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LAMINAR FLOW AT THE INTERFACE OF TWO LIQUIDS

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ABSTRACT

The velocity distribution in the laminar boundary layers at the interface of two liquids in relative rectilinear motion, the thickness of the layers, and the stress at the interface are determined. Numerical results are given for nine cases of liquids in contact, including identical liquids and liquids with varying degrees of dissimilarity in characteristics. The evaluation of the desired quantities is based on Prandtl's boundary-layer theory, and is carried out by a method of successive approximations. The numerical results are those given by the second approximation.

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I. INTRODUCTION

When a liquid of given density is flowing over a liquid of greater density, the latter being at rest, three successive forms of the interface are discernible as the velocity of the moving liquid is increased. To take a concrete example, let us assume that the flow takes place in a closed rectangular channel with a lower pool; the upper liquid is fresh water and the lower liquid is a solution of salt or sugar in water. First, at small velocities, the surface of separation is sharp and distinct, indicating, since the indices of refraction of the liquids are different, that the densities are discontinuous at the interface and the flow at the interface is laminar. The flow in the central part of the upper liquid, on the other hand, may be laminar or turbulent, depending on the size of the channel. Secondly, if the velocity of the upper liquid is gradually increased, at some critical velocity the smoothness of the separating surface first disappears, and then the surface becomes covered with surface waves traveling in the direction of the current and with a velocity slightly smaller than the velocity of the upper current. When the waves first appear, they are longcrested, that is, the length of the crests normal to the direction of motion is greater than the spacing between the crests. The crest lines are parabolic, concave upstream, and highest at the center of the channel, but flattening out toward the channel wall. At this stage of the flow—and this perhaps is significant—the waves are stable and show no tendency to break. The waves travel with practically no deformation, except the tendency to grow slightly larger during their motion. Finally, when the velocity of the upper current exceeds the critical velocity, the waves become sharp-crested and the crests shorter. The waves no longer are stable; that is, portions of the crests break away from the waves and are thrown into the upper current. These portions move forward and upward. In fact, this is the manner by which the mixing of the lower liquid with the moving liquid is brought about. When the velocity of the current is increased, still further within certain limits, the rate of mixing increases. At this stage of the flow, and this also is perhaps significant, the wave length is not affected by changes of velocity. These are the main results of observations made in a laboratory channel where the depth of the upper current is not over 10 centimeters and the length of the channel is not over 10 meters.

A detailed explanation of these phenomena is lacking. A theoretical approach supplementing experimental knowledge should prove helpful. A theoretical investigation should begin with a study of the stability of the interface on the basis of the actual conditions in the neighborhood of the interface. One of these conditions is a discontinuity of density at the surface of separation. Another is a pair of laminar boundary layers on the two sides of the surface of separation. In this paper the thicknesses of these boundary layers and the velocity distributions within them are determined by means of successive approximations, on the assumption that the interface is a smooth, plane surface.

II. FORMULATION OF THE PROBLEM

A liquid of indefinite height and having initially a uniform velocity, U, meets and flows over a still liquid of indefinite depth. In general, the two liquids have different physical characteristics; that is, their viscosities and densities are not the same. It will be assumed that the lower liquid has the greater density. If the pressure is everywhere hydrostatic, the interface or surface of separation is a horizontal plane. When U is sufficiently small the narrow region on both sides of the interface is a region of viscous laminar flow. Thus, the activating forces at the point where the moving liquid comes first in contact with the lower liquid are tangential and act in the horizontal direction; there is no normal force acting on the interface and hence the interface remains horizontal. Every particle of liquid on the interface moves with a constant velocity, u_0. The thickness of the viscous boundary layers in the two liquids is initially zero and increases with the length downstream (see fig. 1). In the upper layer the velocity in a vertical section increases from u_0 to U; in the lower layer it decreases from u_0 to 0. It is proposed to evaluate the variation of the thickness of the laminar layer with the length of the interface, the variation of velocity in the two layers, the shearing stress at the interface and the dependence of these quantities on the physical characteristics of the two liquids.

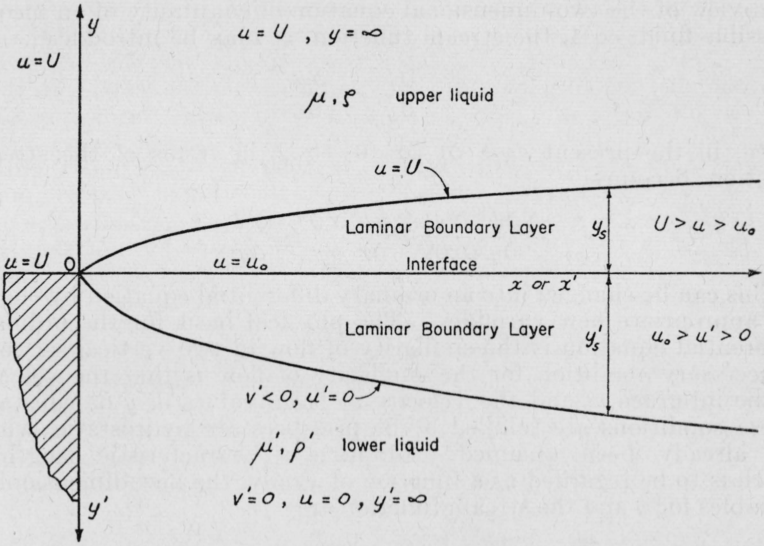

FIGURE 1.—*Laminar boundary layers at the interface of two liquids.*

III. PRANDTL'S BOUNDARY-LAYER EQUATION

In order to avoid negative values of ordinates two distinct sets of axes will be used, x, y for the upper liquid and x', y' for the lower liquid. See figure 1. The x-axes of the two sets coincide and lie in the plane of separation of the two liquids, the positive sense being in the direction of U. The point at which the two liquids first come into contact will be taken as the origin. In the upper liquid, positive y is directed upward; in the lower liquid, positive y' is directed downward. The density, the viscosity, and the kinematic viscosity of the upper liquid will be denoted by ρ, μ and ν, respectively, and the corresponding properties of the lower liquid by ρ', μ', and ν', respectively. The parameter, r, defined as

$$r=\sqrt{[\mu'\rho'/\mu\rho]}, \tag{1}$$

is of special significance in this analysis, as will be seen subsequently.

The basis of these computations is the boundary-layer equation of Prandtl for the two-dimensional flow of an incompressible fluid [1].[1] This equation with the law of hydrostatic pressure and the equation of continuity, when expressed in terms of the upper liquid, becomes

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y}+g=0, \tag{3}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{4}$$

where u and v are the components of velocity in the x and y directions, respectively, and p is the hydrostatic pressure.

[1] The figures in brackets indicate the literature references at the end of this paper.

In view of the two-dimensional equation of continuity of an incompressible fluid, eq 4, the stream function, ψ, may be introduced:

$$u=-\frac{\partial\psi}{\partial y},\ v=\frac{\partial\psi}{\partial x}. \tag{5}$$

Since, in the present case $\partial p/\partial x=0$, eq 2, in terms of the stream function, becomes

$$-\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}+\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\nu\frac{\partial^3\psi}{\partial y^3}. \tag{6}$$

This can be changed into an ordinary differential equation by choosing appropriate new variables. The physical basis for the ordinary differential equation is the similarity of flow in two vertical sections. A necessary condition for the similarity of flow is that the velocity at the interface u_0 and the velocity at large values of y be constant. These conditions are fulfilled if the pressures are hydrostatic, which has already been assumed. Adopting a characteristic length δ, which is to be regarded as a function of x only, the new dimensionless variables for y and the stream function are

$$\eta=ny/\delta, \tag{7}$$

and

$$H(\eta)=-\psi/U\delta, \tag{8}$$

where n is a dimensionless numerical constant as yet not specified. These variables are dimensionless. In terms of these, eq 6 now becomes

$$U\frac{d\delta}{dx}H\frac{d^2H}{d\eta^2}+n\frac{\nu}{\delta}\frac{d^3H}{d\eta^3}=0. \tag{9}$$

Thus, if we select δ to satisfy the relation

$$\delta\frac{d\delta}{dx}=n\frac{\nu}{U},$$

or

$$\delta^2=2n\frac{\nu x}{U}, \tag{10}$$

then, from eq 9, we have

$$\frac{d^3H}{d\eta^3}+H\frac{d^2H}{d\eta^2}=0, \tag{11}$$

which is Blasius' equation for the laminar boundary layer when the pressure is independent of x [2].

