fixed, the ratio $\cos^4 \beta/\cos^4 \beta'$ will, in general, be different for different pairs of conjugate areas in the entrance and exit pupils. It follows that the exit pupil will be a distorted image of the entrance pupil. If, on the other hand, dS and dS' are assumed to be two conjugate elementary areas that contain the axial points of the planes S and S', the rays making angles β and β' with the axis are chief rays. If now the positions of dA and dA' are allowed to vary within the object and image planes, the ratio $\cos^4 \beta/\cos^4 \beta'$ will, in general, not be constant, and this indicates that the areal magnification at the axial point of the exit pupil, with respect to the entrance pupil, is a function of the inclination of the chief ray. These conclusions are in conflict with deductions based on first-order imagery, and the departures are of the same order as the variations of irradiance in the image plane with which we are concerned. Consequently, it is necessary that these departures from first order imagery not be neglected when computing the irradiance in different parts of the field of a photographic objective.

III. Irradiance at Any Point in Image Plane

If either member of eq. 3 is integrated over the area of the corresponding pupil, one obtains the total flux through the elementary area dA' in the image space. Dividing by dA', one obtains the equation

$$E=\frac{B}{e^2M^2}\int^{S}\cos^4 \beta dS=\frac{B}{e'^2}\int^{S'}\cos^4 \beta' dS', \qquad (6)$$

where E is the irradiance. For any point in the field of the lens, for which vignetting does not occur, either integration will give the correct value of the irradiance. The angles β and β' are different for different points in the pupil areas S and S', respectively, and are also functions of the point in the field for which the irradiance is being computed. Care must be used in selecting the appropriate boundary for the area over which the integration is extended. If the pupil over which one integrates is an actual physical diaphragm, the integration is extended over the aperture and the limits of integration are the same for all points of the field. On the other hand, if the pupil is an image of a diaphragm positioned somewhere else in the optical system, eq 4 indicates that the area over which the integration is to be performed, in general, will be a function of the point in the field for which the irradiance is sought. The correct area over which the integration should extend is the cross section of the complete pencil lying in the pupil plane. Equation 6 will now be applied to special cases that admit of general treatment.

IV. Special Illustrative Examples

1. Diaphragm Follows the Lens

For this example it is better to integrate the third member of eq 6 because of the simpler limits for the integration. If the aperture of the diaphragm is circular, of radius a, and if β_0' is the angle between the axis and the chief ray passing through the point under consideration, the integral is seen to be identical with that giving the irradiance produced by a disk of radius a and of uniform brightness B at a point distant e' from the disk and at a distance $e' \tan \beta_0'$ from the axis. The exact value of this integral, as given by Foote [4] may be written in the form

$$E=\frac{\pi B}{2}\left[1-\left(1+\frac{4a^2k^2}{(x^2+k^2-a^2)^2}\right)^{-\frac{1}{2}}\right], \qquad (7)$$

where a is the diameter of the disk, k is the distance of the selected point from the plane of the disk, and x is the distance of the point from the axis. These magnitudes and their equivalents in the

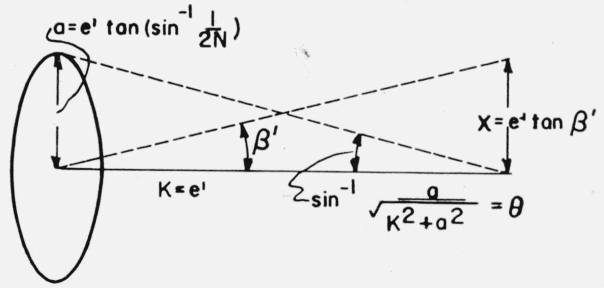

FIGURE 3. *Geometrical quantities involved in computing the irradiance produced by a disk of radius a and of uniform radiance at a distance e' from the disk and at a distance e' tan β' from its axis.*

present notation are shown in figure 3. If θ is defined as $\tan^{-1}a/k$ and β_0' is defined as $\tan^{-1}x/k$, eq 7 can be written

$$E=\frac{\pi B}{2}\left[1-\left(1+4\cos^4 \beta_0' \frac{\tan^2 \theta}{(1-\tan^2 \theta \cos^2 \beta_0')^2}\right)^{-\frac{1}{2}}\right]. \qquad (8)$$

[4] Bul. BS **12**, 583 (1915) S263.

