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Theory for Axial Rigidity of Structural Members Having Ovaloid or Square Perforations

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Formulas are derived for computing the over-all lengthening (or shortening) of a tension (or compression) member having a uniform gross cross section and a series of similar perforations of approximately ovaloid or approximately square shape uniformly distributed along the length.

I. Introduction

In a previous paper [1] [1] the author demonstrated a method for computing the axial rigidity of a tension or compression member having a uniform gross cross section and a series of similar perforations of circular or elliptical shape uniformly distributed along the length. The axial-rigidity factor K is defined so that KEA_g is the rigidity that should be used in place of EA_g in the ordinary formula for computation of the extension of the member. Here E is Young's modulus of elasticity, and A_g is the gross cross-sectional area. The present paper extends this method to other shapes of perforation, originally described in reference [2]. The boundary of the hole has the parametric equation

$$x=p \cos \beta+r \cos 3\beta, \; y=q \sin \beta-r \sin 3\beta. \quad (1)$$

Equation 1 represents a closed curve having symmetry about the x-axis and about the y-axis, and which, for appropriate values of p, q, and r, does not cross itself. By adjustment of the values of p, q, and r, a variety of curves is obtained, including a good approximation to an ovaloid (a square with a semicircle erected on each of two opposite sides) and a good approximation to a square with rounded corners, as well as exact ellipses ($r=0$) of any eccentricity. The approximate ovaloid obtained by taking

$$p=2.063, \; q=1.108, \; r=-0.079, \quad (2)$$

is shown compared to the actual ovaloid in figure 1. The approximate square obtained by taking

$$p=q=1, \; r=-0.14 \quad (3)$$

is shown in figure 2. The sides of the square are parallel to the axes of coordinates. By taking

$$p=q=1, \; r=0.14, \quad (4)$$

[1] Numbers in brackets indicate literature references at the end of this paper.

Contents

	Page
I. Introduction	157
II. Displacements	158
1. Displacement-function method	158
2. Mushelisvili's method	159
III. Axial rigidity	160
1. General	160
2. Ovaloid; load parallel to long axis	160
3. Ovaloid; load parallel to short axis	161
4. Square with rounded corners	161
IV. Summary	161
V. References	161

the same square, but with the diagonals parallel to the axes of coordinates, is obtained. The radius of curvature at the midpoint of the fillet is about 0.086 times the length of the side of the square.

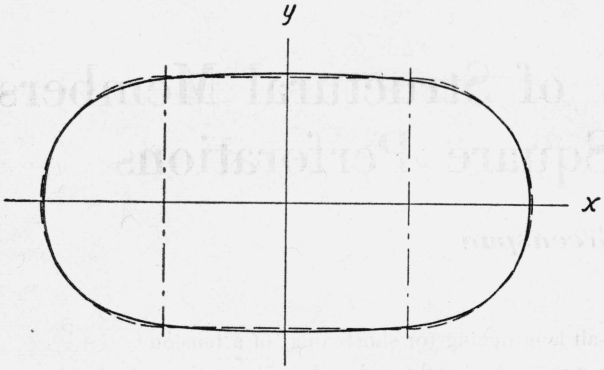

Figure 1.—*Actual and approximate ovaloids.*

The dashed line represents the actual ovaloid and the full line the approximate ovaloid of equations 1 and 2

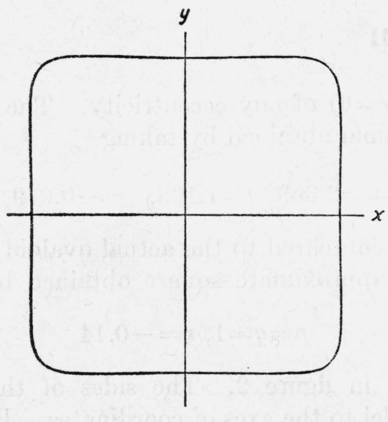

Figure 2.—*The approximate square of equations 1 and 3.*

It is shown in [1, eq 2][2] that for a member having perforations spaced $2l$ apart along the x-axis (parallel to the load), the axial rigidity factor K is given by

$$\frac{1}{K}=1+\frac{E}{Pl}\int u_p dydz, \tag{5}$$

in which u_p is the x-component of the displacement of the boundary of the hole, and the surface integral is extended over half the boundary of the perforation. The values of the load P and the displacement u_p may be assumed, for sufficiently large values of l and of A_g compared to the dimensions of the perforation, to be the same as in an infinite plate under uniform stress S_x disturbed by a single perforation.