In eq 10 the dependence of δ on x is given except for the numerical value of n. We determine this by considering the velocity at large values of y. Now,

$$u=-\frac{\partial\psi}{\partial y}=nU\frac{dH}{d\eta},$$

hence if we specify that $dH/d\eta=1$ for large values of η, and this is a boundary condition for the upper fluid, we must then take $n=1$. These values we now adopt.

Summarizing, we have for the upper liquid

$$L(\eta)=\frac{d^3H}{d\eta^3}+H\frac{d^2H}{d\eta^2}=0, \tag{12}$$

where

$$H=-\frac{\psi}{U\delta}, \eta=\frac{y}{\delta}, \tag{13}$$

and

$$\delta=\sqrt{\frac{2\nu x}{U}}. \tag{14}$$

A similar set of equations may be established for the lower liquid by using primes to distinguish the variables,

$$L(\eta')=\frac{d^3H'}{d\eta'^3}+H'\frac{d^2H'}{d\eta'^2}=0, \tag{15}$$

where

$$H'=-\frac{\psi'}{U\delta'}, \eta'=\frac{y'}{\delta'}, \tag{16}$$

and

$$\delta'=\sqrt{\frac{2\nu' x}{U}}. \tag{17}$$

The actual velocities and their first derivatives in terms of the dimensionless variables are

$$u=U\frac{dH}{d\eta}, \tag{18}$$

$$\frac{\partial u}{\partial y}=\frac{U}{\delta}\frac{d^2H}{d\eta^2}, \tag{19}$$

for the upper liquid; and

$$u'=U\frac{dH'}{d\eta'}, \tag{20}$$

$$\frac{\partial u'}{\partial y'}=\frac{U}{\delta'}\frac{d^2H'}{d\eta'^2}, \tag{21}$$

for the lower liquid.

For the simultaneous solution of the two basic differential equations, eq 12 and 15, six boundary conditions are needed. These we now consider. In the upper liquid, u equals U for $y=\infty$; in the lower liquid, u' vanishes for $y'=\infty$. Accordingly,

$$\frac{dH}{d\eta}=1, \quad \eta=\infty; \tag{22}$$

and

$$\frac{dH'}{d\eta'}=0, \quad \eta'=\infty. \tag{23}$$

The stream function, ψ, may be interpreted as the flux per unit width between a point y of the upper liquid and the interface; and ψ' as the flux per unit width between a point y' of the lower liquid and the interface. As we have supposed that the interface is the horizontal plane $y=y'=0$, we must have

$$H=0,\ \eta=0; \tag{24}$$

and

$$H'=0,\ \eta'=0. \tag{25}$$

The velocities are continuous at the interface; hence

$$\frac{dH}{d\eta}=\frac{dH'}{d\eta'},\ \eta=\eta'=0. \tag{26}$$

Again, the shearing forces are continuous at the interface; hence

$$\mu\frac{\partial u}{\partial y}=-\mu'\frac{\partial u'}{\partial y'},\ y=y'=0,$$

$$\frac{\mu}{\sqrt{\nu}}\frac{d^2H}{d\eta^2}=-\frac{\mu'}{\sqrt{\nu'}}\frac{d^2H'}{d\eta'^2},\ \eta=\eta'=0,$$

or, using eq 1,

$$\frac{d^2H}{d\eta^2}=-r\frac{d^2H'}{d\eta'^2},\ \eta=\eta'=0. \tag{27}$$

These are the six boundary conditions of the problem.

IV. APPROXIMATE EXPRESSIONS FOR THE STREAM FUNCTIONS

To solve the differential equations, eq 12 and 15, we shall resort to a method of approximations which is a modification of the method used by Pohlhausen for the solution of the Blasius plate problem [3].

In this method certain approximate boundary conditions for large values of η and η' are required. First, consider the upper moving liquid. Select the value η_s of η such that

$$1-\frac{dH}{d\eta}\leqq\epsilon,\ \text{if}\ \eta\geqq\eta_s, \tag{28}$$

where ϵ is a small quantity, say 1/100 or any other smaller fraction. Denote the value of H corresponding to η_s by H_s. Now, writing eq 12 in the form

$$\frac{d^2H}{d\eta^2}=-\frac{1}{H}\frac{d^3H}{d\eta^3}, \tag{29}$$

and integrating between $\eta=\eta_s$ and $\eta=\infty$, we have

$$1-\left(\frac{dH}{d\eta}\right)_{\eta_s}=-\int_{\eta_s}^{\infty}\frac{1}{H}\frac{d^3H}{d\eta^3}\,d\eta.$$

Integrating the right-hand member by parts, and neglecting the term multiplied by ϵ, and making an obvious substitution from eq 29, using eq 28, we obtain

$$1-\left(\frac{dH}{d\eta}\right)_{\eta_s}=\frac{1}{H_s}\left(\frac{d^2H}{d\eta^2}\right)_{\eta_s}+\int_{\eta_s}^{\infty}\frac{1}{H^3}\frac{d^3H}{d\eta^3}\,d\eta.$$

Treating the integral in the right-hand member again in the same manner, we obtain

$$1-\left(\frac{dH}{d\eta}\right)_{\eta s}=\left(\frac{d^2H}{d\eta^2}\right)_{\eta s}\left(\frac{1}{H_s}-\frac{1}{H_s^3}\right)-3\int_{\eta s}^{\infty}\frac{1}{H^5}\frac{d^3H}{d\eta^3}\,d\eta.$$

If ϵ is sufficiently small, that is, if H_s is sufficiently large, the integral in the latter equation may be discarded, leaving

$$1-\left(\frac{dH}{d\eta}\right)_{\eta s}=\left(\frac{d^2H}{d\eta^2}\right)_{\eta s}\left(\frac{1}{H_s}-\frac{1}{H_s^3}\right). \tag{30}$$

This is the approximate relation which exists between the first and second derivatives of H when $\eta=\eta_s$.

If we now suppose that H_s is so large that 1 can be neglected compared to the square of H_s, then, in view of eq 28, 29, and 30, we must have

$$\left.\begin{aligned}\frac{dH}{d\eta}&=1-\epsilon, & \eta&=\eta_s,\\ \frac{d^2H}{d\eta^2}&=\epsilon H_s, & \eta&=\eta_s,\\ \frac{d^3H}{d\eta^3}&=-\epsilon H_s^2, & \eta&=\eta_s.\end{aligned}\right\} \tag{31}$$

Consider now the lower still liquid. We select the value η_s' of η', such that

$$\frac{dH'}{d\eta'}\leqq\epsilon',\ \text{if}\ \eta'\geqq\eta_s', \tag{32}$$

where ϵ' has the same general meaning as ϵ. Denote the value of H' corresponding to η_s' by H_s'. For values of η' larger than η_s', eq 15 may be written in the form

$$\frac{d^3H'}{d\eta'^3}=-H_s'\frac{d^2H'}{d\eta'^2}.$$

Obviously, then

$$\left.\begin{aligned}\frac{dH'}{d\eta'}&=\epsilon', & \eta'&=\eta_s',\\ \frac{d^2H'}{d\eta'^2}&=-\epsilon' H_s', & \eta'&=\eta_s',\\ \frac{d^3H'}{d\eta'^3}&=\epsilon' H_s'^2, & \eta'&=\eta_s'.\end{aligned}\right\} \tag{33}$$

Now, in the method employed here, the end conditions for the large values of η and η' will be taken from eq 31 and 33 after assigning definite values to ϵ and ϵ'. The values assigned to ϵ and ϵ' have a bearing on the accuracy of the approximation and also govern the course of the computations. If ϵ and ϵ' are assigned values different from zero, the accuracy of the approximation is increased, but the analysis becomes more complicated. In order to simplify computations, we shall take $\epsilon=\epsilon'=0$ and adopt the following end conditions. For the upper liquid, we have

$$\frac{dH}{d\eta}=1, \; \frac{d^2H}{d\eta^2}=0, \; \frac{d^3H}{d\eta^3}=0, \; \eta=\eta_s, \tag{34}$$

and for the lower liquid,

$$\frac{dH'}{d\eta'}=0, \; \frac{d^2H'}{d\eta'^2}=0, \; \frac{d^3H'}{d\eta'^3}=0, \; \eta'=\eta_s'. \tag{35}$$

These now replace the conditions implied by eq 22 and 23. With this choice of the end conditions, η_s and η_s' have simple geometric interpretations. Let y_s and y_s' be the values of the distance from the interface corresponding to η_s and η_s', respectively; that is, let

$$y_s=\eta_s\sqrt{(2\nu x/U)}, \tag{36}$$

and

$$y_s'=\eta_s'\sqrt{(2\nu' x/U)}, \tag{37}$$

then y_s and y_s' are the thicknesses of the two laminar boundary layers at the interface.