With this notation the aperture ratio of the lens is $1:N$, where $N=1/(2 \sin \theta)$. For a point on the axis, $\beta_0'=0$, and eq 8 reduces to the familiar equation

$$E=\pi B \sin^2 \theta=\pi B(\tfrac{1}{2}N)^2. \qquad (9)$$

If eq 8 is expanded with reference to the exponent $-1/2$ by the binomial theorem and the first two terms of the expansion retained, one obtains the equation

$$E=\pi B \cos^4 \beta_0' \left(\frac{\tan \theta}{1-\tan^2 \theta \cos^2 \beta_0'}\right)^2 + \ldots \qquad (10)$$

Equation 10 is not recommended for computational purposes, but it does show that the irradiance produced by a uniformly irradiated disk at a point off the axis varies approximately as the fourth power of the cosine of the angular displacement of the point from the axis. By the argument of the preceding paragraph, it follows that when a photographic objective has the diaphragm between it and the image plane, the irradiance of the field varies approximately as the fourth power of the cosine of the angle between the axis and the chief ray in the *image* space.

An estimate of the degree of this approximation is afforded by table 1.

TABLE 1.—*Comparative values of irradiance (or effective exposure) in the field of a photographic objective as a function of relative aperture and angular distance β_0' from the center of field when the diaphragm is between the lens and the image plane*

β_0'	Aperture ratio		
	1:8	1:4	1:2
0°	1.000	1.000	1.000
20°	1.000 cos⁴ 20°	1.007 cos⁴ 20°	1.027 cos⁴ 20°
30°	1.002 cos⁴ 30°	1.013 cos⁴ 30°	1.052 cos⁴ 30°
40°	1.004 cos⁴ 40°	1.018 cos⁴ 40°	1.076 cos⁴ 40°

In this table, the values of the irradiance have been computed by means of eq. 7, which is exact, for aperture ratios 1:8, 1:4, and 1:2 and for the values 0, 20, 30, and 40 degrees. The computed values are expressed as the product of $\cos^4 \beta_0'$ and a numerical coefficient. It will be noted that the coefficients of $\cos^4 \beta_0'$ are greater than 1 for points not on the axis, indicating that the irradiance of the image falls off less rapidly than $\cos^4 \beta_0'$. If a lens is so designed that the second principal plane lies between the lens and the focal plane, the diaphragm may be placed in the second principal plane, and β will equal β_0'. In such a case, one has a lens in which the irradiance of the image decreases from the center of the field outward at a rate somewhat less than that indicated by the cosine-fourth-power law, although the departure is not of sufficient magnitude to be of great importance photographically. It does, however, show conclusively that the cosine-fourth-power law does not represent a limiting condition of maximum attainable uniformity of image irradiance. If the diaphragm precedes the second principal plane, β is greater than β_0', and the departure from the cosine-fourth-power law will be still greater.

2. Diaphragm Precedes the Lens

For this example the first integral of eq 6 is the one more convenient to apply. This integral represents the irradiance produced by a disk of radius b, where b is the radius of the entrance pupil and of brightness B/M^2 at a distance e^2. In most cases encountered in practice, the object will be at a distance of several focal lengths in front of the lens, and consequently the numerical aperture of the incident pencil will be much less (N greater) than the values assumed in the preceding case. Referring to table 1, it is evident that the irradiance will be more nearly proportional to the fourth power of the angle (measured in the object space) than for the preceding example. If, for example, the object is 10 focal lengths' distant, the aperture ratio is approximately $1:10N$. For a 1:2 lens, the aperture ratio in the object space, therefore, becomes 1:20, and it is clear that the departures from the cosine-fourth-power law will be much less than those tabulated for the preceding example.

3. Diaphragm Precedes the Lens, Object at Infinite Distance

This is a particular limiting case of the preceding example. When the first integral of eq 6 is applied, it is discovered that e becomes infinite and M becomes zero in the limit as the distance to the object increases. The limiting value of the product Me is required. If c denotes the distance from the first principal focus to the plane of the entrance pupil,

$$M=f/(e+c) \qquad (11)$$

and

$$Me=fe/(e+c). \qquad (12)$$

It follows that

$$\lim_{e \doteq \infty} Me = f \tag{13}$$

With the object at infinity, β is constant over the entrance pupil. From eq 6

$$E = \frac{BS}{f^2} \cos^4 \beta, \tag{14}$$

and this example presents an instance in which the cosine-fourth-power law, as stated in the introduction to this treatment, is precisely followed.

4. Diaphragm in Plane of First Principal Focus, Object at Infinite Distance

This is a particular case of example 3. The imagery is telecentric in the image space, and all chief rays are normal to the focal plane. As β_0' equals zero for all parts of the image surface, it has at times been mistakenly assumed that the irradiance will be uniform. It is evident that there is nothing in eq 6 to support this conclusion. The application of the first integral of eq 6 gives the result that has been obtained in the preceding example and indicates the irradiance varies as $\cos^4\beta$. If the second integral is applied, it must be remembered that the exit pupil is infinitely large and at an infinite distance. The value of e' also becomes infinite and the integral remains finite. Although it is true that β_0' is zero for all the pencils, test measurements will indicate that the solid angles included by the pencils at different distances from the center of the field are not equal but vary in accordance with eq 5, and consequently a correct evaluation of the two integrals of eq 6 gives identical results.