For a member having circular holes, and of gross cross-sectional area A_g and net cross-sectional area A_n,

$$P=S_xA_gC(n)=S_xA_g\left(1-\frac{1}{2n^2}-\frac{1}{2n^4}\right), \tag{6}$$

where S_x is the mean stress on the gross cross-sectional area and $n=A_g/(A_g-A_n)$. Equation 6 is derived in [1]. $C(n)$ may be considered to be a correction factor arising from the finite area of the member. It has been shown [3][3] that $C(n)$ is nearly independent of eccentricity in case of an elliptical hole, and since it is nearly unity in any practical case, eq 6 may be used without serious error for a hole of any shape.

The displacement u_p may be calculated from the stress function given in [2] or by the method of Mushelisvili.

[2] This notation is used throughout for the numbered equations in the references.

[3] See p. 545 of reference [3].

II. Displacements

1. Displacement-function method

This method [4] presupposes a knowledge of the Airy stress-function, ϕ. The form of ϕ is exhibited in [2] in terms of curvilinear coördinates (α, β) such that eq 1 of the boundary of the hole reduces to $\alpha=\alpha_0$. The parameters a, b, and c of [2], which, together with α_0, define the shape of the hole, are related to the p, q, and r of eq 1 by

$$e^{\alpha_0}=\frac{p+q}{2},\ abe^{-\alpha_0}=\frac{p-q}{2},\ ac^3e^{-3\alpha_0}=r. \tag{7}$$

[4] See p. 130 of [3].

The function ψ defined, in the absence of body forces, by

$$\frac{\partial^2\psi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\nabla^2\phi, \tag{8}$$

with ψ adjusted by means of the arbitrary functions of integration (of the form $f_1(x)+f_2(y)$) so that $\nabla^2\psi=0$, is called the displacement function. The x-component of the displacement is given by

$$u=\frac{1}{E}\frac{\partial\psi}{\partial y}-(1+\nu)\frac{\partial\phi}{\partial x}, \tag{9}$$

where μ is Poisson's ratio. In the present case the only values of u required are u_p, those on the boundary of the perforation. On this (free) boundary, $\partial\phi/\partial x$ is constant, and hence zero, since ϕ is even in x and in y. Hence from eq 9,

$$Eu_p=\frac{\partial\psi}{\partial y}\ (\alpha=\alpha_0). \tag{10}$$

According to [2, eq 18],

$$\phi=C_1\phi_1+C_2\phi_2+C_3\phi_3+C_4\phi_4+ C_5\phi_5+C_6\phi_6+C_7\phi_7+C_8\phi_8.$$

Of these terms, all but those in ϕ_1, ϕ_2, and ϕ_6 are harmonic and need not be considered for purposes of eq 8. Further, $C_6=0$ for no shear, as shown by [2, eq 23]. Thus only the terms in ϕ_1 and ϕ_2 are required. From paragraph 5 of [2] ϕ_1 and ϕ_2 are, omitting harmonic terms, which disappear for operation ∇^2,

$$\phi_1=4ye^\alpha \sin\beta-4abye^{-\alpha}\sin\beta-x^2-y^2,$$

and

$$\phi_2=2ye^{-\alpha}\sin\beta,$$

so that

$$\nabla^2\phi_1=8\frac{\partial}{\partial y}(e^\alpha\sin\beta-abe^{-\alpha}\sin\beta)-4,$$

