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Synchronization of Oscillators

By Robert D. Huntoon and Albert Weiss

A theory is presented which predicts the behavior of any self-limiting oscillator in the presence of an injected sinusoidal voltage or current of small but constant magnitude. The internal mechanism responsible for synchronization is not needed, and the theory is thus applicable to any source of alternating current. Experimental verification of the theory is presented for the case of a low-power Hartley oscillator operating at 11.5 megacycles.

The theory is extended to include the mutual synchronization of two oscillators of arbitrary properties, and a method of treating the mutual interaction of several oscillators is outlined.

The theory is applied to a number of examples to indicate briefly the properties of a synchronized oscillator when used as (1) a linear voltmeter for small voltages, (2) a field-intensity meter, (3) a linear amplitude modulation demodulator for small signals, (4) a frequency modulation demodulator, (5) a synchronous amplifier limiter. The use of a synchronized oscillator for these applications is of particular interest because microwave generators can be used in addition to the more conventional triode oscillators.

I. Introduction

The early experiments of Vincent,[1] followed by Appleton's [2] theoretical treatment, have led to a considerable interest in possible practical applications of the synchronization of oscillators.[3] Since the publication of these early papers there has been a continually growing literature on the subject,[4] with attention now primarily centered on (a) the use of an oscillator as a synchronous-amplifier-limiter for frequency modulation reception, and (b) the use of a chain of synchronous oscillators to drive a linear accelerator for the production of high-energy atomic particles. There are, of course, numerous other applications, some of which are discussed in terms of the theory that is the subject of this paper.

Following Appleton, theoretical treatments of oscillator synchronization have been concerned with the mechanism within a triode oscillator that accounts for synchronization. The phenomenon of synchronization with a disturbance impressed from an external source is not limited to triode oscillators. Rather, any source of alternating electromotive force whose frequency depends upon the load impedance attached to it (magnetron, for example) will exhibit similar behavior. It should thus be possible to discuss certain general features of synchronization without reference to the internal mechanism which accounts for it. The theory so derived will be generally applicable to all types of oscillators.

In a recent paper Adler [5] has developed a differential equation whose solution accounts for many of the observed phenomena of synchronization. Again, the triode oscillator mechanism has been the basis of the discussion. However, the scheme used by Adler can be extended in a manner

[1] J. H. Vincent, On some experiments in which two neighboring maintained oscillatory circuits affect a resonating circuit, Proc. Roy. Soc. (London) **32**, [B], 84 (1920).

[2] E. V. Appleton, The automatic synchronization of triode oscillators, Proc. Cambridge Phil. Soc. **21**, 231 (1922).

[3] The term "oscillator" as used here means a source of harmonic vibration whose steady-state amplitude is limited to a finite value by some internal nonlinear characteristic.

[4] See, for example, Robert Adler, A study of locking phenomena in oscillators, Proc. Inst. Radio Engrs. **34**, 351 (1946).

[5] Robert Adler, A study of locking phenomena in oscillators, Proc. Inst. Radio Engrs. **34**, 351 (1946).

that does not involve the particular generator. The result is a differential equation similar to his but more general. In addition, amplitude behavior, as well as frequency behavior, can be included.

The development proceeds in a manner somewhat similar to Adler's and is based upon similar assumptions regarding time constants, etc. The performance of the oscillator is specified in terms of a set of compliance coefficients that show how amplitude and frequency depend upon the load impedance. The frequency coefficient is closely related to the conventional pulling parameter. The values of the coefficients are not derived here but are assumed to be given as constants of the problem. They may be derived theoretically or measured for the particular oscillator. Briefly, the scheme of the argument is that the given source of electromotive force is connected to a known load impedance [6] into which the outside disturbance is to be coupled. The frequency of oscillation F and some amplitude parameter A, such as d-c plate current, a-c plate swing, d-c grid bias, or output voltage are measured or calculated as a function of the load impedance. Each of the above parameters is expanded in a Taylor series about the operating value of the load impedance. Only first-order terms in the series are kept, so the theory is strictly true for small disturbances only. Actually, large disturbances are treated with reasonable accuracy.

The voltage (or current) coupled into the load impedance from the external source is treated as an added impedance (or admittance) in accordance with the compensation theorem.[7] This is valid if (1) the disturbance is so small that it does not materially alter the magnitudes of voltage or current established in the load by the oscillator, (2) both source and disturbance are free of harmonics, (3) both have the same frequency. The first limits the size of disturbance for which the theory is valid. The second does not ordinarily need consideration when circuit Q's are large. If harmonics are present to any great extent, a second-order theory will be needed. The third is avoided when the frequencies are not the same by saying that the impedance equivalent to the impressed disturbance varies with time in such a manner that its phase angle accounts for the difference frequency. This is a good enough approximation when the conditions established by Adler are met, i. e., when the difference frequency is small compared with the band width of the circuits involved.

By using the compliance coefficients and the impedance equivalent to an impressed disturbance, the general properties of synchronization can be deduced.

II. Synchronization by an Impressed Voltage

In the discussion to follow, complex quantities will be represented by bold-faced type; quantities not so designated will denote absolute magnitudes. The factor $e^{j\omega t}$ will usually be omitted.

1. Compliance Coefficients

Let figure 1 represent an energy source of the type that converts d-c energy to a-c energy, such as a typical triode oscillator or magnetron.

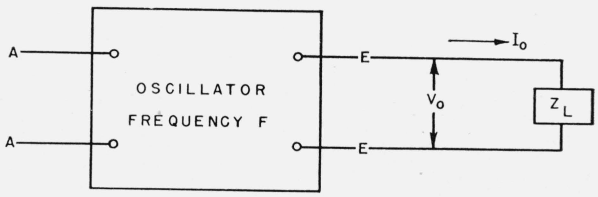

FIGURE 1.—*Self-limiting oscillator.*

For the discussion, we are interested in two pairs of terminals. The terminals marked E are assumed to be the output terminals of the device for delivering a-c power to a load impedance Z_L or admittance Y_L.

Terminals marked A represent a pair of terminals that give a d-c or a-c indication of the amplitude of oscillation, such as grid bias, d-c plate current, or the reading of a diode voltmeter across the output circuit.

Assume that there are also available, if necessary, instruments which indicate either the voltage V_0 across the load or current I_0 through it.

Let $\boldsymbol{V}_0$ and $\boldsymbol{I}_0$ be the initial or undisturbed value of these quantities when the oscillator is feeding its load circuit. Similarly, let F represent the frequency of the oscillator, and F_0 its undisturbed value.

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[6] This load impedance may be a part of the oscillator circuit, such as the plate tank of a triode oscillator.

[7] F. E. Terman, Radio Engineers Handbook, p. 198 (McGraw-Hill Book Co., Inc., New York, N. Y., 1943).