We shall also need the following end conditions from eq 24, 25, 26, for the upper liquid,

$$H=0, \; \frac{dH}{d\eta}=a_0, \; \eta=0, \tag{38}$$

and for the lower liquid,

$$H'=0, \; \frac{dH'}{d\eta'}=a_0', \; \eta'=0, \tag{39}$$

where

$$a_0=a_0'=u_0/U.$$

We are now ready to develop the general expressions for the stream functions H and H'. Since from eq 15 and 34, $d^3H/d\eta^3$ vanishes at $\eta=0$ and $\eta=\eta_s$, then it must be possible to express this function of H as a summation of polynomials where each polynomial vanishes for $\eta=0$ and $\eta=\eta_s$. Suitable polynomials may be selected in various ways. In order to extend Pohlhausen's computations [3], we select the following oscillating algebraic functions as the expansion polynomials for this analysis:

$$N_1=-\alpha(\alpha-1)=\alpha-\alpha^2,$$

$$N_2=+\alpha(\alpha-1)(2\alpha-1)=2\alpha^3-3\alpha^2+\alpha,$$

$$N_3=-\alpha(\alpha-1)(3\alpha-1)(3\alpha-2)=2\alpha-11\alpha^2+18\alpha-9\alpha^4,$$

and, in general,

$$N_n=(-1)^n\alpha(\alpha-1)(n\alpha-1)(n\alpha-2)(n\alpha-3)(\;.\;.\;.\;)(n\alpha-n+1),$$

where

$$\alpha=\eta/\eta_s.$$

Using only the first three functions, we put

$$\frac{dH^3}{d\eta^3}=\frac{a_1}{\eta_s}(-6N_1+32k_2N_2-30k_3N_3). \tag{40}$$

Here a_1, η_s, k_2, and k_3 are constants to be determined. Integrating once with respect to η, we have

$$\frac{d^2H}{d\eta^2}=a_1(J_1+k_2J_2+k_3J_3), \tag{41}$$

where J_1, J_2 and J_3 are functions of α:

$$J_1=1-3\alpha^2+2\alpha^3,$$

$$J_2=16\alpha^2-32\alpha^3+16\alpha^4,$$

$$J_3=1-30\alpha^2+110\alpha^3-135\alpha^4+54\alpha^5.$$

It must be remarked that J_1 is derived from N_1 by integrating with respect to η and selecting the constant of integration so that $J_1=0$ for $\alpha=1$, and similarly for J_2 and J_3. Equation 41 thus obviously satisfies the second end conditions in eq 34. Integrating eq 41 with respect to η, we have

$$\frac{dH}{d\eta}=a_0+a_1\eta_s(G_1+k_2G_2+k_3G_3), \tag{42}$$

where G_1, G_2, and G_3 are functions of α:

$$G_1=\alpha-\alpha^3+\frac{1}{2}\alpha^4,$$

$$G_2=\frac{16}{3}\alpha^3-8\alpha^4+\frac{16}{5}\alpha^5,$$

$$G_3=\alpha-10\alpha^3+\frac{55}{2}\alpha^4-27\alpha^5+9\alpha^6.$$

In this case, G_1 is derived from J_1 by integrating the latter with respect to η and selecting the constant of integration so that $G_1=0$ when $\eta=0$, and similarly for G_2 and G_3. The constant a_0 is introduced in order to satisfy the second end conditions in eq 38. Integrating eq 42 with respect to η, we have

$$H=a_0\eta_sF_0+a_1\eta_s^2(F_1+k_2F_2+k_3F_3), \tag{43}$$

where F_0, F_1, F_2, and F_3 are functions of α:

$$F_0=\alpha,$$

$$F_1=\frac{1}{2}\alpha-\frac{1}{4}\alpha^4+\frac{1}{10}\alpha^5,$$

$$F_2=\frac{4}{3}\alpha^4-\frac{8}{5}\alpha^5+\frac{8}{15}\alpha^6,$$

$$F_3=\frac{1}{2}\alpha^2-\frac{5}{2}\alpha^4+\frac{11}{12}\alpha^5-\frac{9}{2}\alpha^6+\frac{9}{7}\alpha^7,$$

where F_1 is derived from G_1 by integrating the latter with respect to η and selecting the constant of integration so that $F_1=0$ when $\eta=0$, and similarly for F_0, F_2, and F_3. It is obvious that the first of the end conditions in eq 38 is satisfied. The functions N, J, G, and F are tabulated in tables 1, 2, 3, and 4.

TABLE 1.—*Values of the N functions*

α	$N_1(\alpha)$	$N_2(\alpha)\times 10$	$N_3(\alpha)\times 10$
0.00	0.0000	0.0000	0.0000
.05	.0475	.4275	.7030
.10	.0900	.7200	1.0710
.15	.1275	.0925	1.0869
.20	.1600	.9600	0.8960
.25	.1875	.9375	.5859
.30	.2100	.8400	.2310
.35	.2275	.6825	−.1081
.40	.2400	.4800	−.3840
.45	.2475	.2475	−.5631
.50	.2500	−.0000	−.6250
.55	.2475	−.2475	−.5631
.60	.2400	−.4800	−.3840
.65	.2275	−.6825	−.1081
.70	.2100	−.8400	.2310
.75	.1875	−.9375	.5859
.80	.1600	−.9600	.8960
.85	.1275	−.8925	1.0869
.90	.0900	−.7200	1.0710
.95	.0475	−.4275	0.7030
1.00	.0000	−.0000	.0000

TABLE 2.—*Values of the J functions*

α	$J_1(\alpha)$	$J_2(\alpha)$	$J_3(\alpha)$
0.00	1.0000	0.0000	1.0000
.05	0.9929	.0344	0.9451
.10	.9720	.1296	.7971
.15	.9393	.2632	.6312
.20	.8960	.4096	.4842
.25	.8437	.5624	.3690
.30	.7840	.7056	.3075
.35	.7183	.8280	.2979
.40	.6480	.9216	.3366
.45	.5747	.9800	.4105
.50	.5000	1.0000	.5001
.55	.4253	0.9800	.5904
.60	.3520	.9216	.6633
.65	.2817	.8280	.7008
.70	.2160	.7056	.6942
.75	.1563	.5624	.6309
.80	.1040	.4096	.5280
.85	.0607	.2632	.3675
.90	.0280	.1296	.2037
.95	.0073	.0344	.0615
1.00	.0000	.0000	.0000

TABLE 3.—*Values of the G functions*

α	$G_1(\alpha)$	$G_2(\alpha)$	$G_3(\alpha)$
0.00	0.0000	0.0000	0.0000
.05	.0499	.0011	.0480
.10	.0990	.0045	.0927
.15	.1469	.0144	.1270
.20	.1928	.0299	.1569
.25	.2363	.0552	.1760
.30	.2770	.0869	.1942
.35	.3146	.1254	.2093
.40	.3488	.1691	.2255
.45	.3793	.2168	.2444
.50	.4062	.2669	.2641
.55	.4294	.3165	.2932
.60	.4488	.3640	.3244
.65	.4646	.4077	.3590
.70	.4771	.4464	.3937
.75	.4863	.4781	.4272
.80	.4928	.5024	.4559
.85	.4969	.5191	.4783
.90	.4990	.5288	.4925
.95	.4999	.5326	.4999
1.00	.5000	.5333	.5000