5. Diaphragm Precedes Lens, Object at Infinite Distance, Distortion Present

Referring to the first integral of eq 6, when there is distortion the areal magnification, M^2 is a function of β and must remain within the integral. If one writes

$$M^2 = M_0^2 \cos^4 \beta, \tag{15}$$

where M_0^2 is the areal magnification at the center of the field, eq 6 becomes

$$E = BS/e^2 M_0^2, \tag{16}$$

and the field is uniformly irradiated. In the presence of distortion, the linear magnification is not the square root of the areal magnification. For a system possessing rotational symmetry, it can be shown that eq 15 is satisfied, provided

$$r' = M_0 e \sin \beta, \tag{17}$$

where r' is the radial distance from the center of the image plane to the image of an object point distant $e \tan \beta$ from the axis. Applying the condition that e becomes infinite, eq 16 becomes

$$E = BS/f^2, \tag{18}$$

where f is the focal length corresponding to the scale of the image in the neighborhood of the axial point. Equation 17 can therefore be written

$$r' = f \sin \beta. \tag{19}$$

For an undistorted image, the corresponding equation is

$$r'' = f \tan \beta. \tag{20}$$

Consequently, for an object point at the angular distance β from the axis, the linear distortion is $f(\sin \beta - \tan \beta)$, which is negative and corresponds to "barrel shaped" distortion. The ratio of radial magnification of corresponding points in the distorted and undistorted images is $dr'/dr'' = \cos^3\beta$. In general, when negative distortion is present, even though it does not follow eq 16 precisely, the irradiance over the field is more nearly uniform than when distortion is absent.

6. Diaphragm Between Components of Lens

Most photographic objectives are designed with the diaphragm within the optical system, and eq 6 cannot be applied in the general manner of the preceding examples because neither the entrance nor exit pupil is a physical diaphragm, and the region over which either integration must be extended is a function of the position chosen in the image field. It will be assumed that the lens system is divided into two parts, a and b of focal lengths f_a and f_b, the subscript a referring to the part of the lens between the object and diaphragm, and b to the part following the diaphragm. Equation 6 will be applied to the part of the lens consisting of the diaphragm and the part b. The equation becomes

$$E = B/e^2 \int^{S_d} \frac{\cos^4 \beta_d}{M_b} dS, \tag{21}$$

where S_d refers to the area of the diaphragm, β_d is the angle between a ray and the lens axis in the space occupied by the diaphragm, and M_b is the areal magnification for the part of the system under consideration. (It is assumed that the image formed by the first part of the system, which serves as the object for the second part, is stigmatic.) If the rays passing through the diaphragm and proceeding to any given point in the image field are parallel, β_d is a constant over the area of the diaphragm, and eq 21 becomes

$$E = B/e^2 \cos^4 \beta_d \int^{S_d} \frac{1}{M_b} dS. \qquad (22)$$

Even if the rays passing through the diaphragm are not parallel, for most lens systems in use, the intermediate image is remote, and the aperture ratio of the beam is small. Consequently, as is shown by reference to table 1, eq 22, while not rigorously exact, will be a good approximation.

The areal magnification M_b will be a constant if part b of the lens system introduces no distortion. In general, however, part b is a positive lens system preceded by a diaphragm, and the distortion for such a system is frequently negative. As has been mentioned, the effect of negative distortion is to increase the irradiance of the peripheral part of the image as compared with the central part. For most photographic systems β_d is greater than β. Consequently, the factor $\cos^4 \beta_d$ indicates a more rapid decrease of irradiance from the center outward than would be predicted by the factor $\cos^4 \beta$. Therefore, in eq 22, the factors $1/M_b$ and $\cos^4 \beta_d$ indicate departures from the cosine-fourth-power law in opposite directions, and a system with an internal diaphragm must be carefully analyzed before it is possible to say in which direction the irradiance deviates from the cosine-fourth-power law. In particular, if the portion of the lens that follows the diaphragm is free from distortion, M_b is a constant and, as is shown by eq 22, the irradiance varies as the fourth power of the angle between the chief ray and the lens axis in the space occupied by the diaphragm.

V. Effect of Light Losses Introduced by Reflection or Absorption

In the preceding examples it has been assumed that the lens is free from losses by reflection and absorption. In an uncoated lens system such losses are seldom less than 30 percent and may be considerably greater. Actually, the losses by reflection and absorption for any element of a lens system are functions of the angle at which the pencil of light is incident. These losses, therefore, are functions of β or β' of eq 6, and if they are to be correctly taken into account, the transmission coefficient should be within the integral sign. Deriving an analytical expression for the loss by absorption or reflection as a function of the angle of incidence, involving the curvatures of the several surfaces and the variations in thickness of the lens elements introduces so many complications, however, that it is usual to consider the transmittance as independent of β or β' and to write it in front of the integral sign, as a factor multiplied into B. This gives results satisfactorily accurate for most photometric questions that arise in connection with a photographic objective. When the transmission factor is written in front of the integral sign of eq 6, the magnitude of the factor does not affect the indicated variations in the irradiance of the field that have been derived in the several examples of section IV. Neither does the neglect of the effects of absorption and reflection introduce any error in the geometric relations that have been derived between entrance and exit pupils.