We proceed to evaluate the performance of the oscillator by adding small impedances, $\boldsymbol{z}$, in series with the load. We determine the oscillator performance by measuring the effect of $\boldsymbol{z}$ as shown by changes in the reading across A–A. Suppose that the information is presented in the form of a d-c voltage, A, between terminals A–A, grid bias, for example. The useful parameters of the oscillator will be specified by the following compliance coefficients:

$$A_r = \left.\frac{\partial A}{\partial r}\right|_{r=0} \qquad A_x = -\left.\frac{\partial A}{\partial x}\right|_{x=0} \tag{1}$$

$$F_r = \left.\frac{\partial F}{\partial r}\right|_{r=0} \qquad F_x = -\left.\frac{\partial F}{\partial x}\right|_{x=0}, \tag{2}$$

where

$$\boldsymbol{z} = r + jx. \tag{3}$$

As already indicated, we proceed by expanding A and F in a Taylor expansion about A_0 and F_0, keeping only first-order terms. This gives

$$A - A_0 = rA_r - xA_x \tag{4}$$

$$F - F_0 = rF_r - xF_x. \tag{5}$$

There will also be occasion to use two complex compliance coefficients, an amplitude coefficient, $\boldsymbol{C}_A$, and a frequency coefficient, $\boldsymbol{C}_F$. They are defined by

$$\boldsymbol{C}_A = C_A e^{+j\alpha} = A_r + jA_x = \sqrt{A_r^2 + A_x^2}\, e^{+j\alpha} \tag{6}$$

$$\boldsymbol{C}_F = C_F e^{+j\beta} = F_r + jF_x = \sqrt{F_r^2 + F_x^2}\, e^{+j\beta}. \tag{7}$$

Throughout the discussion only series impedance will be used; the argument can be based upon admittances in a similar manner.

2. Synchronization Equation

Suppose that a small voltage is induced in the load circuit from an outside source. Assume the voltage is small enough so that the change in I can be neglected, and that we can, with sufficient accuracy, represent I by its initial value I_0. By the compensation theorem we can replace the induced voltage $\boldsymbol{v}$ by a small impedance z, where

$$\boldsymbol{z} = \frac{v}{I_0} e^{j\phi}. \tag{8}$$

We may thus write (keeping only real parts)

$$A - A_0 = \boldsymbol{C}_A \boldsymbol{z} = \frac{C_A v}{I_0} \cos(\phi + \alpha) \tag{9}$$

$$F - F_0 = \boldsymbol{C}_F \boldsymbol{z} = \frac{C_F v}{I_0} \cos(\phi + \beta), \tag{10}$$

where

$$\tan \alpha = \frac{A_x}{A_r}, \qquad \tan \beta = \frac{F_x}{F_r}, \tag{11}$$

and it will be remembered that in typical oscillators $A_x \ll A_r$, while $F_r \ll F_x$. Thus α is usually very small, and β is nearly $\pi/2$. There is, however, a special, interesting case where this is not true. It will be discussed later.

We now proceed with eq. 10, as done by Adler, to establish the synchronization equation. Assume that the injected voltage $\boldsymbol{v}$ has the frequency F^1 and that the instantaneous frequency of the oscillator is F. We then have

$$\frac{1}{2\pi}\frac{d\phi}{dt} = F^1 - F = (F^1 - F_0) - (F - F_0). \tag{12}$$

If $F^1 - F$ is not too large, we can consider the equivalent impedance $\boldsymbol{z}$ to be varying with time as its phase angle changes. If the frequency changes are not too rapid,[8] the oscillator frequency will follow the impedance changes in the manner predicted by eq. 10. We may thus write

$$\frac{1}{2\pi}\frac{d\phi}{dt} = F^1 - F_0 - \frac{C_F v}{I_0} \cos(\phi + \beta), \tag{13}$$

an equation similar to that obtained by Adler. F^1 and F_0 are constant frequencies, the latter being the free-running frequency of the oscillator. Equation 13 is a differential equation for ϕ, and its solution shows how the beat frequency $(1/2\pi)\,(d\phi/dt)$, varies with time.

[8] See remarks by Robert Adler, A study of locking phenomena in oscillators, Proc. Inst. Radio Engineers **34**, 351 (1946). Also note:

$$\boldsymbol{I} = I e^{j2\pi F t}$$

$$\boldsymbol{v} = v e^{j2\pi F^1 t}$$

$$\boldsymbol{z} = \frac{v}{I} e^{j2\pi(F^1 - F)t} = \frac{v}{I} e^{j}$$

If we let

$$F^1 - F_0 = f$$

and

$$\frac{C_F v}{I_0} = Kv, \tag{14}$$

eq. 13 becomes

$$\frac{1}{2\pi}\frac{d\phi}{dt} = f - Kv \cos(\phi + \beta). \tag{15}$$

This makes f the beat frequency that would exist if the oscillator did not react to the injected voltage, i. e., the undisturbed beat frequency.

It is evident immediately from eq 15 that the solution $\phi(t)$ is of a complicated periodic form when

$$f^2 > K^2 v^2 \tag{16}$$

and reduces exponentially to a steady value of ϕ when

$$f^2 < K^2 v^2. \tag{17}$$

Equation 17 corresponds to synchronization between the injected voltage and the oscillator current at a fixed phase angle, ϕ. As we are interested primarily in synchronization, the solution of eq 15, subject to eq 17, is needed. It is

$$\frac{\cos\psi - \cos(\phi+\beta)}{1 - \cos(\phi + \beta - \psi)} = \text{const}\ e^{-2\pi t\sqrt{K^2v^2 - f^2}}, \tag{18}$$

where

$$\cos\psi = \frac{f}{Kv}, \tag{19}$$

from which we observe that the steady state value of ϕ for large t is given by

$$\cos(\phi+\beta) = \frac{f}{Kv}. \tag{20}$$

The equilibrium value is approached in such a manner that the time constant is approximately

$$\tau \simeq \frac{1}{2\pi\sqrt{K^2v^2 - f^2}} = \frac{1}{2\pi Kv \sin\psi}. \tag{21}$$

There are two values of $(\phi+\beta)$ that satisfy eq 20. One corresponds to stable equilibrium, the other to unstable equilibrium. From eq 15

$$\frac{1}{2\pi}\frac{d}{a\phi}\left(\frac{d\phi}{dt}\right) = Kv \sin(\phi+\beta). \tag{15a}$$

For stability $(d/d\phi)(d\phi/dt)$ must be negative Thus only values of $(\phi+\beta)$ such that $\sin(\phi+\beta)$ is negative lead to stable synchronization.

Equation 20 differs from the result obtained by Adler in the phase angle β. The size of β is important where the exact phase of lock-in is important, as in the application to the linear accelerator.

Equation 20 shows that synchronization can be obtained over a range of f, such that

$$-Kv < f < Kv,$$

or over a band of frequencies

$$\Delta f = 2Kv. \tag{22}$$

3. Amplitude changes

The quantity $a = A - A_0$ expresses the change of some convenient amplitude parameter, such as radio-frequency voltage, d-c grid bias, or d-c plate current, in the presence of the injected signal. It is evident from eq 10 that a and f are functionally related through the parameter ϕ. As f is varied from one edge of the synchronization band to the other, ϕ varies through the range

$$-\beta > \phi > -\pi - \beta. \tag{23}$$

This variation in ϕ produces variations in a, so that it exhibits a characteristic wave from which $(\alpha - \beta)$ can be readily deduced. By defining new quantities,

$$\begin{aligned} \delta &= (\phi+\beta) \\ \rho &= (\alpha - \beta). \end{aligned} \tag{24}$$

we can write eq 10 in terms of the dimensionless variables (U, W)

$$U = \frac{fI_0}{C_F v} = \cos\delta \tag{25}$$

$$W = \frac{aI_0}{C_A v} = \cos(\delta + \rho), \tag{26}$$

from which it is evident that the form of the a versus f wave is independent of I_0 and v (for small disturbances). Also the relation between U and W is an ellipse which degenerates into a line when $\rho = 0$ or π, and into a circle when $\rho = \pm\pi/2$.