and

$$\nabla^2\phi_2=-4\frac{\partial}{\partial y}e^{-\alpha}\sin\beta.$$

Equation 8 becomes

$$\frac{\partial^2\psi}{\partial x\partial y}=C_1\left[8\frac{\partial}{\partial y}(e^\alpha\sin\beta-abe^{-\alpha}\sin\beta)-4\right] -4C_2\frac{\partial}{\partial y}e^{-\alpha}\sin\beta,$$

from which

$$\frac{\partial\psi}{\partial x}=C_1\left[8(e^\alpha\sin\beta-abe^{-\alpha}\sin\beta)-4y\right] -4C_2e^{-\alpha}\sin\beta+Dx+E, \tag{11}$$

in which $Dx+E$ is the most general function of integration of the form $f_1(x)$ which preserves the harmonic character of $\partial\psi/\partial x$. Because ψ is harmonic by definition, $\partial\psi/\partial y$ as well as $\partial\psi/\partial x$ is harmonic, and application of the Cauchy-Riemann test to the expression

$$\frac{\partial\psi}{\partial y}+i\frac{\partial\psi}{\partial x}$$

shows it to be a function of the complex variable $x+iy$. Thus $\partial\psi/\partial y$ is the real part of the function of which $\partial\psi/\partial x$ is the imaginary part, so that from eq 11

$$\frac{\partial\psi}{\partial y}=C_1(8e^\alpha\cos\beta+8abe^{-\alpha}\cos\beta-4x) +4C_2e^{-\alpha}\cos\beta-Dy+E'.$$

The constants D and E^1 are obtained by substituting for $\partial\psi/\partial y$ in eq 9, which gives

$$Eu=C_1(8e^\alpha\cos\beta+8abe^{-\alpha}\cos\beta-4x) +4C_2e^{-\alpha}\cos\beta-Dy+E'-(1+\nu)\frac{\partial\phi}{\partial x}.$$

For $x=0(\beta=\pm\pi/2)$, $u=0$ for all y, and $\partial\phi/\partial x=0$ by symmetry. Therefore, $D=E'=0$, and

$$\frac{\partial\psi}{\partial y}=C_1(8e^\alpha\cos\beta+8abe^{-\alpha}\cos\beta-4x) +4C_2e^{-\alpha}\cos\beta. \tag{12}$$

Substitution of eq 12, with C_1 and C_2 replaced by their values from [2, eq 23] into eq 10 gives

$$\frac{Eu_p}{S_x}=-x_0+\frac{2e^{\alpha_0}\cos\beta}{1-ac^3e^{-4\alpha_0}}(2-ac^3e^{-4\alpha_0}-a^2bc^3e^{-6\alpha_0}),$$

where x_0 is the value of x on the boundary $\alpha=\alpha_0$, which, together with eq 7, gives

$$\frac{Eu_p}{S_x}=-x_0+2\frac{(p+q)^2-2pr}{p+q-2r}\cos\beta. \tag{13}$$

2. Mushelisvili's method

This method, which is treated in detail by Sokolnikoff [4], expresses the stresses, displacements, and the boundary conditions in terms of two analytic functions, ϕ and ψ (not the same as those of the preceding section), of the complex variable $z=x+iy$. The functions ϕ and ψ, which satisfy the problem of [2], have been given by Morkovin [5], who uses the notation

$$s=(p+q)/2,\ t=(p-q)/2,\ r=r, \tag{14}$$

and

$$\zeta=e^{\alpha-\alpha_0}+i\beta, \tag{15}$$

so that the function

$$z=\omega(\zeta)=s\zeta+t/\zeta+r/\zeta^3 \tag{16}$$

maps the boundary $\alpha=\alpha_0$ onto the unit circle $|\zeta|=1$. The value of ζ on the unit circle is denoted by σ; thus $\bar{\sigma}$, the conjugate of σ, is $1/\sigma$. The notation

$$\phi(z)=\phi[\omega(\zeta)]=\phi(\zeta),$$

is used, and similarly for ψ.