One can use a filter that is denser in the center and becomes more transparent from the center outward. If such a filter is placed some distance from a pupil plane, the transmitted beams irradiating the central parts of the field pass through the denser parts of the filter while the more oblique beams that irradiate the peripheral parts of the field pass through the outer less dense parts of the filter. Such a filter acts selectively to make the irradiance of the field more uniform. Unless one has an abundance of light this is an undesirable method for increasing the uniformity of irradiance because it reduces the average irradiance and therefore increases the length of the required exposure. To compute the effect of such a filter, the analytical expression for its transmittance, as a function of β (or β') should be introduced into the integrals of eq 6.

VI. Experimental Method for Measuring Pupils

When a pupil is the image of a diaphragm formed by a portion of the lens system, its size and shape can be most readily determined by a photographic method. If, for example, the size and shape of the entrance pupil are to be determined, a camera may be placed in front of the lens of which the pupil is to be measured and the diaphragm photographed through the parts of the lens that come between the actual physical diaphragm and the object space. If the camera system is centered on the axis of the lens and directed axially toward the diaphragm, one obtains the pupil corresponding to $\beta=0$. If, now, the camera remains fixed and the lens under test is rotated through the angle β_0 about an axis in the plane of the entrance pupil and at the right angles to the optical axis of the lens, one obtains the pupil corresponding to $\beta=\beta_0$. The images of the entrance pupils are measured on the resulting negatives by means of a planimeter and multiplied by a suitable scale factor to determine the area of the entrance pupil. The diaphragm aperture can be irradiated to be photographed by a ground glass and lamp placed back of the lens. This method of measurement gives the area of the entrance pupil as affected by vignetting if present. To similarly measure the exit pupil, the diaphragm is photographed through the back component of the lens under test.

The method of the preceding paragraph is satisfactory for most purposes but is not rigorously accurate because, for the oblique positions, different parts of the pupil are at different distances from the lens and are reproduced to different scales resulting in a distortion. This can be eliminated by using a telecentric system as shown in figure 4. The lens for which the entrance pupil is to be determined is shown at A. The entrance pupil, which is the image of the diaphragm formed by the first two elements of the system, is represented at B. Actually, for the case illustrated, the entrance pupil would be within the lens, but it is shown here in front of the lens to emphasize its position in the object space. At C there is a lens, preferably corrected for chromatic aberration, which has the small diaphragm D in its second focal plane. Lens G and image plane H schematically indicate the camera with which the photograph is made. The diaphragm at D should in reality be the entrance pupil of the lens G. In the object space of lens C the chief rays are parallel, and therefore the different parts of the entrance pupil are reproduced to the same scale. This arrangement gives a correct determination of the entrance pupil when the object is at an infinite distance.

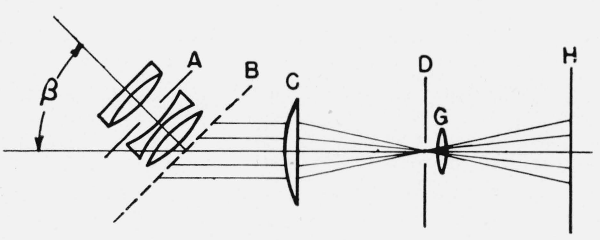

FIGURE 4. *Diagrammatic sketch of arrangements of parts for experimental determination of the dimensions of an entrance pupil of a photographic objective.*

VII. Computational Method for Determining Diameters of Pupils

To determine the size of either entrance or exit pupil by computation is a tedious process. If the entrance pupil is to be determined for the angle β_0, a point in the object plane distant $e \tan \beta_0$ from the axis is chosen, and rays from this point are traced into the lens as far as the particular space in which the physical diaphragm is located. By successive tests a series of rays are finally found that intersect the edge of the diaphragm. The tracing must be done trigonometrically and, skew rays are required. After a sufficient number of rays intersecting the edge of the diaphragm have been traced, one can construct the surface that encloses all the rays of the transmitted pencil. The cross section formed by the intersection of this surface with the plane of the entrance pupil defines the entrance pupil corresponding to β_0. If there is vignetting, the rays must be traced through the entire system and the size of the maximum transmitted beam determined. To determine the exit pupil, one traces rays in a similar manner, proceeding from the selected image point in the reverse direction through the lens.

WASHINGTON, May 7, 1947.