Figure 2 shows the form of the synchronization amplitude pulse in terms of the variables U and W for several values of ρ. A study of figure 2 and eq 25 and 26 shows the following:

1. When $\rho = \pm\pi/2$, a is zero at the ends of the synchronization band, and the a-curve is symmetrical.

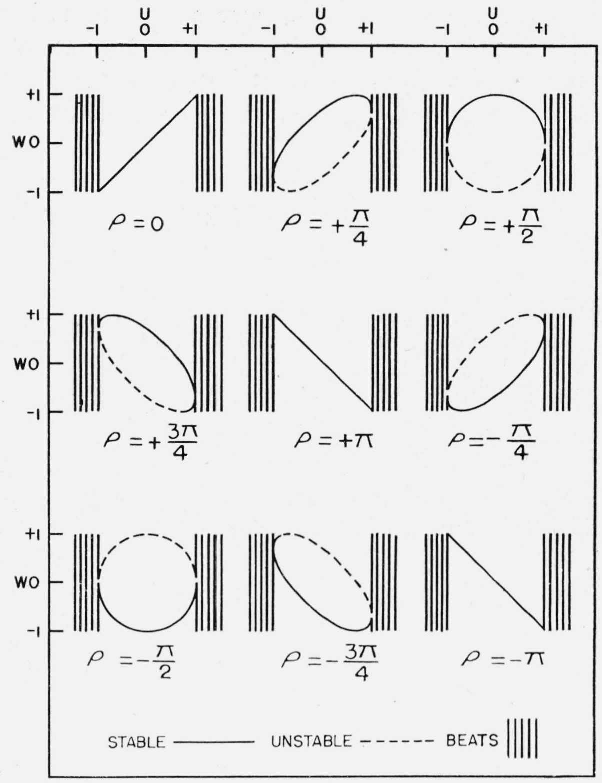

FIGURE 2.—*Forms of the U-W curve in the region of synchronization.*

2. When $\rho \neq \pm\pi/2$, the a-wave is not symmetrical, and a is not zero at the edge of the synchronization band. Thus at the instant beats cease, the phase between the oscillator and the injected voltage is such that there is a resistance component of loading and a has a finite value.

3. When $\rho=0$ or π, there is a linear relation a and f. The a-curve is completely asymmetric and a has its largest value at the edge of the synchronization band.

4. The maximum value of a is the same for all curves, and has the value

$$a_{\max}=\frac{C_A v}{I_0}. \tag{27}$$

5. At the high-frequency edge of the synchronization band

$$a=a_{\max}\cos\rho. \tag{28}$$

By using this relation, ρ can be determined directly from the a-curve as well as by measurement of the compliance coefficients.

6. Equations 25 and 26 show that the phase between injected voltage and undisturbed current is $\pm\beta$ at the edge of the synchronization band. For a simple oscillator circuit β is usually nearly $\pi/2$.

7. From eq 22, 25, and 26 we see that

$$\frac{\Delta f}{a_{\max}}=\frac{2C_F}{C_A}, \tag{29}$$

which shows that the synchronization band width per volt injected is a constant. As a is proportional to v, it is often convenient to use a directly as a measure of v without bothering to measure v. This obviates the need for knowledge of I_0 as the synchronization band can be predicted directly from eq 29, as is done in section III.

III. Experimental measurements

In order to check the foregoing theory, experimental measurements were made on a small Hartley oscillator operating at 11.5 megacycles Radio frequency voltage for injection was supplied by a push-pull power oscillator operating at 10 times the plate voltage of the small oscillator and very loosely coupled to it inductively. Frequencies were read on a receiver arranged with panoramic adapter and vernier-vernier tuning control. Any pulling of the power oscillator by the small oscillator is thus removed from the frequency readings. As will be seen in section IV, the pulling of the driving oscillator does play a part in the observed bank of synchronization, and the power level of this oscillator must be kept as high as practicable.

Figure 3 is a circuit diagram of the test oscillator showing the method of voltage injection and a

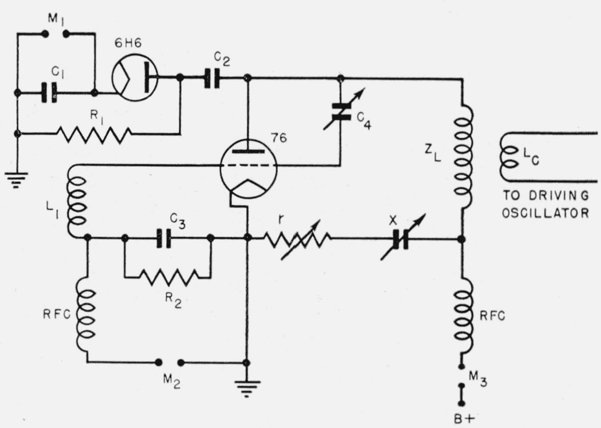

FIGURE 3.—*Circuit of test oscillator.*

Z_L, 3.9 μh; L_1, 5.5 μh; L_c, coupling coil; C_1, 0.001 μf; C_2, 0.001 μf; C_3, 0.00007 μf; C_4, 0.00002 μf; R_1, 1.0 megohm; R_2, 15,000 ohms; M_1, meter to indicate radio-frequency plate voltage; M_2, grid-voltage meter; M_3, plate-current meter; r, series resistance; x, series reactance; RFC, 2.5 mh.

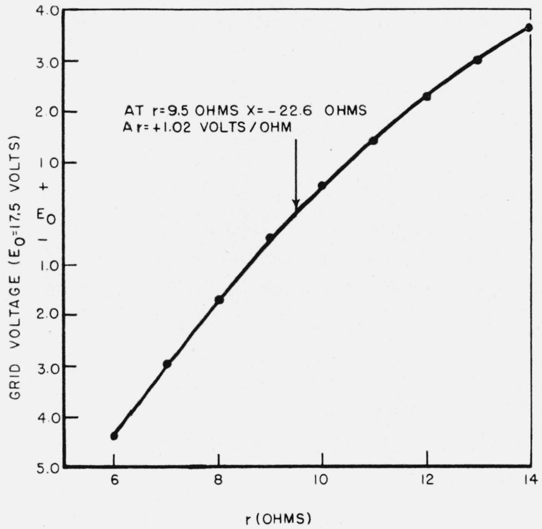

FIGURE 4.—*Experimental curve for evaluation of* A_r.

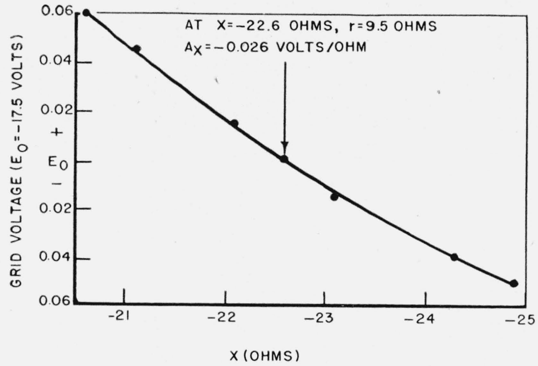

FIGURE 5.—*Experimental curve for determination of* A_x.

diode for measuring radio frequency plate swing. It will be noted that the plate coil has been used for the load, Z_L, and that the synchronizing voltage is injected in this coil. The additional diode circuit is referred to in section IV.

The compliance coefficients were measured by inserting capacitors x and resistors r in series with the plate-tank coil. To allow measurement on both sides of the operating point, this point was specified to be $r=9.5$ ohms, $x=-22.6$ ohms.