TABLE 4.—*Values of the F functions*

α	$F_1(\alpha)$	$F_2(\alpha)$	$F_3(\alpha)$
0.00	0.0000	0.0000	0.0000
.05	.0012	.0000	.0012
.10	.0050	.0001	.0047
.15	.0112	.0006	.0106
.20	.0196	.0016	.0176
.25	.0303	.0037	.0261
.30	.0432	.0073	.0351
.35	.0580	.0127	.0449
.40	.0746	.0200	.0556
.45	.0928	.0297	.0692
.50	.1125	.0415	.0807
.55	.1334	.0562	.0995
.60	.1554	.0733	.1096
.65	.1782	.0925	.1267
.70	.2018	.1138	.1457
.75	.2258	.1370	.1661
.80	.2505	.1618	.1881
.85	.2753	.1874	.2118
.90	.3001	.2134	.2357
.95	.3750	.2398	.2608
1.00	.3500	.2666	.2857

In these expressions there appear two types of constants. The constants k_2 and k_3 are of one type and may be referred to as the expansion constants, since these are introduced because of N_2 and N_3. The constants a_0, a_1, and η_s are of the other type and may be referred to as the basic constants, since a_0 relates to the interfacial velocity u_0, a_1, to the local shear and η_s to the thickness of the laminar boundary layer in the upper liquid.

Reasoning in a similar manner, we have for the lower, still liquid,

$$\frac{d^3H'}{d\eta'^3}=\frac{a_1'}{\eta_s'}(-6N_1+32k_2'N_2-30k_3'N_3), \tag{44}$$

$$\frac{d^2H'}{d\eta'^2}=a_1'(J_1+k_2'J_2+k_3'J_3), \tag{45}$$

$$\frac{dH'}{d\eta'}=a_0'+a_1'\eta_s'(G_1+k_2'G_2+k_3'G_3), \tag{46}$$

and

$$H'=a_0'\eta_s'F_0+a_1'\eta_s'^2(F_1+k_2'F_2+k_3'F_3), \tag{47}$$

where N, J, G, and F are the same functions as before but now depend on the independent variable of $\alpha'=\eta'/\eta_s'$. Again, k_2' and k_3' are the expansion constants, and a_0', a_1', and η_s' are the basic constants for the laminar boundary layer in the lower, still liquid.

Equations 43 and 47 are the approximate expressions for the stream functions H and H'. The unknown elements are the basic and expansion constants, which we now proceed to determine.

V. EVALUATION OF THE CONSTANTS

There are a total of six basic constants and four expansion constants in the expressions for H and H', and thus 10 relations are required for their determination. Four such relations are obtained immediately by considering the first of the end conditions in eq 34 and 35 and the continuity conditions, eq 26 and 27. These give, first,

$$a_0+a_1\eta_s A_0=1, \tag{48}$$

where

$$A_0=\frac{1}{2}\left(1+\frac{16}{15}k_2+k_3\right); \tag{49}$$

second,

$$a_0'+a_1'\eta_s'A_0'=0, \tag{50}$$

where

$$A_0'=\frac{1}{2}\left(1+\frac{16}{15}k_2'+k_3'\right); \tag{51}$$

third,

$$a_0=a_0'; \tag{52}$$

and fourth,

$$a_1=-ra_1'M; \tag{53}$$

where

$$M=(1+k_3')/(1+k_3), \quad r^2=(\mu'\rho')/(\mu\rho). \tag{54}$$

The remaining six relations may be obtained from eq 12 and 15 by using one of two methods.

In the first method the left-hand members of eq 12 and 15 are multiplied successively by N_1, N_2, and N_3, and integrated with respect to α or α' between the limits 0 and 1 and set equal to zero, since L is identically equal to zero. That is, putting

$$L=\frac{d^3H}{d\eta^3}+H\frac{d^2H}{d\eta^2}, \tag{55}$$

and

$$L'=\frac{d^3H'}{d\eta'^3}+H'\frac{d^2H'}{d\eta'^2}, \tag{56}$$

the desired six relations are

$$\int_0^1 LN_i\,d\alpha=0,\ (i=1,\,2,\,3); \tag{57, 58, 59}$$

and

$$\int_0^1 L'N_i\,d\alpha'=0,\ (i=1,\,2,\,3). \tag{60, 61, 62}$$

A simpler method would be to use Q_1, Q_2, and Q_3 instead of N_1, N_2, and N_3, where

$$\begin{array}{ll} Q_1=+1, & 0\leqq\alpha \text{ or } \alpha'\leqq 1; \\ & \\ Q_2=+1, & 0\leqq\alpha \text{ or } \alpha'\leqq 1/2; \\ Q_2=-1, & 1/2\leqq\alpha \text{ or } \alpha'\leqq 1; \\ & \\ Q_3=+1, & 0\leqq\alpha \text{ or } \alpha'\leqq 1/3; \\ Q_3=-1, & 1/3\leqq\alpha \text{ or } \alpha'\leqq 2/3; \\ Q_3=+1, & 2/3\leqq\alpha \text{ or } \alpha'\leqq 1. \end{array}$$

The second method, which is the more familiar one, utilizes the following six relations [4]:

$$\int_0^1 L\,d\alpha=0; \tag{63}$$

$$\int_0^1 L'\,d\alpha=0; \tag{64}$$

$$\frac{dL}{d\eta}=0,\ \eta=\eta_s; \tag{65}$$

$$\frac{dL}{d\eta}=0,\quad \eta=0; \tag{66}$$

$$\frac{dL'}{d\eta'}=0,\quad \eta'=\eta_s'; \tag{67}$$

$$\frac{dL'}{d\eta'}=0,\quad \eta'=0. \tag{68}$$

The constants of the present problem have been determined, by using both of the above methods, and the results agreed to the same order of accuracy. The results obtained from eq 63 to 68 will be given here.

From eq 40, 41, and 43, we find from eq 55,

$$\begin{aligned} L(\eta)=&\frac{a_1}{\eta_s}(-6N_1+32k_2N_2-30k_3N_3) \\ &+a_0a_1\eta_s(J_1F_0+k_2J_2F_0+k_3J_3F_0) \\ &+a_1^2\eta_s^2(J_1F_1+k_2[J_2F_1+F_2J_1]+k_3[J_3F_1+F_3J_1]) \\ &+a_1^2\eta_s^2(k_2^2J_2F_2+k_2k_3[J_2F_3+F_2J_3]+k_3^2J_3F_3). \end{aligned}$$

Substituting in eq 63 and integrating, we find that

$$a_1^2=A_1a_0a_1^2\eta_s^2+A_2a_1^3\eta_s^3, \tag{69}$$

where

$$A_1=(a_{01}+a_{02}k_2+a_{03}k_3)/(1+k_3), \tag{70}$$

and

$$A_2=(a_{11}+a_{12}k_2+a_{13}k_3+a_{22}k_2^2+a_{23}k_2k_3+a_{33}k_3^2)/(1+k_3). \tag{71}$$

The values of the coefficients with the double subscripts occurring in eq 70 and 71 are

$$a_{01}=0.15000, \quad a_{11}=0.029365, \quad a_{22}=0.0308,$$
$$a_{02}=0.26667, \quad a_{12}=0.07619, \quad a_{23}=0.0758,$$
$$a_{03}=0.21425, \quad a_{13}=0.07625, \quad a_{33}=0.0400.$$

By treating eq 64 in a similar manner, we find that

$$a_1'^2=A_1'a_0'a_1'^2\eta_s'^2+A_2'a_1'^3\eta_s'^3, \tag{72}$$

where

$$A_1'=(a_{01}+a_{02}k_2'+a_{03}k_3')/(1+k_3'), \tag{73}$$

and

$$A_2'=(a_{11}+a_{12}k_2'+a_{13}k_3'+a_{22}k_2'^2+a_{23}k_2'k_3'+a_{33}k_3'^2)/(1+k_3'). \tag{74}$$