The functions ϕ and ψ as found by Morkovin are

$$\phi(\zeta)=sB\zeta+\frac{a_1}{\zeta}-\frac{Br}{\zeta^3},$$

$$\psi(\zeta)=-\left(\frac{s}{\zeta}+t\zeta+r\zeta^3\right)\left(\frac{sB\zeta^4-a_1\zeta^2+3rB}{s\zeta^4-t\zeta^2-3r}\right)-\frac{sB}{\zeta}$$
$$+s(B'+iT_{xy})\zeta-a_1\frac{r}{s}\zeta+Br\zeta^3+Bt\left(1+\frac{r}{s}\right)\zeta,$$

where

$$a_1=Bt\frac{r+s}{r-s}+\frac{B's^2}{r-s}+\frac{iT_{xy}s^2}{r+s},$$

and

$$4B=S_x+S_y,\ 2B'=S_y-S_x.$$

In the present case, $S_y=T_{xy}=0$, giving

$$\phi(\zeta)=\frac{S_x}{4}\left(s\zeta+\frac{a}{\zeta}-\frac{r}{\zeta^3}\right), \tag{17}$$

$$\psi(\zeta)=\frac{S_x}{4}\left[-\left(\frac{s}{\zeta}+t\zeta+r\zeta^3\right)\left(\frac{s\zeta^4-a\zeta^2+3r}{s\zeta^4-t\zeta^2-3r}\right)\right.$$
$$\left.-\frac{s}{\zeta}-2s\zeta-\frac{ar}{s}\zeta+r\zeta^3+t\left(1+\frac{r}{s}\right)\zeta\right], \tag{18}$$

where

$$a=\frac{4a_1}{S_x}=\frac{t(r+s)-2s^2}{r-s}.$$

The rectangular components, u and v, of the displacement are expressed in terms of the functions ϕ and ψ by means of the relation

$$E(u+iv)=(3-\nu)\phi(\zeta)-(1+\nu)\left[\frac{z\overline{\phi'(\zeta)}}{\overline{\omega'(\zeta)}}+\overline{\psi}(\overline{\zeta})\right],$$

so that on the boundary $\zeta=\sigma$,

$$E(u_p+iv_p)=(3-\nu)\phi(\sigma)$$
$$-(1+\nu)\left[\frac{z\overline{\phi'}(\overline{\sigma})}{\overline{\omega'}(\overline{\sigma})}+\overline{\psi}(\overline{\sigma})\right]. \tag{19}$$

Substitution of the functions ω, ϕ, and ψ from eq 16, 17, and 18 in eq 19 yields, after considerable reduction,

$$\frac{E}{S_x}(u_p+iv_p)=-z_0+2s\sigma+\frac{2s^2-2rt}{s-r}\frac{1}{\sigma},$$

where z_0 is on $\alpha=\alpha_0$, and therefore,

$$\frac{Eu_p}{S_x}=-x_0+\frac{4s^2-2sr-2rt}{s-r}\cos\beta, \tag{20}$$

since $\sigma=e^{i\beta}$ from eq 15.

In terms of the original constants (eq 14), eq 20 becomes,

$$\frac{Eu_p}{S_x}=-x_0+2\frac{(p+q)^2-2pr}{p+q-2r}\cos\beta,$$

which is the same as eq 13.

III. Axial Rigidity

1. General

Substitution of P from eq 6 and u_p from eq 13 into eq 5 gives

$$\frac{1}{K}=1+\frac{t}{A_g l C(n)}\left[\int xdy\right.$$
$$\left.+2\frac{(p+q)^2-2pr}{p+q-2r}\int\cos\beta dy\right], \tag{21}$$

where t, the thickness of the plate in which the perforation occurs, is $\int dz$. The two line integrals in eq 21, which are to be taken along half the boundary $\alpha=\alpha_0$, are obtained from eq 1, as

$$\int xdy=\frac{1}{2}A_0=\frac{\pi}{2}(pq-3r^2),\ \int\cos\beta dy=\frac{\pi q}{2},$$

where A_0 is the area of the perforation.