Figures 4 and 5 show the experimental curves from which A_r and A_x can be obtained. From them we observe that the value of the compliance coefficients at the operating point are

$$A_r=+1.02 \text{ v/ohm}$$

$$A_x=-0.026 \text{ v/ohm}$$

$$C_A=\sqrt{A_r^2+A_x^2}=1.02 \text{ v/ohm}$$

$$\alpha=-1.5 \text{ degrees.}$$

Figure 6 shows similar curves for evaluating the frequency compliance parameters. The appropriate values at the operating point are

$$F_r=-2.74 \text{ kc/ohm}$$

$$F_x=+10.5 \text{ kc/ohm}$$

$$C_F=\sqrt{F_r^2+F_x^2}=10.8 \text{ kc/ohm}$$

$$\beta=+105 \text{ degrees.}$$

From the above values the band width of synchronization can be computed. Substitution into eq. 29 gives

$$\frac{\Delta f}{a_{\max}}=\frac{2C_F}{C_A}=21.2 \text{ kc/v.}$$

Figure 7 shows the band width of synchronization as a function of $a_{\max}$ for several values of injected voltage. The slope of the line at the origin is 20.5 kc/v, in good agreement with the above prediction. Note also that the curve is linear over a considerable range of injected voltage.

To check upon the linearity of the relation

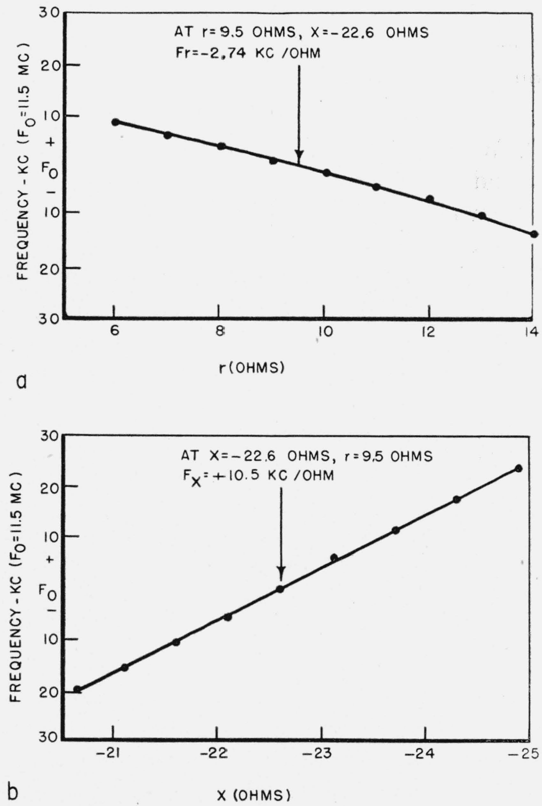

FIGURE 6.—*Experimental curves for determination of* F_r *and* F_x.
a, F_r; b, F_x.

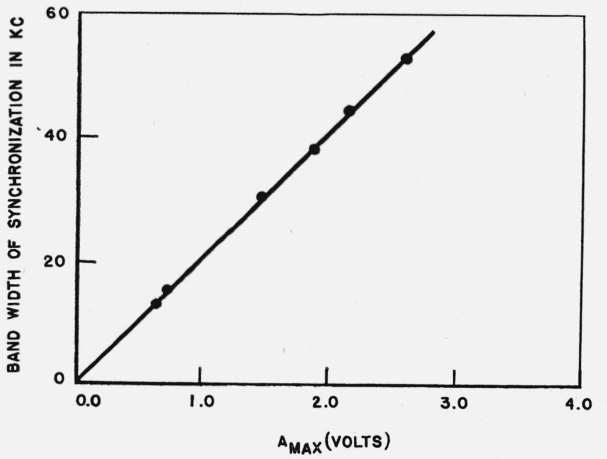

FIGURE 7.—*Experimental determination of band width of synchronization in terms of injected voltage as measured by* a_{max}. *Slope of the curve 20.5 kc/v.*

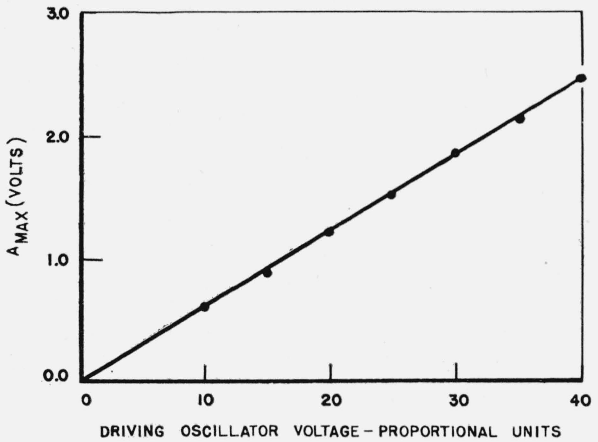

FIGURE 8.— *Relation between* a_{max} *and injected voltage.*

between a_{max} and v, a_{max} was measured as a function of v. The power-oscillator voltage was measured by a diode to give a number proportional to v; v was not measured directly. Figure 8 shows the result. Note that a is proportional to v over a surprisingly large range of values. The observed linearity of a_{max} as a function of v implies that f is also linear in v, as f was linear in a_{max}.

The theory also predicts that the curve of U versus W for this oscillator should be very nearly a semicircle. Figure 9, a, shows the measured U versus W curve. The solid curve is the theoretical one for $\rho=(-106.5°)$. Solid dots represent experimental values obtained by sweeping the power oscillator from high to low frequency through the synchronization band. Crosses represent the data when the frequency is swept in the reverse direction. There is no noticeable evidence of any hysteresis effect here. The simple theory would not indicate that there should be any, but there is some mention of such an effect in Appleton's papers.

The curve was also run for a large injected signal. The result is shown by circles and triangles in figure 9, b. The large signal gives poorer fit, but there is no evidence of hysteresis.

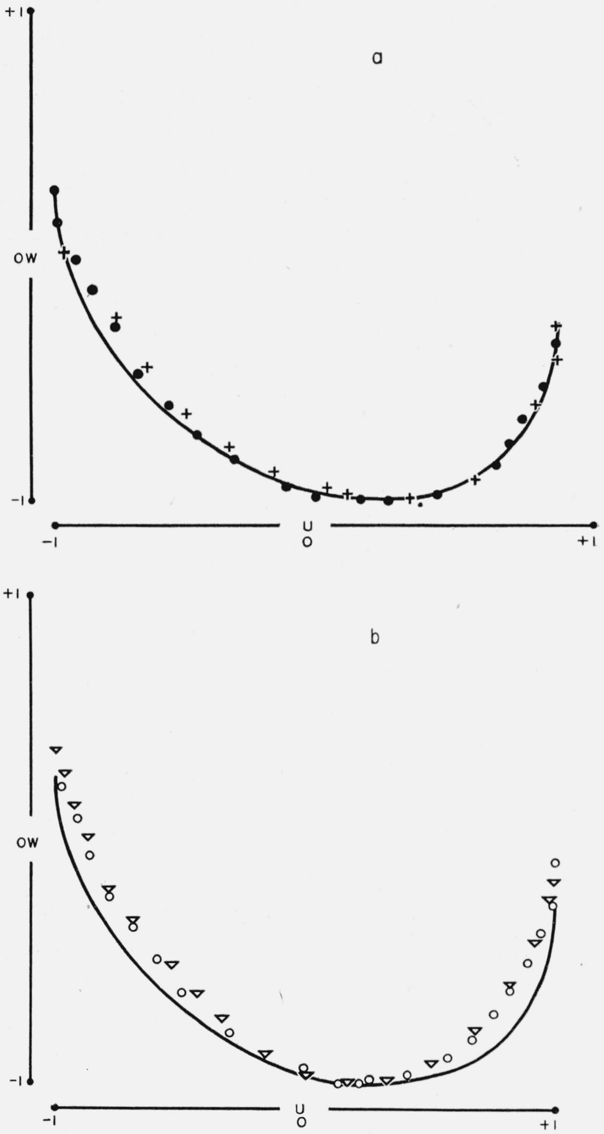

FIGURE 9.—*Experimental and theoretical U-W curves for the test oscillator.*

a, Solid line, theoretical curve for $\rho=(-106.5°)$; solid dots, experimental values sweeping from high to low frequency; crosses, experimental values sweeping from low to high frequency; $a_{max}=0.92$ v; b, experimental and theoretical *U-W* curves for the test oscillator. Solid line, theoretical curve for $\rho=(-106.5°)$; circles, experimental values sweeping from high to low frequency; triangles, experimental values, sweeping from low to high frequency; $a_{max}=4.9$ v.