Since from eq 55

$$\frac{dL}{d\eta}=\frac{d^4H}{d\eta^4}+\frac{dH}{d\eta}\frac{d^2H}{d\eta^2}+H\frac{d^3H}{d\eta^3},$$

and

$$\frac{dN_1}{d\alpha}=1, \quad \frac{dN_2}{d\alpha}=1, \quad \frac{dN_3}{d\alpha}=2, \quad \eta=0, \quad \alpha=0,$$
$$\frac{dN_1}{d\alpha}=-1, \quad \frac{dN_2}{d\alpha}=1, \quad \frac{dN_3}{d\alpha}=-2, \quad \eta=\eta_s, \quad \alpha=1,$$

we find from eq 65

$$\frac{a_1}{\eta_s}(6+32k_2+60k_3)=0, \tag{75}$$

and from eq 66

$$\frac{a_1}{\eta_s^2}(-6+32k_2-60k_3)=-a_0a_1(1+k_3). \tag{76}$$

Adding eq 75 to eq 76, we have

$$64k_2=-a_0\eta_s^2(1+k_3). \tag{77}$$

Since η_s is finite and a_1 does not vanish, we have, from eq 75,

$$60k_3=-6-32k_2. \tag{78}$$

We have from eq 77, using eq 48, the relation

$$64k_2=-\frac{4a_0(1-a_0)^2}{a_1^2}\left[(1+k_3)/\left(1+\frac{16}{15}k_2+k_3\right)^2\right], \tag{79}$$

where η_s does not appear. It may be emphasized that eq 78 and 79 are two relations expressing the expansion constants k_2 and k_3 in terms of the basic constants a_0 and a_1.

In a similar manner we obtain from eq 67 and 68 the two relations

$$60k_3'=-6-32k_2', \tag{80}$$

$$64k_2'=-\frac{4a_0'^3}{a_1'^2}\left[(1+k_3')/\left(1+\frac{16}{15}k_2'+k_3'\right)^2\right]. \tag{81}$$

Equations 48, 50, 52, 53, 69, 72, 78, 79, 80, and 81 are the desired 10 relations for determining the 10 constants appearing in the expressions for H and H' given by eq 43 and 47, respectively. In principle the determination of the constants in a direct manner is possible, but since this is tedious, it may be carried out by successive approximations. The successive approximations to the values of the constants give the successive approximations to the present boundary-layer problem.

VI. FIRST APPROXIMATE SOLUTION FOR THE LAMINAR BOUNDARY LAYERS

In the first approximate solution it is assumed that the expansion constants k_2, k_3, k_2', and k_3' are zero.

Consider the relations

$$a_0+a_1\eta_sA_0=1, \tag{48a}$$

$$a_0'+a_1'\eta_s'A_0'=0, \tag{50a}$$

$$a_0=a_0', \tag{52a}$$

$$a_1=-ra_1'M, \quad [r^2=(\mu'\rho')/(\mu\rho)], \tag{53a}$$

$$a_1^2=A_1a_0a_1^2\eta_s^2+A_2a_1^3\eta_s^3, \tag{69a}$$

and

$$a_1'^2=A_1'a_0'a_1'^2\eta_s'^2+A_2'a_1'^3\eta_s'^3. \tag{72a}$$

Here the constants A_0, A_0', M, A_1, A_1', A_2 and A_2' are functions of k_2, k_3, k_2', and k_3' and may be determined from eq 49, 51, 54, 70, 71, 73, and 74. For k_2, k_3, k_2', and k_3' all equal to zero, we obtain

$$A_0=A_0'=0.50000,$$
$$A_1=A_1'=0.15000,$$
$$A_2=A_2'=0.023936,$$
$$M=1.$$

Now, the determination of the unknown quantities a_0, a_1, η_s and a_0', a_1', η_s' from the above six relations, 47a to 72a, is an easy task once a specific value is assigned to r. Separate determinations were made by selecting nine values for r, namely, $r^2=0.00$, 0.01, 0.10, 0.3162, 1, 3.162, 10.0, 100, and ∞. The results are shown in table 5.

TABLE 5.—*First approximate values of the constants a_0, a_1, a_1', η_s and η_s'*

$r^2 = (\mu'\rho')/(\mu\rho)$

r^2	a_1	a_1', a_0	or a_0'	η_s	η_s'	H_s	H_s'
0.00	----------	0.6038	1.000	----------	3.309	----------	0.995
.01	0.0540	.5402	0.9285	2.611	3.433	2.552	.959
.10	.1388	.4390	.8085	2.744	3.678	2.582	.893
.3162	.2031	.3614	.7100	2.845	3.923	2.595	.838
1.0	.2727	.2724	.5910	2.995	4.331	2.616	.772
3.162	.3397	.1911	.4646	3.146	4.851	2.630	.679
10	.3916	.1238	.3479	3.324	5.605	2.672	.590
100	.4494	.0449	.1773	3.655	7.844	2.750	.423
∞	.4839	------	.000	4.124	------	2.884	------

The expressions for H and H' from eq 43 and 47, respectively, wherein the constants k and k' are put equal to zero and the values of the basic constants are taken from table 5, constitute the first approximate solution of the laminar boundary layers at the interface of two liquids, one of which is still and the other moving. More specifically, in the upper liquid

$$H = a_0\eta_s F_0(\eta/\eta_s) + a_1{}^2\eta_s F_1(\eta/\eta_s), \quad \frac{dH}{d\eta} = a_0 + a_1\eta_s G_1(\eta/\eta_s),$$

and in the lower liquid

$$H' = a_0'\eta_s' F_0(\eta'/\eta_s') + a_1'^2\eta_s'^2 F_1(\eta'/\eta_s'), \quad \frac{dH'}{d\eta'} = a_0' + a_1'\eta_s' G_1(\eta'/\eta_s').$$

The numerical values of F_0, F_1, and G_1 are given in tables 3 and 4.

It may be of interest to note that the case $r^2 = \infty$ can be physically interpreted as representing the case in which the lower liquid has an infinite viscosity, which is obviously the case for a liquid flowing over a plate. The solution obtained here for $r^2 = \infty$ is identical with Pohlhausen's fourth approximation. In the present analysis this constitutes only the first approximate solution.

VII. SECOND APPROXIMATE SOLUTION OF THE LAMINAR BOUNDARY LAYERS

Inserting the first approximate values of a_0, a_1, and a_0', a_1' from table 5 in eq 78, 79, 80, and 81, we obtain a set of values for k_2, k_3, k_2', and k_3', after discarding the squares and the higher powers of these constants. The results are given in table 6 for the various values of r^2.

TABLE 6.—*Second approximate values of the expansion constants k_2, k_3, k_2', and k_3'*

$r^2 = (\mu'\rho')/(\mu\rho)$

r^2	k_2	k_3	k_2'	k_3'	M
0.00	----------	----------	−0.2600	+0.0382	----------
.01	−0.1354	−0.0278	−.2600	+.0382	1.0681
.10	−.1259	−.0328	−.2600	+.0382	1.0737
.3162	−.1162	−.0380	−.2600	+.0382	1.0816
1.0	−.1037	−.0446	−.2600	+.0382	1.0869
3.162	−.0897	−.0522	−.2600	+.0382	1.0956
10	−.0734	−.0608	−.2600	+.0382	1.1057
100	−.0429	−.0771	−.2600	+.0382	1.1252
∞	.0000	−.1000	--------	--------	--------

With these, corresponding values of the coefficients A_0, A', A_1, A_1', A_2, A_2', and M are computed. The results are shown in table 7. With the values of the coefficients thus known, the second approximate values of the basic constants are determined by the same equations as for the first approximate solution, that is, eq 48a to 72a, inclusive. The results of this determination are given in table 8.

Expressions for H and H' from eq 43 and 47, respectively, where the expansion constants are given in table 6 and the basic constants are given in table 8, constitute the second approximate solution of the laminar layers boundary problem for the interface between two liquids.

