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METHOD OF LAGRANGIAN CURVILINEAR INTERPOLATION[1]

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ABSTRACT

This report describes a simplified method of computing Lagrangian coefficients for curvilinear interpolation, which may be used when tables of Lagrangian coefficients are not available or when tables are available but the coefficients are not tabulated for the exact fraction of the interval to which the interpolation is to be made.

CONTENTS

I. INTRODUCTION

The common formulas of curvilinear interpolation,[3] such as those of Newton, Gauss, Stirling, Bessel, and Lagrange[4] when carried to completion, are all equivalent in that they approximate the function by a polynomial of suitable degree (the number of points required is one greater than the degree of the polynomial assumed). All of these formulas except that of Lagrange require, as a first step, the computation of a "table of differences" for the function, to an order equal to the degree of the polynomial. Lagrangian interpolation, on the other hand, is carried out by multiplying each of the tabulated values on which the interpolation is to be based by a suitable coefficient—the Lagrangian coefficients—and taking the sum of these products. This "cumulative" multiplication can be carried out as a continuous operation on a modern calculating machine, and when such a machine is available Lagrangian interpolation is usually more rapid than the other methods referred to.

Very extensive tables of Lagrangian interpolation coefficients are now available[5] (references to other, less complete, tables are given in this work).

[1] This investigation was performed at the National Bureau of Standards as part of the work of the American Petroleum Institute Research Project 44 on the "Collection and Analysis of Data on the Properties of Hydrocarbons."

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[3] The discussion in this report is restricted to interpolation in tables in which a function is tabulated at equally spaced intervals of the independent variable.

[4] David, Gibb, Edinburgh Mathematical Tracts No. 2, Interpolation and numerical integration (G. Bell & Sons, Ltd., London, 1915).

[5] Tables of Lagrangian interpolation coefficients, Mathematical Tables Project, Works Project Administration, Federal Works Agency (National Bureau of Standards; Columbia University Press, New York, N. Y., 1944).

In this report there is described a simplified method of computing the Lagrangian coefficients, which may be used when tables of Lagrangian coefficients are not available, or when tables are available, but the coefficients are not tabulated for the exact fraction of the interval to which the interpolation is to be made. This method requires only a single division for the computation of each coefficient, for interpolation on the basis of any number of tabulated values. The coefficients, as thus computed, are not normalized—they may be normalized in the usual way, although it is usually more convenient to interpolate by means of the non-normalized Lagrangian coefficients.

Lagrangian interpolation, with the use of tables of Lagrangian coefficients (see footnote 5), or by the method described in this report, is recommended as a rapid and generally useful method of interpolation in those tables in which a function, or property, is tabulated at several equally spaced values of the independent variable. The number of tabulated values, or the degree of the polynomial, on which the interpolation is based must be chosen to represent the function with sufficient accuracy in the interval in question.

II. GENERAL METHOD

The form of the modified Lagrangian interpolation formula will be stated, and then the derivation of the formula will be given.

Let the $(n+1)$ tabulated values of the function $y(x)$ on which the interpolation is to be based be $y_0, y_1, \ldots, y_k, \ldots, y_n$, and let the corresponding values of the independent variable—equally spaced at intervals of h — be $x_0, x_1, \ldots, x_k, \ldots, x_n$. The $(n+1)$ values of y determine a polynomial of the nth degree in x, $P(x)$, which is an approximation to the function $y(x)$ in the range x_0 to x_n. The value of $P(x)$, and therefore, approximately, of y, at a given value of x, is given by the modified Lagrangian formula

$$P(x)=\sum_{k=0}^{n}a_k y_k/\sum_{k=0}^{n}a_k, \tag{1}$$

where the coefficients, a_k, are given by

$$a_k=\pm\binom{n}{k}/|(x_k-x)/h|. \tag{2}$$

$|(x_k-x)/h|$ is the absolute value of $(x_k-x)/h$, and $\binom{n}{k}$ is the binomial coefficient