IV. Mutual Synchronization of Oscillators

Consider two oscillators of the form shown in figure 1, and let them be coupled by a mutual impedance

$$\boldsymbol{Z}_{12}=Z_{12}e^{j\phi_{12}}.$$

Let the two systems be identified by subscripts 1 and 2. The coupling should be so arranged that the coupled voltages are induced in the load impedances $\boldsymbol{Z}_L$ of each system.

Both Z_{12} and ϕ_{12} will, in general, be functions of frequency. To simplify the present discussion, we assume that this dependence can be neglected over the narrow range of frequencies covered by the synchronization band.

As we are interested only in synchronization, we assume that both oscillators are synchronized at frequency F, and that their undisturbed frequencies are F_{01} and F_{02}, respectively.

In order to specify phases, we refer all phases to the current $\boldsymbol{I}_1$ in the load of oscillator 1. We will seek the value of the phase angle θ_{12} between the currents $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$. We write (omitting the term $e^{j2\pi Ft}$)

$$\left.\begin{aligned}\boldsymbol{I}_1&=I_1e^{j0}\\ \boldsymbol{I}_2&=I_2e^{j\theta_{12}}\\ \boldsymbol{V}_1&=v_1e^{j\phi_1}\\ \boldsymbol{v}_2&=v_2e^{j(\theta_{12}+\phi_2)}\\ \boldsymbol{Z}_{12}&=Z_{12}e^{j\phi_{12}}\end{aligned}\right\} \qquad (30)$$

Now,

$$\boldsymbol{v}_1=\boldsymbol{I}_2\boldsymbol{Z}_{12}=I_2Z_{12}e^{j(\theta_{12}+\phi_{12})}, \qquad (31)$$

from which

$$\left.\begin{aligned}\boldsymbol{v}_1&=I_2Z_{12}\\ \theta_{12}+\phi_{12}&=\phi_1+2n\pi,\end{aligned}\right\} \qquad (32)$$

also

$$\boldsymbol{v}_2=\boldsymbol{I}_1\boldsymbol{Z}_{12}=I_1Z_{12}e^{j\phi_{12}}, \qquad (33)$$

from which

$$\left.\begin{aligned}v_2&=I_1Z_{12}\\ \phi_{12}&=\theta_{12}+\phi_2+2n\pi\end{aligned}\right\} \qquad (34)$$

We will drop the $2n\pi$, since it has no further interest.

Each of the oscillators will react to the coupled voltage independently of the other oscillator. Thus we write two equations like eq 10 and get

$$F-F_{10}=\frac{C_{F1}v_1}{I_{01}}\cos\,(\phi_1+\beta_1) \qquad (35)$$

$$F-F_{20}=\frac{C_{F2}v_2}{I_{02}}\cos\,(\phi_2+\beta_2). \qquad (36)$$

These we can combine with the aid of eq 31, 32, 33, and 34 to get

$$F_{20}-F_{10}=\frac{C_{F1}v_1}{I_{01}}\left[\cos\,(\theta_{12}+\phi_{12}+\beta_1)-\frac{C_{F2}}{C_{F1}}\left(\frac{I_{01}}{I_{02}}\right)^2\cos\,(-\theta_{12}+\phi_{12}+\beta_2)\right], \qquad (37)$$

which is an equation involving θ_{12} as the only unknown.

We observe immediately from eq 37 that both oscillators contribute to the band width of synchronization. To see the effect more clearly, we write

$$\left.\begin{aligned}\Phi&=(\theta_{12}+\phi_{12}+\beta_1)\\ \epsilon_1&=(-\beta_2-\beta_1-2\phi_{12})\\ k&=\frac{C_{F2}}{C_{F1}}\left(\frac{I_{01}}{I_{02}}\right)^2\end{aligned}\right\} \qquad (38)$$

and get

$$F_{20}-F_{10}=\frac{C_{F1}v^1}{I_{01}}\,[\cos\,\Phi-k\,\cos\,(\Phi+\epsilon_1)] \qquad (39)$$

$$=\frac{C_{F1}v_1}{I_{01}}\,[\sqrt{1+k^2-2k\,\cos\,\epsilon_1}\,\cos\,(\Phi+\epsilon_2)] \qquad (40)$$

where

$$\tan\,\epsilon_2=\frac{-k\,\sin\,\epsilon_1}{1-k\,\cos\,\epsilon_1}.$$

From eq 40 we see that the two oscillators synchronize over a band of frequencies Δf_{12} given by

$$\Delta f_{12}=\Delta f_1\sqrt{1+k^2-2k\,\cos\,\epsilon_1}. \qquad (41)$$

If oscillator 2 is much more powerful than oscillator 1 and otherwise identical, k will be very small and Δf_{12} becomes equal to Δf_1.

From this it can be seen that is is important to have the driving oscillator more powerful than the test oscillator when making synchronization measurements. If the two are identical, k will be 1, and the band of synchronization can vary from 0 to $2\Delta f_1$, depending on ϵ_1.

The allowed values of Φ, and hence of θ, can be obtained from eq 37, 38, 39, and 40 when the necessary parameters are given.

The problem of N oscillators can be set up in a manner similar to that for two oscillators. We proceed to indicate the operations required. Let the oscillators be indicated by subscripts 1 to N, and assume them to be synchronized. As before, we seek the $N-1$ phase angles $\theta_{12}\ldots\theta_{1N}$ relating the phase of the current in the tank of each oscillator to that in oscillator 1.

The mutual-impedances coupling the oscillators will be specified by

$$\boldsymbol{Z}_{xy}=Z_{xy}e^{j\phi_{xy}}. \tag{42}$$

There are the additional relations

$$\boldsymbol{I}_1=I_1e^{j0};\ \boldsymbol{I}_2=I_2e^{j\phi_{12}}\ \ldots\ \boldsymbol{I}_N=I_Ne^{j\theta_{1N}} \tag{43}$$

$$\boldsymbol{v}_1=v_1e^{j\phi_1}\ \ldots\ \boldsymbol{v}_N=v_NE_N{}^{j\phi} \tag{44}$$

and the system of equations expressing the interaction

$$\begin{array}{l} \boldsymbol{v}_1=Z_{12}I_2e^{j(\phi_{12}+\theta_{12})}\ \ldots\ Z_{1N}I_Ne^{j(\phi_{1N}+\theta_{1N})} \\ \vdots \\ \boldsymbol{v}_N=Z_{1N}I_1e^{j(\phi_{1N})}\ \ldots\ Z_{N(N-1)}I_{N-1}e^{j(\phi_{N(N-1)}+\theta_{1(N-1)})} \end{array} \tag{45}$$

Note that all the diagonal elements in eq 45 are missing. These can be omitted as it is assumed that the I's are given and are not appreciably disturbed by the coupled voltages v_N.