TABLE 7.—*Second approximate values of the auxiliary constants A_0, A_1, A_2, A_0', A_1', and A_2'*

$r^2=(\mu'\rho')/(\mu\rho)$

r^2	A_0	A_1	A_2	A_0'	A_1'	A_2'
0.00	----------	----------	----------	0.3806	0.08562	0.01336
.01	0.4139	0.1110	0.01830	.3806	.08562	.01336
.10	.4164	.1131	.01871	.3806	.08562	.01336
.3162	.4190	.1155	.01917	.3806	.08562	.01336
1.00	.4224	.1181	.01970	.3806	.08562	.01336
3.162	.4260	.1212	.02038	.3806	.08562	.01336
10	.4304	.1250	.02107	.3806	.08562	.01336
100	.4382	.1321	.02245	.3806	.08562	.01336
∞	.4600	.1427	.02460	------	--------	--------

TABLE 8.—*Second approximate values of the basic constants a_0, a_1, a_1', η_s, and η_0*

$r^2=(\mu'\rho')/(\mu\rho)$

r^2	a_1	a_1'	a_0 or a_0'	η_s	η_s'	H_s	H_s'
0.00	----------	−0.5906	1.0000	----------	4.449	3.012	1.041
.01	0.0567	−.5329	0.9282	3.069	4.576	3.004	0.993
.10	.1456	−.4287	.8082	3.163	4.953	2.991	.936
.3162	.2144	−.3527	.7088	3.241	5.280	2.990	.875
1.0	.2899	−.2668	.5888	3.358	5.793	2.991	.798
3.162	.3611	−.1854	.4619	3.498	6.545	2.996	.707
10	.4174	−.1194	.3446	3.648	7.582	3.016	.602
100	.4817	−.0428	.1739	3.913	10.672	3.055	.434
∞	.5196	--------	.0000	4.277	------	------	------

VIII. QUALITY OF THE APPROXIMATIONS

The quality of the approximations may be studied conveniently by substituting the approximate expressions for H and H' and for their derivatives into the differential equations, eq 12 and 15, and then considering the magnitude of the remainders. Let the remainders be denoted by ΔL and $\Delta L'$:

$$\Delta L=\frac{d^3H}{d\eta^3}+H\frac{d^2H}{d\eta^2},$$

and

$$\Delta L'=\frac{d^3H'}{d\eta'^3}+H'\frac{d^2H'}{d\eta'^2}.$$

If the ratios

$$\Delta L/(d^3H/d\eta^3)_{\max} \text{ and } \Delta L'/(d^3H'/d\eta'^3)_{\max}$$

become smaller in the successive approximations, the magnitude of decrease is a measure of the improvements in the approximations.

We have computed these remainders only for the two extreme

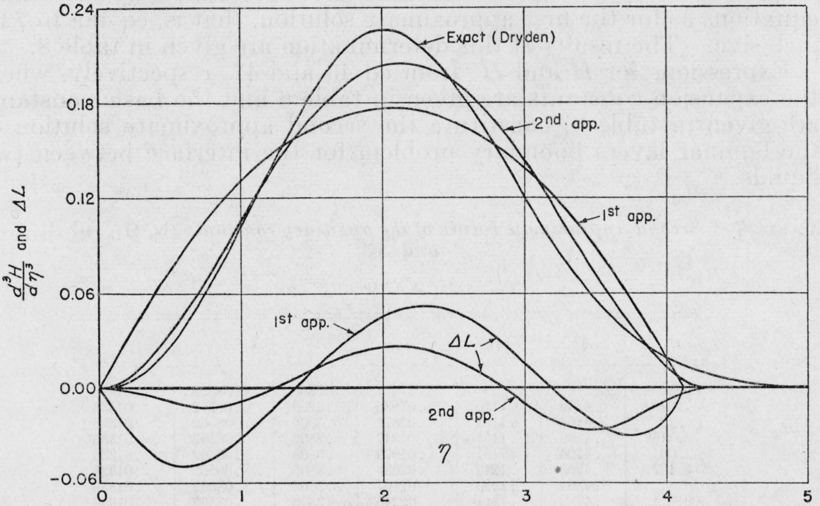

FIGURE 2—*Remainders of the first and of the second approximations for the case* $r^2=\infty$.

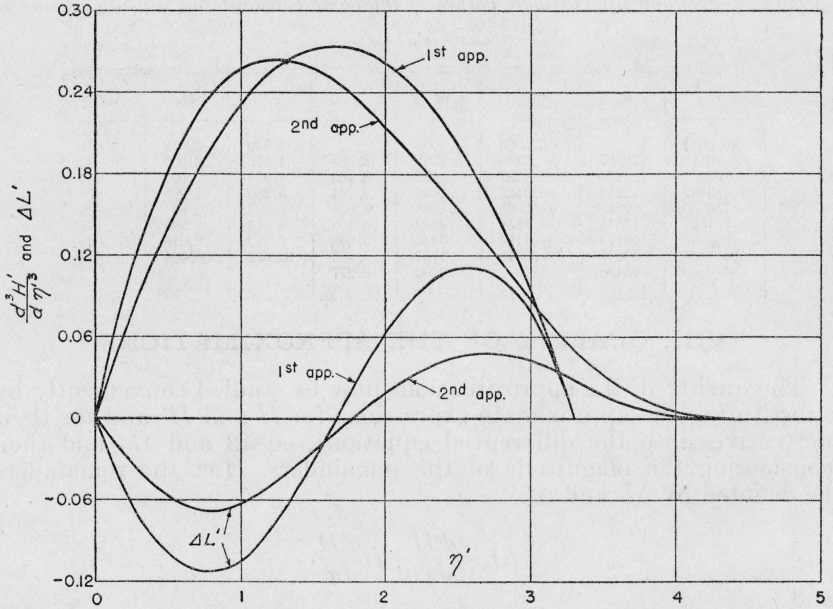

FIGURE 3—*Remainders of the first and of the second approximations for the case* $r^2=0$.

cases represented by $r^2=\infty$ and $r^2=0$. The values of the remainders together with the values of the third derivatives of H and H', are given in figures 2 and 3, respectively. The case of a liquid of finite viscosity moving over a liquid of infinite viscosity is represented by $r^2=\infty$. As mentioned before, this is mathematically identical with the Blasius problem of flow over a plate. In examining the curves in figure 2, we see that the amplitude of the oscillation of the remainder in the second approximate solution is only about half as large as in the first approximation, indicating that the improvement in the second approximate solution is appreciable. In the second approximate solution, the maximum value of the remainder is 0.027; the maximum value of the third derivative, 0.25. It is thus seen that the remainders are relatively small. In this same figure we give also the curve of the third derivative of H from the exact solution by Dryden [5]. The agreement between the second approximate solution and the exact solution is rather close. The velocity gradient at $y=0$, from the second approximate solution, is

$$\frac{1}{U}\frac{\partial u}{\partial y}=0.3307\sqrt{U/(\nu x)}.$$

In the exact solution the corresponding coefficient has the value 0.3320, thus indicating that the error in the value of this quantity from the second approximate solution is only 0.39 percent.

A liquid of infinite viscosity moving over a liquid of finite viscosity is represented by $r^2=0$. Physically, it would seem that this is the case of a solid surface moving in a still liquid. The curves in figure 3 also show that in this case the remainder of the second approximate solution has been reduced and has become about half as large as the corresponding quantity in the first approximate solution. But the accuracy obtained is not as satisfactory as in the case $r^2=\infty$, since the remainders are relatively larger. Apparently to obtain the same degree of accuracy as was obtained in the case $r^2=\infty$, and with the second approximation it will be necessary to introduce the expansion functions k_4N_4 and k_5N_5.

IX. PHYSICALLY SIGNIFICANT QUANTITIES OF THE LAYERS

Among the quantities of the laminar layers at the interface of the two liquids, one at rest and the other moving over it, the following are of practical significance: (1) The intensity of the viscous stresses at the surface of separation, (2) the thickness of the layers, and (3) the distribution of the velocities in the layers. The computation of these quantities will be made, using the second approximate solution of the analysis.