$$\binom{n}{k}=n!/k!(n-k)!. \tag{3}$$

The signs of the coefficients, a_k, are determined as follows: If x lies between x_m and x_{m+1},

$$x_m<x<x_{m+1},$$

then the coefficients, $\ldots a_{m-2}, a_m, a_{m+1}, a_{m+3}, \ldots$, are positive, and the coefficients, $\ldots, a_{m-3}, a_{m-1}, a_{m+2}, a_{m+4}, \ldots$, are negative. The pattern of signs is thus of the type

$$\ldots \; - \; + \; - \; + \; - \; \underset{\substack{\uparrow \\ \boxed{\text{POINT OF INTERPOLATION}}}}{+ \quad +} \; - \; + \; - \; + \; - \; \ldots$$

The coefficients, a_k, are proportional to the Lagrangian interpolation coefficients but are not normalized. In fact, the sum of the a_k's is given by the expression

$$\sum_{k=0}^{n} a_k = n!/\prod_{k=0}^{n} |(x_k - x)/h|, \tag{4}$$

which may be used as a check on the calculation of the a_k's.

If several interpolations to the same fraction of an interval are to be made, it is more convenient to put eq 1 in the form

$$P(x) = \sum_{k=0}^{n} A_k y_k, \tag{5}$$

where the A_k's are the normalized Lagrangian coefficients,

$$A_k = a_k / \sum_{k=0}^{n} a_k, \tag{6}$$

which have been tabulated elsewhere (see footnote 5). As used here, "normalized" indicates that the algebraic sum of the coefficients is unity.

These results may be derived in the following way: $P(x)$ may be written in the form

$$P(x) = \sum_{k=0}^{n} \left[\frac{(x-x_0)(x-x_1) \quad \ldots \quad (x-x_{k-1})(x-x_{k+1}) \quad \ldots \quad (x-x_n)}{(x_k-x_0)(x_k-x_1) \quad \ldots \quad (x_k-x_{k-1})(x_k-x_{k+1}) \quad \ldots \quad (x_k-x_n)} \right] y_k \tag{7}$$

as $P(x)$ (by definition) and the expression on the right of eq 7 are both polynomials of the nth degree in x, and are equal to y_k when $x = x_k$, for $k = 0, 1, \ldots, n$. But two polynomials of the nth degree which coincide at $(n+1)$ points must be identical, which establishes eq 7. Rearrangement of eq 7 yields

$$P(x) =$$

$$\left[\left(\frac{x-x_0}{h}\right)\left(\frac{x-x_1}{h}\right) \ldots \left(\frac{x-x_n}{h}\right) \Big/ n!\right] \sum_{k=0}^{n} \left[n! \Big/ \left(\frac{x_k-x_0}{h}\right)\left(\frac{x_k-x_1}{h}\right) \ldots \right.$$

$$\left. \left(\frac{x_k-x_{k-1}}{h}\right)\left(\frac{x_k-x_{k+1}}{h}\right) \ldots \left(\frac{x_k-x_n}{h}\right)\right] y_k \Big/ \left(\frac{x-x_k}{h}\right) \tag{8}$$

The expression in brackets under the summation sign is equal in absolute value to the binomial coefficient, $\binom{n}{k}$. Therefore if the signs of the terms are temporarily disregarded, as indicated by the $\pm$ sign, eq 8 may be rewritten as

$$P(x)=\left[\frac{1}{n!}\prod_{k=0}^{n}\left|(x-x_k)/h\right|\right]\sum_{k=0}^{n}\left[\pm\binom{n}{k}\Big/\left|(x_k-x)/h\right|\right]y_k. \tag{9}$$

Inspection of eq 7 shows that the signs of the terms follows the rule already given for the signs of the coefficients a_k. In view of the definition of a_k, eq 2, therefore, eq 9 becomes

$$P(x)=\left[\frac{1}{n!}\prod_{k=0}^{n}\left|(x-x_k)/h\right|\right]\sum_{k=0}^{n}a_ky_k. \tag{10}$$

In the special case in which $y_k=1$, for $k=0, 1, \ldots, n$, the polynomial $P(x)$ must have the value unity for all values of x. It follows that

$$\left[\frac{1}{n!}\prod_{k=0}^{n}\left|(x-x_k)/h\right|\right]\sum_{k=0}^{n}a_k=1. \tag{11}$$

Equations 10 and 11 lead immediately to eq 1, and eq 11 is equivalent to eq 4.