By equating members of eq 45 with corresponding members of eq 44, we get N equations that are to be solved for the $N\phi_N$ in terms of $\theta_{12}\ldots\theta_{1N}$ and the ϕ_{xy}.

We can now write the N equations of synchronization

$$\left.\begin{array}{l} F-F_{01}=\dfrac{C_{F1}v_1}{I_{01}}\cos(\phi_1+\beta_1) \\ \vdots \\ F-F_{0N}=\dfrac{C_{FN}v_N}{I_{0N}}\cos(\phi_N+\beta_N). \end{array}\right\} \tag{46}$$

The ϕ_N are known from the solution of eq 45 as functions of ϕ_{xy} and θ_{1N}; the C_F's and I_{N0}'s are specified. We substitute for $\phi_1\ldots\phi_N$ their values in terms of the ϕ_{xy} and the $N-1$ quantities θ_{1N}. Then results a system of N equations to be solved for the $N-1$ quantities $\theta_{12}\ldots\theta_{1N}$ and F. Solution of eq 46 completes the problem.

V. Applications

Several interesting applications of the synchronized oscillator, some of which have been described elsewhere, can be studied with the aid of this theory. In what follows no attempt has been made to make an exhaustive study of any particular application but rather to indicate as a basis for further investigation some interesting applications of the synchronized oscillator.

1. Linear Radio-Frequency Voltmeter

From eq 27 and the argument which led to it, it can be seen that $a_{\max}$ is proportional to the injected voltage v. Moreover, it has been shown that the value of $a_{\max}$ does not depend upon C_F, α, or β. Thus the properties of the device as a voltmeter can be determined by a substitution of resistors and condensers in its load circuit where the voltage to be measured is injected. The indication of voltage can be obtained from the dc grid bias, or dc plate current or from a diode that reads the radio-frequency voltage across some part of the circuit.

If V is the radio-frequency voltage (peak) on the load impedance Z_L, then

$$V=I_0Z_L,$$

and

$$a_{\max}=C_AZ_L\frac{v}{V}. \tag{47}$$

Typical values measured on the experimental oscillator are $C_A=+1.02$ v/ohms; $Z_L=236$ ohms; $V=47.0$ v. From which

$$v=\frac{a_{\max}}{5.1}. \tag{48}$$

If the oscillator is running stably, it will be found very nearly that

$$C_A=SV, \tag{49}$$

where S is a constant of the oscillator. Figure 10 shows a curve of C_A versus V, which demonstrates the approximately linear dependence of C_A on V. S has the dimensions of a conductance and has the value 0.02 mho for the oscillator tested.

Because of this property, we can write

$$v=\frac{a_{\max}}{SZ_L}, \tag{50}$$

and note particularly that v is essentially independent of V. This is important as it means that the constant of the device used as a voltmeter does not depend upon the voltage of the power supply

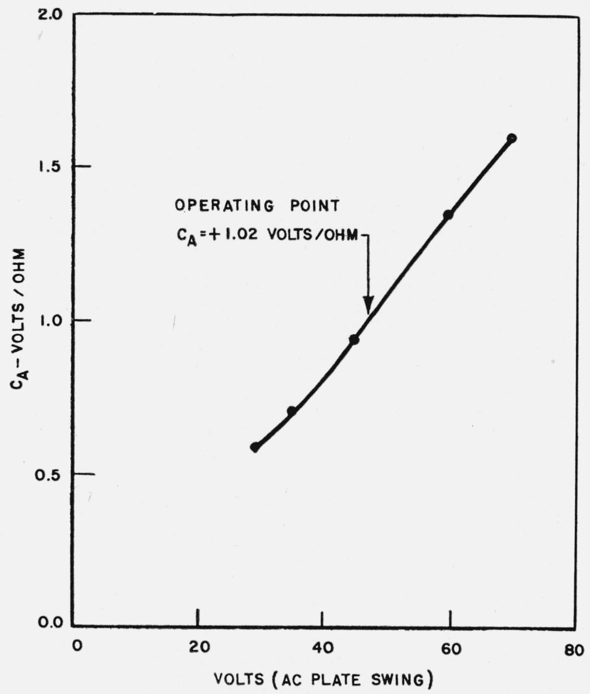

FIGURE 10.—*C_A as a function of test oscillator radio-frequency voltage at the plate.*

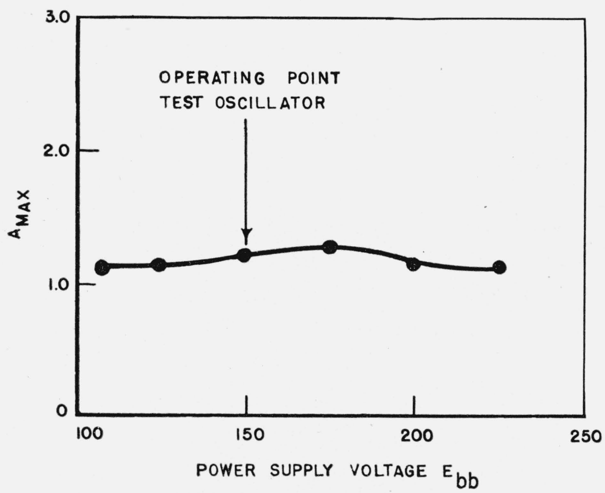

FIGURE 11.—*a_{max}, as a function of test oscillator, E_{bb}, for fixed injected voltage.*

driving it. To demonstrate this point, the test oscillator was used to measure a fixed injected voltage while the power supply driving it was varied from E_{bb}=110 to 225 v. Note the essential independence of a_{max} on E_{bb} shown in figure 11.

It will be remembered that the whole theory upon which this is based assumes that v/V is small. To a first approximation, the error will be v/V of the actual reading. Actually, figure 7 shows that the device is linear for values of a_{max} up to 2½ v.

For the measurement of large voltages a diode is sufficiently good in the frequency range in in which it is applicable. At higher frequencies, including the microwave region, the synchronized oscillator is a good device for measuring voltage as it can be calibrated with d-c instruments and known impedances. For the measurement of small voltages the linearity of the oscillator is an advantage. The lower limit at which voltage measurements can be made is set by the noise from the oscillator.

An additional advantage of the oscillator-voltmeter is its gain. It will be observed in eq 48 that there is an amplification of about five in the oscillator tested.

One large disadvantage in some applications is that the measuring oscillator couples energy into the circuit to which it is attached.

It should also be noted that the synchronized oscillator can be used, as done by Appleton, to measure small voltages by determining the band width of synchronization which is also linearly related to v by the relation

$$\Delta f = 2C_F \frac{fZ_L}{V}. \tag{51}$$

However, C_F is relatively independent of V, and therefore V must be known accurately in order to make a measurement of v by a frequency-variation method.

If the frequency of the injected voltage cannot be varied across the synchronization band of the voltmeter, the frequency of the voltmeter can be varied across the synchronization band by a small variable condenser. The d-c grid bias or other amplitude indicator can be coupled through a blocking condenser to a peak voltmeter. As the voltmeter-oscillator is wobbled back and forth across the frequency of the injected voltage to be measured, a pulse will be observed whose peak value is a_{max}. From this pulse the size of the injected voltage can be calculated.