1. LOCAL STRESS AT THE SURFACE OF SEPARATION

The stress in question is determined from the velocity gradient of the upper liquid at the surface of separation:

$$\tau=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}.$$

From eq 19

$$\tau=\mu\,\frac{U}{\delta}\left(\frac{d^2H}{d\eta^2}\right)_0,$$

or

$$\tau = \frac{2\nu}{\delta U}\left(\frac{d^2H}{d\eta^2}\right)_0 \frac{\rho}{2} U^2.$$

Substituting from eq 14,

$$\tau = \sqrt{2}\left(\frac{d^2H}{d\eta^2}\right)_0 \left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}} \frac{\rho}{2} U^2.$$

If we use the velocity gradient of the lower liquid, we obtain in the same manner,

$$\tau = -\sqrt{2}\left(\frac{d^2H'}{d\eta'^2}\right)_0 \left(\frac{Ux}{\nu'}\right)^{-\frac{1}{2}} \frac{\rho'}{2} U^2.$$

Putting

$$\sqrt{2}\left(\frac{d^2H}{d\eta^2}\right)_0 = \sqrt{2} a_1 (1+k_3) = s,$$

and

$$-\sqrt{2}\left(\frac{d^2H'}{d\eta'^2}\right)_0 = -\sqrt{2} a_1' (1+k_3') = s',$$

we have the local stresses

$$\tau = s\left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}} \frac{\rho}{2} U^2, \tag{82}$$

or

$$\tau = s'\left(\frac{Ux}{\nu'}\right)^{-\frac{1}{2}} \frac{\rho'}{2} U^2. \tag{83}$$

The computed values of s and s' as functions of r^2 are given in table 9.

TABLE 9.—*Values of the constants s and s′ in the expressions of the local stress at the interface, eq 82 and 83*

$r^2 = (\mu'\rho')/(\mu\rho)$

r^2	s	s'
0.00	----------	0.8672
.01	0.0782	.7825
.10	.1992	.6295
.3162	.2917	.5178
1.0	.3917	.3918
3.162	.4840	.2722
10	.5543	.1753
100	.6287	.0628
∞	.6612	----------

2. THICKNESSES OF THE LAYERS

Since the velocities in the laminar boundary layers approach the limiting values in an asymptotic manner, the definition of the thickness of the layers becomes a matter of convention. Of particular utility are the following definitions. See figure 4.

$$2\int_0^{y_s} (U-u)dy = (U-u_0)\delta_1, \tag{84}$$

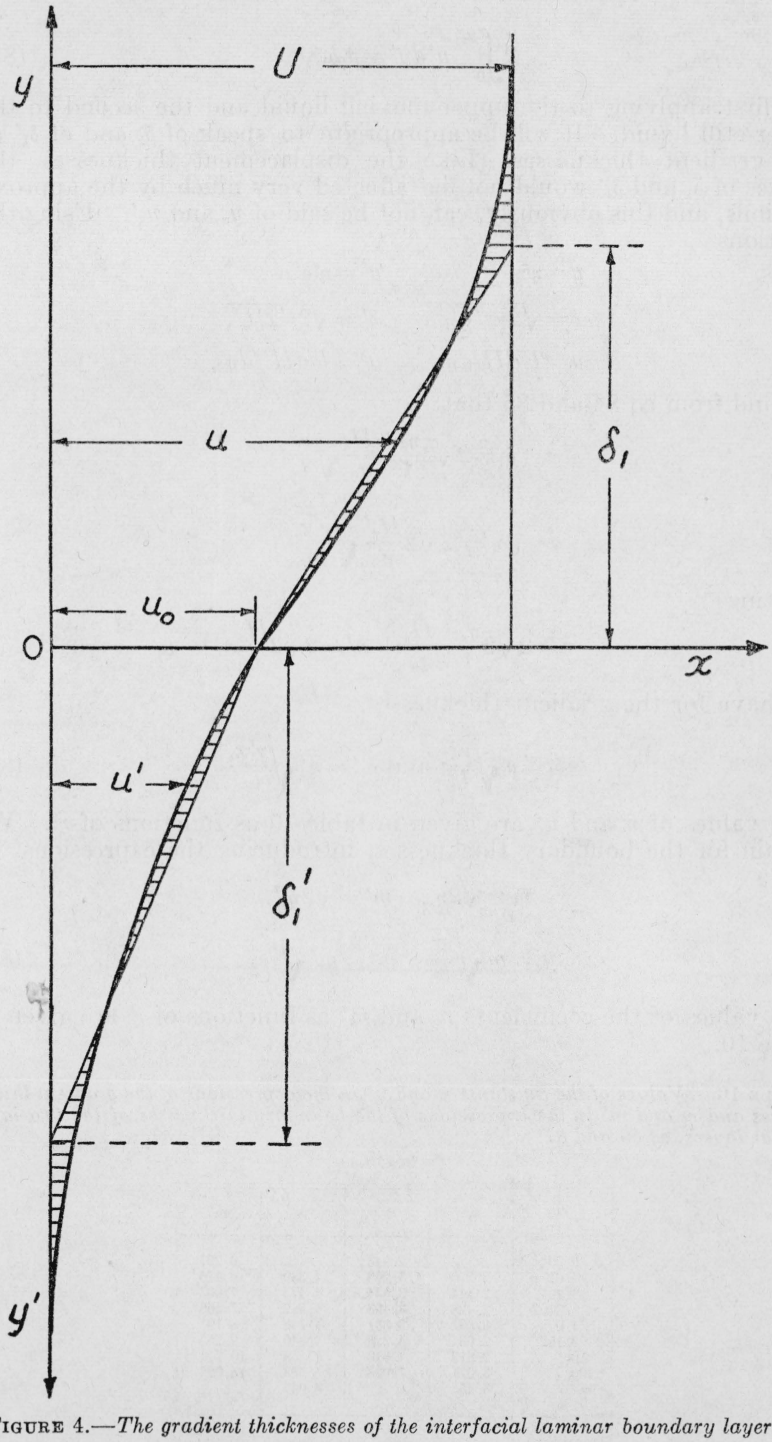

FIGURE 4.—*The gradient thicknesses of the interfacial laminar boundary layers.*

and

$$2\int_0^{y_s'} u'dy' = u_0\delta_1', \tag{85}$$

the first applying to the upper moving liquid and the second to the lower still liquid. It will be appropriate to speak of δ_1 and of δ_1' as the gradient thicknesses. Like the displacement thicknesses, the values of δ_1 and δ_1' would not be affected very much by the approximations, and this obviously, can not be said of y_s and y_s'. Using the relations

$$y = \eta\delta, \qquad y' = \eta'\delta',$$

$$\delta = \sqrt{2\nu x/U}, \qquad \delta' = \sqrt{2\nu' x/U},$$

$$u = UdH/d\eta, \qquad u' = UdH'/d\eta',$$

we find from eq 84 and 85 that

$$\delta_1 = 2\sqrt{2}\frac{\eta_s - H_s}{1-a_0}\sqrt{\frac{\nu x}{U}},$$

and

$$\delta_1' = 2\sqrt{2}\frac{H_s'}{a_0}\sqrt{\frac{\nu' x}{U}}.$$

Putting

$$n = 2\sqrt{2}\frac{\eta_s - H_s}{1-a_0}, \quad n' = 2\sqrt{2}\frac{H_s'}{a_0},$$

we have for the gradient thicknesses,

$$\delta_1 = n\sqrt{\frac{\nu x}{U}}, \text{ and } \delta_1' = n'\sqrt{\frac{\nu' x}{U}}. \tag{86}$$

The values of n and n' are given in table 10 as functions of r^2. We obtain for the boundary thicknesses, introducing the expressions

$$m = \sqrt{2}\eta_s, \quad m' = \sqrt{2}\eta_s':$$

$$y_s = m\sqrt{\frac{\nu x}{U}}, \quad y_s' = m'\sqrt{\frac{\nu' x}{U}}. \tag{87}$$

The values of the coefficients m and m' as functions of r^2 are given in table 10.