Equation 21 becomes

$$\frac{1}{K}=1+\frac{V_0}{V_gC(n)}\ \frac{q\,(p+q)(p+2q)-2pqr+3r^2\,(p+q-2r)}{(p+q-2r)(pq-3r^2)}, \tag{22}$$

where $V_0=A_0t$ is the volume of the perforation and $V_g=2A_gl$ is the gross volume of one bay of the member.

For an ellipse, $p=a$, $q=b$, $r=0$, where a and b are the semiaxes (a parallel to the load) and eq 22 reduces to

$$\frac{1}{K}=1+\frac{1+\frac{2b}{a}}{C(n)}\frac{V_0}{V_g},$$

which is the same as [1, eq 48].

It remains to compute the coefficient of V_0/V_g in eq 22 for the various cases of interest.

2. Ovaloid; load parallel to long axis

Here $p=2.063$, $q=1.108$, $r=-0.079$ (eq 2). Equation 22 becomes

$$\frac{1}{K}=1+\frac{2.048}{C\,(n)}\frac{V_0}{V_g}. \tag{23}$$

In practical numerical cases, eq 23 gives substantially the same results as [1, eq 13], derived by a more approximate method. However [1, eq 13] may be used for an ovaloid of which the rectangular base is not square whereas eq 23 applies only to the shape shown in figure 1.

3. Ovaloid; load parallel to short axis

In this case, $p=1.108$, $q=2.063$, $r=0.079$. Equal 23 becomes

$$\frac{1}{K}=1+\frac{4.968}{C(n)}\frac{V_0}{V_g}. \tag{24}$$

This case and the following cases are not covered by previous work.

4. Square with Rounded Corners

For the approximate square of eq 3 and figure 2, $p=q=1$, $r=-0.14$. Eq. 22 becomes

$$\frac{1}{K}=1+\frac{2.989}{C(n)}\frac{V_0}{V_g}, \tag{25}$$

for load parallel to the side of the square.

For load parallel to the diagonal of the square (eq 4) $p=q=1$, $r=0.14$, and eq 22 becomes

$$\frac{1}{K}=1+\frac{3.596}{C(n)}\frac{V_0}{V_g}. \tag{26}$$

IV. Summary

Approximate formulas have been developed for the computation of the axial rigidity of a long tension or compression member containing a plate of constant thickness uniformly perforated with a series of similar holes of various shapes. The axial rigidity factor K is defined so that KEA_g is the rigidity that should be used in place of EA_g in the ordinary formula for computation of the extension of the member.

In many cases of interest the axial rigidity factor K is given by an equation of the form

$$\frac{1}{K}=1+\frac{f}{C(n)}\frac{V_0}{V_g}, \tag{27}$$

where

$n=A_g/(A_g-A_n)$, A_g being the gross and A_n the net cross-sectional area of the member

$C(n)=1-\frac{1}{2n^2}-\frac{1}{2n^4}$

V_0= the volume of the perforation

V_g= the gross volume of one bay of the member

f= a constant depending on the shape of the perforation and the direction of the applied load.

Values of the constant f for various cases are given in table 1.

TABLE 1.—*Values of f in equation 27*

Perforation	Load parallel to—	f
Ellipse of semiaxes a and b	Major axis, a	$1+(2b/a)$
Do	Minor axis, b	$1+(2a/b)$
Ovaloid (fig. 1)	Long axis	2.048
Do	Short axis	4.968
Square (fig. 2)	Side	2.989
Do	Diagonal	3.596

V. References

[1] Martin Greenspan, Axial rigidity of perforated structural members, J. Research NBS **31,** 305 (1943) RP1568.

[2] Martin Greenspan, Effect of a small hole on the stresses in a uniformly loaded plate, Quart. Applied Math. **II,** 60 (1944).

[3] E. G. Coker and L. N. G. Filon, A treatise on photoelasticity (Cambridge, 1931).

[4] I. S. Sokolnikoff, Mathematical theory of elasticity, (McGraw-Hill Book Co., Inc., New York, N. Y. 1946).

[5] Vladimir Morkovin, Effect of a small hole on the stresses in a uniformly loaded plate, Quart. Applied Math. **II,** 350 (1945).

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