2. Field-Intensity Meter

The voltmeter properties of the synchronized oscillator lend themselves nicely to the measurement of field intensity at any frequency for which an oscillator is available. Appleton used the synchronization band width of an oscillator to measure field intensities. It is proposed here to use the voltage changes directly instead of the synchronization band largely because the power-supply variation no longer enters the calculation and frequency measurements are not needed.

Assume that a small oscillator like that of figure 1 or 2 is available and that the grid bias is to be used as the indicating voltage. The first step is to determine V_g (grid-bias voltage) as a function of resistance in series with load impedance Z_L. An antenna is added across Z_L and its length adjusted so that a reasonable match is obtained. The antenna load should drop V_g to about half its unloaded value. Simultaneously the reactance of the load, Z_L, should be changed to compensate for the reactance of the antenna. This can be done by keeping the oscillator frequency constant.

From the curve of V_g as a function of r, the series-radiation resistance of the antenna R_s can be determined. (Ohmic losses in the antenna are neglected in this treatment.)

If the antenna is in a radio-frequency field of strength E (peak volts per meter), whose strength is to be measured, the field will induce a voltage, v (as already defined), in the load impedance, Z_L, of which the antenna is now a part. The magnitude of v can be shown to be

$$v=\frac{\lambda}{\pi}E\sqrt{\frac{R_sG}{120}}f(\theta), \tag{52}$$

in which G is the gain referred to an isotropic radiator, and $f(\theta)$ is the normalized radiation pattern of the antenna. If the antenna is alined with the electric field, $f(\theta)=1$. The gain, G, will be very nearly 1.5, as it will be found that quite a short dipole will load the oscillator adequately.

C_A and S should be measured about an operating load, including the R_s of the antenna to be used. If a tuning condenser in the oscillator is wobbled back and forth through the synchronization region, a pulse of peak value $a_{\max}$ will be observed as in the case of the voltmeter. From its height the strength of the field, E, can be calculated. It will be

$$E=\frac{\pi}{\lambda}\frac{a_{\max}}{SZ_L}\sqrt{\frac{120}{R_sG}}. \tag{53}$$

The device can be used for the measurement of field intensities from about 10μv/m to about 1v/m. The useful range will depend on λ, but it appears that the device should work for microwaves as well as for more conventional frequencies. At microwave frequencies the antenna gain, G, can be increased to compensate for loss due to smaller λ.

As has already been mentioned, the calculation is essentially independent of the power-supply voltage. The device can be made quite simple at ordinary frequencies.

3. Linear Amplitude Modulation Detector

Inasmuch as the synchronized oscillator is a linear voltmeter it is also a linear amplitude modulation detector. Instead of reading the changes in a on a meter they can be fed directly into an audio frequency amplifier.

In this application it will be best to operate with a circuit which has $\rho=\pm\pi/2$ so that the U-W curve is a semicircle. This will be true if the signal is injected into the plate or grid circuit of a class C oscillator and the output is read from the d-c grid bias. It will be necessary to inject enough signal so that the band width of synchronization will be wide enough to include all side bands.

As the band width of synchronization is proportional to the injected voltage, the use of a synchronized oscillator poses an interesting problem. It is a linear device whose region of linear response varies instantaneously with the voltage injected. It is not immediately clear how to express the band width characteristics of such a system.

It appears reasonable to require that the time constant of the device be short compared to the shortest period of the modulation to be received. We have seen in eq 21 that an approximate time constant is

$$\tau=\frac{1}{2\pi\sqrt{\left(\frac{\Delta f}{2}\right)^{2}-f^{2}}}=\frac{1}{\pi\Delta f\sin\psi}. \tag{54}$$

Sin ψ is unity near the center of lock-in where f is nearly zero. Thus the requirement that τ be short compared to $1/f_{\max}$, in which $f_{\max}$ is the highest modulation frequency to be reproduced, means that

$$\frac{1}{\pi\Delta f}<<\frac{1}{f_{\max}}$$

or that

$$\pi\Delta f>>f_{\max}. \tag{55}$$

Thus the signal used at the demodulator must be large enough to give a band width of about 30 kc to give faithful reproduction of 10-kc modulation.

If this turns out to be reasonable and the synchronized oscillator is used, it can give a demodulation amplification of about 5 to 10. It will also provide a measure of automatic frequency control as it will follow a deviation in carrier frequency in the intermediate frequency channel.

When nearly 100-percent modulation is used, the device will lead to distortion of a peculiar form because synchronization may be lost when the signal is small near the peak of modulation. However, the synchronized oscillator demodulator appears to present interesting possibilities worthy of further investigation.

4. Frequency Modulation Discriminator

If the oscillator circuit is arranged so that $\rho=0$ or π, the synchronized oscillator can be used as a frequency modulation discriminator demodulator. Reference to figure 1 shows that under these conditions the a curve is a straight line with $a=0$ at center frequency.

One way of achieving this is to couple an auxiliary resonant circuit to the test oscillator and inject the synchronizing signal into this auxiliary circuit. The output can be taken from the d-c grid bias of the oscillator or from a diode connected across the resonant circuit. Figure 12 shows the auxiliary resonant circuit and the coupling to the driving oscillator.

It will be assumed that the voltage is injected in series with Z_L as before. Let R represent the total resistance of the tuned circuit L, C, Z_L, and let X represent its total reactance. The resistance,

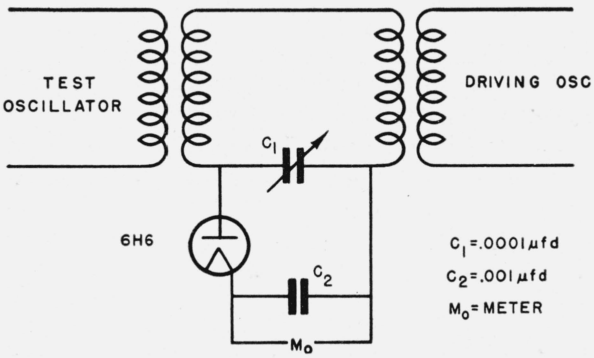

FIGURE 12.—*Auxiliary resonant circuit to obtain behavior characteristic of $\rho=0$.*

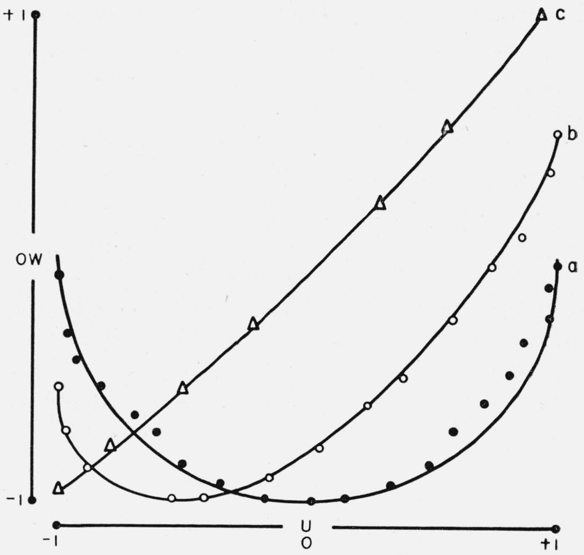

FIGURE 13.—*Experimental U-W curves for auxiliary tuned circuit.*

a, Auxiliary circuit is tuned to exact resonance; solid curve, theoretical for $\rho=0$; dots, experimental points; b, Auxiliary circuit is detuned to 90 percent of resonant voltage. c, Desired result with auxiliary circuit detuned to 70 percent of resonant voltage.