TABLE 10.—*Values of the constants n and n′ in the expressions of the gradient thickness and m and m′ in the expressions of the boundary thicknesses of the two laminar layers, eq 86 and 87*

$r^2 = (\mu'\rho')/(\mu\rho)$

r^2	n	n'	m	m'
0.00	----------	2.943	----------	6.291
.01	2.270	3.026	4.339	6.470
.10	2.344	3.304	4.472	7.003
.3162	2.428	3.493	4.582	7.466
1.0	2.531	3.831	4.748	8.191
3.162	2.664	4.328	4.946	9.255
10	2.813	4.940	5.158	10.721
100	3.071	7.063	5.533	15.090
∞	3.455	------	6.047	------

3. DISTRIBUTION OF THE VELOCITIES IN THE LAYERS

These velocities are determined from eq 42 and 46. The results of the computations for the upper liquid are given in table 11, where u/D is tabulated as a function of y/y_s for various values of r^2. Similar results for the lower liquid are given in table 12.

TABLE 11.—*Velocity distribution in the interfacial laminar layer of the upper, moving liquid*

The boundary thickness y_s is given by eq 87

$r^2=(\mu'\rho')/(\mu\rho)$

y/y_s \ r^2	0.010	0.100	0.316	1.000	3.16	10	100	∞
	u/U							
0.0	0.9282	0.8082	0.7088	0.5888	0.4619	0.3446	0.1739	0.0000
.1	.9449	.8521	.7747	.6807	.5808	.4862	.3466	.1994
.2	.9602	.8929	.8363	.7667	.6917	.6204	.5122	.3935
.3	.9735	.9279	.8892	.8412	.7891	.7387	.6609	.5723
.4	.9838	.9557	.9316	.9015	.8685	.8359	.7850	.7271
.5	.9913	.9758	.9626	.9457	.9273	.9089	.8797	.8440
.6	.9961	.9889	.9826	.9747	.9660	.9573	.9432	.9252
.7	.9988	.9962	.9933	.9910	.9878	.9848	.9801	.9727
.8	.9998	.9991	.9985	.9980	.9973	.9966	.9960	.9938
.9	.9999	.9999	.9998	.9998	.9997	.9996	.9996	.9995
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE 12.—*Velocity distribution in the interfacial laminar layer of the lower, still liquid*

The boundary thickness y_s' is given by eq 87

$r^2=(\mu'\rho')/(\mu\rho)$

y'/y_s' \ r^2	0.00	0.010	0.100	0.316	1.000	3.16	10	100
	u'/U							
0.0	1.0000	0.9282	0.8082	0.7088	0.5888	0.4619	0.3446	0.1739
.1	.7337	.6811	.5930	.5200	.4320	.3389	.2527	.1276
.2	.4980	.4624	.4023	.3530	.2932	.2301	.1716	.0866
.3	.3121	.2897	.2524	.2213	.1838	.1441	.1075	.0542
.4	.1765	.1639	.1427	.1250	.1039	.0814	.0608	.0306
.5	.0886	.0822	.0717	.0628	.0522	.0404	.0305	.0154
.6	.0369	.0343	.0298	.0262	.0217	.0171	.0131	.0064
.7	.0119	.0112	.0098	.0092	.0070	.0055	.0041	.0020
.8	.0026	.0025	.0022	.0021	.0016	.0011	.0009	.0004
.9	.0007	.0006	.0005	.0004	.0003	.0002	.0001	.0001
1.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

A more suggestive picture of the velocity distribution is obtained by plotting u/U against y/y_s, y now being the distance from the surface of separation to a point, above or below the surface, and y_s being the boundary thickness of the layer situated in the upper liquid. Figure 5 gives the velocities for identical liquids, $\mu=\mu'$ and $\rho=\rho'$. Figure 6 gives the velocities for $\nu'=7.85\nu$ and $\rho'=1.14\rho$. This case will be realized if we take water for the upper moving liquid and aqueous solution of 56 percent glycerol for the lower, still liquid, both being at 20° C. These two figures demonstrate the very large influence that the increased viscosity of the lower liquid has on the velocity distribution.

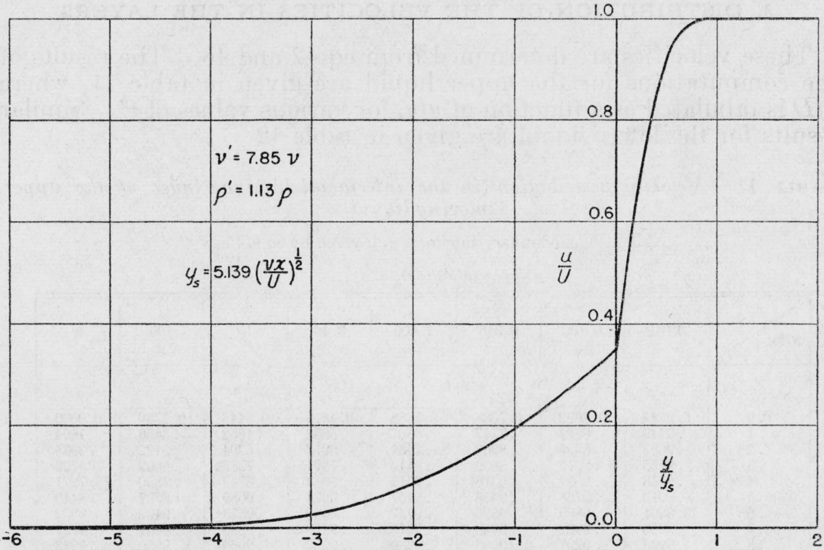

FIGURE 5.—*Interfacial laminar velocity distribution for the case $r^2=1$.*

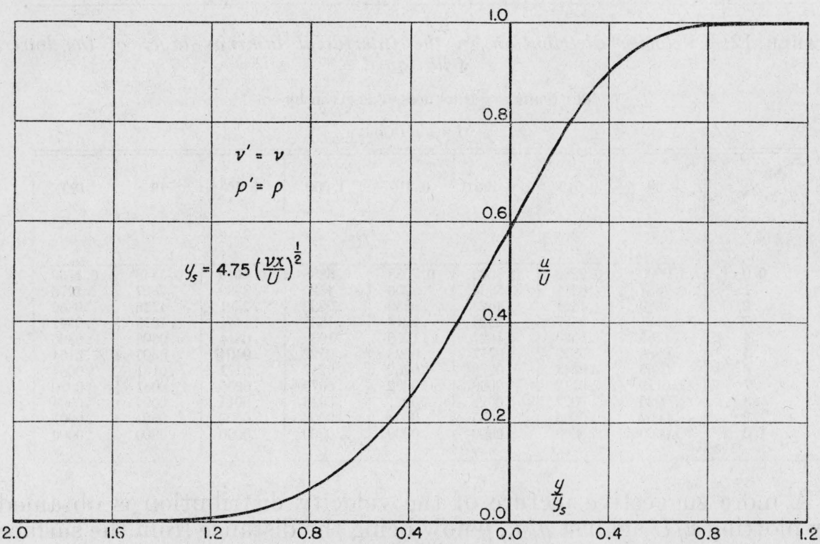

FIGURE 6.—*Interfacial laminar velocity distribution for the case $r^2=10$.*

X. TRANSVERSAL VELOCITIES OF THE LAYERS

The expressions of the transversal velocities, v and v', in terms of H and η, may be derived conveniently from the condition of continuity eq 4. Since at the interface, v and v' vanish, we have

$$v = -\int_0^y \frac{\partial u}{\partial x} dy.$$

This yields, using eq 13, 14, and 18, the relation

$$\frac{\delta v}{\nu} = \int_0^\eta \frac{d^2H}{d\eta^2} \eta d\eta.$$

Integrating by parts and noting that $H=0$ when $\eta=0$, we obtain

$$\frac{\delta v}{\nu} = \frac{dH}{d\eta} \eta - H, \tag{88}$$

which is the expression for the transversal velocities at the point (η, δ).

The corresponding expression for the lower fluid is

$$\frac{\delta' v'}{\nu'} = \frac{dH'}{d\eta'} \eta' - H'. \tag{89}$$

Thus, at the points of the lower liquid and in the region outside of the layer, $\eta' > \eta_s'$, there is a weak current moving normally upward toward the interface. The strength of the current is given by

$$v' = -\frac{\nu' H_s'}{\delta'}. \tag{90}$$

In a sense, the boundary layer of the upper liquid acts as a pump, raising the small portions of the lower liquid to the level of the interface, then causing these portions to move horizontally. So long as these motions as required by the theory are not interfered with, the steadiness of the layer will be assured.

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