III. NUMERICAL EXAMPLE

Given the following (equally spaced) values of the descending exponential function, e^{-x}, find by interpolation the value of e^{-x} at $x=0.54316$:

x	$y=e^{-x}$
$x_0=0.52$	$y_0=0.594\ 520\ 548\ 0$
$x_1=0.53$	$y_1=0.588\ 604\ 969\ 7$
$x_2=0.54$	$y_2=0.582\ 748\ 252\ 4$
$x_3=0.55$	$y_3=0.576\ 949\ 810\ 4$
$x_4=0.56$	$y_4=0.571\ 209\ 063\ 8$

Solution: The interval, h, of the independent variable, x, is 0.01. The values of the binomial coefficients for $n=4$ are 1, 4, 6, 4, and 1. The value $x=0.54316$ lies between $x_2=0.54$ and $x_3=0.55$. Then the values of the a_k's, or the non-normalized Lagrangian coefficients, are given by eq 2:

$$\begin{aligned}
a_0&=+\tbinom{4}{0}/|(x_0-x)/h|=+1/2.316=+\ \ 0.431\ 778\ 9\\
a_1&=-\tbinom{4}{1}/|(x_1-x)/h|=-4/1.316=-\ \ 3.039\ 513\ 7\\
a_2&=+\tbinom{4}{2}/|(x_2-x)/h|=+6/0.316=+18.987\ 341\ 8\\
a_3&=+\tbinom{4}{3}/|(x_3-x)/h|=+4/0.684=+\ \ 5.847\ 953\ 2\\
a_4&=-\tbinom{4}{4}/|(x_4-x)/h|=-1/1.684=-\ \ 0.593\ 824\ 2
\end{aligned}$$

$$\sum_{k=0}^{4} a_k=21.633\ 736\ 0$$

A check may be obtained from eq 4:

$$4!\Big/\prod_{k=0}^{4}\left|(x_k-x)/h\right|=24/(2.316)(1.316)(0.316)(0.684)(1.684)$$

$$=21.633\ 736\ 0.$$

The value of $P(x)$ at $x=0.54316$ may now be calculated from eq 1 (the cumulative multiplication involved may be performed as a continuous operation on the calculating machine):

$$P(x)=\sum_{k=0}^{4}a_ky_k\Big/\sum_{k=0}^{4}a_k$$

$$=\left(\frac{12.567\ 246\ 5}{21.633\ 736\ 0}\right)$$

$$=0.580\ 909\ 673.$$

The true value of $y=e^{-x}$, at $x=0.54316$, correct to 10 decimal places, is

$$y=0.580\ 909\ 674\ 4.$$

When the same coefficients are to be used several times, it is convenient to compute the normalized Lagrangian coefficients, or the A_k's, from eq 6:

$$\begin{aligned}A_0&=+0.019\ 958\ 592\\ A_1&=-\ \ .140\ 498\ 789\\ A_2&=+\ \ .877\ 672\ 807\\ A_3&=+\ \ .270\ 316\ 380\\ A_4&=-\ \ .027\ 448\ 990\\ \hline \sum_{k=0}^{4}A_k&=\quad 1.000\ 000\ 000\end{aligned}$$

These coefficients differ by one or two units in the last place from the correct values (see p. 232 of the reference in footnote 5) because the a_k's were computed to only seven decimal places. The value of $P(x)$ at $x=0.54316$ may now be calculated from eq 5:

$$P(x)=\sum_{k=0}^{4}A_ky_k$$

$$=0.580\ 909\ 674.$$

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