R_c, and reactance, X_c, coupled into the oscillator are then

$$R_c=\frac{(2\pi FM)^2R}{R^2+X^2}. \tag{56}$$

$$X_c=-\frac{(2\pi FM)^2X}{R^2+X^2}. \tag{57}$$

The injection of a small voltage into Z_L is equivalent to small changes in R and X in that circuit. If the circuit be detuned to the point where $R=X$ (70% of resonant voltage) small reactance changes reflect purely resistance changes, and small resistance changes reflect pure reactance changes into the oscillator.

This simple discussion neglects the changes in reactance in the auxiliary circuit due to changes in oscillator frequency. Consideration of this effect leads to important corrections but does not alter the essential argument.

If in the original oscillator $A_x=0$ and $F_r=0$, the whole system, including the tuned circuit, will behave as if $A\rho=0$ and $F_x=0$, thus making $_r=0$ or π instead of $\pm\pi/2$, which is the desired result.

Figures 13 shows a series of curves taken from the diode across the resonant circuit. Figure 13, a, shows the $U-W$ curve for exact resonance. Figure 13, b, shows an intermediate case, and figure 13 c shows the linear relation sought at approximately 70-percent detuning.

When the arrangement described above is used as a discriminator, a will be zero at the center frequency, and the device is thus insensitive to amplitude modulation in a manner similar to a balanced discriminator.

5. Frequency modulation synchronous amplifier limiter

In this application the oscillator is locked to a frequency modulation signal. It follows the frequency variations without serious amplitude change and hence becomes a combined amplifier and limiter. It has been discussed widely in the literature.[9]

If the synchronized oscillator is capable of following frequency deviations according to the criteria established by Adler, the response to a frequency modulation signal of the form

$$f(t)=f_0 \sin 2\pi f_m t \tag{58}$$

will be a solution of

$$\frac{1}{2\pi}\frac{d\phi}{dt}+\frac{C_F v(t)}{I_0}\cos(\phi+\beta)=f(t), \tag{59}$$

in which $v(t)$ represents any amplitude modulation of v that may be present. Direct integration of eq 59 is complicated and need not be performed to the approximation needed here. It will be recalled that ϕ responds to changes in f and v with a time constant τ given by eq 21. If the changes in f or v occur in a time long compared with τ, the oscillator is essentially in equilibrium at each instant, and a succession of steady state solutions for various fixed f is a good enough approximation to the actual solution for varying f. If the injected voltage, v, is always so large that

$$Kv=nf_0;\ n>1, \tag{60}$$

then

$$\tau \leqq \frac{1}{2\pi f_0\sqrt{n^2-1}}. \tag{61}$$

As it is standard practice to have $f_0>5f_m$, it is evident that the time constant is short compared with the frequency modulation period, $1/f_m$, and the equilibrium solution, eq 20, is a reasonable approximation. Similar arguments hold for changes in v, but we are not interested in amplitude modulation here, and will henceforth assume v to be constant.

We see from eq 20 that changes in f will produce changes in ϕ so that an additional phase modulation will be added to the impressed signal. This implies $(d\phi/dt)\neq 0$ in contradiction to the original assumptions made in solving eq 13 to get eq 20. The correction will be small if the frequency variations are slow, and we proceed with the approximation to write

$$\frac{d\phi}{dt}=\frac{-f}{Kv\sin(\phi+\beta)}, \tag{62}$$

and

$$F^1-F=\frac{1}{2\pi}\frac{d\phi}{dt}=-\frac{f_0 f_m \cos 2\pi f_m t}{Kv\sin(\phi+\beta)}. \tag{63}$$

This makes the output-frequency deviation, f^1, of the oscillator have the form

$$f^1=f_0\left(\sin 2\pi f_m t-\frac{f_m}{Kv\sin(\phi+\beta)}\cos 2\pi f_m t\right). \tag{64}$$

This indicates a negligible distortion when $Kv>2f_0$, as $\sin(\phi+\beta)$ is then always near unity and $f_m\ll Kv$.

Amplitude changes in v will also lead to phase modulation. However, if Kv is kept large with respect to f_0 the phase is relatively insensitive to voltage changes and the distortion arising from amplitude modulation is thereby minimized.

If the criterion $Kv=2f_0$ is set as a design center, then

$$v=\frac{2f_0}{K}=\frac{2f_0 V}{C_F Z_L}, \tag{65}$$

and the voltage amplification achieved by the use of the synchronized oscillator will be

$$\frac{V}{v}=\frac{C_F Z_L}{2f_0}. \tag{66}$$

To estimate the order of magnitude of the gain that may safely be used assume (1) that a single LC circuit is controlling the oscillator, (2) that the voltage, V, is the one across the entire inductance of the oscillating circuit, and (3) that τ is injected into this inductance. Then

$$C_F=\frac{1}{4\pi L} \tag{67}$$

[9] See, for example, C. W. Carnahan and H. P. Kalmus, Synchronized oscillators as frequency-modulation receiver limiters, Electonics **17**, 108 (1944).

and C_F is practically independent of the power-supply voltage. This gives

$$\frac{V}{v}=\frac{F}{4f_0}. \tag{68}$$

If the oscillator frequency is 10 mc and f_0 is 100 kc, the maximum voltage amplification of the device can be about 25. Of course, if the voltage across a part of the tank inductance is used as in the experiments already described, the gain is correspondingly reduced. However, gains of 15 or more should be readily obtainable. Also the voltage, v, may be injected into the grid circuit of the oscillator and the gain of the tube used to increase the voltage seen in the tank circuit. Equation 68 refers only to the voltage, v, injected into the oscillating circuit, which controls the frequency.

Unless there is some amplitude regulating device on the synchronized oscillator, there will also be an amplitude modulation in its output. The magnitude of the effect can be calculated from eq 9 and 25. It is usually small enough to be neglected.

VI. Conclusion

Although the theory and experiments just described have been discussed in terms of a conventional self-limiting source of alternating electromotive force, the concepts involved are quite general, and with appropriate redefinition of symbols, the equations can equally well apply to any source of harmonic disturbance—electrical, electromagnetic, mechanical, or acoustical, singly or in combination. The self-limitation implies that some nonlinear element is present to limit the amplitude of oscillation. A truly linear oscillator will not exhibit synchronization effects. It will, however, not appear in practice.

We may then conclude that any source of harmonic disturbance whose steady state frequency is a function of the load applied to it, and whose frequency can change with sufficient rapidity, will exhibit synchronization behavior when a harmonic disturbance is impressed upon it from an external source. If the amplitude of the device is also a function of its load, then it will exhibit a characteristic amplitude variation in the synchronization region.

We may also conclude that the source will synchronize with an impressed disturbance, however small, if the frequency of the outside disturbance is close enough to that of the undisturbed source.

It is important to remember that the properties of the external source supplying the synchronizing signal and the coupling impedance are important in determining the band width and phase of synchronization. If there is a considerable disparity in the power output of the two sources, the weaker determines the synchronization properties of the system. If the power outputs are nearly equal, both sources contribute almost equally to its synchronization properties.

The authors are indebted to B. J. Miller and S. Lachenbruch for many helpful contributions in the preparation of this paper, and to Robert Adler of the Zenith Radio Corporation for an opportunity to study and discuss his recent paper, A Study of Locking Phenomena in Oscillators, prior to its publication.

WASHINGTON, October 7, 1946.